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Coherent state realizations of su$(n+1)$ on the n-torus

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We obtain a new family of coherent state representations of SU$(n+1)$, in which the coherent states are Wigner functions over a subgroup of SU$(n+1)$. For representations of SU$(n+1)$ of the type $(\lambda, 0, 0,...)$, the basis functions are simple products of n exponential. The corresponding coherent state representations of the algebra su$(n+1)$ are also obtained, and provide a polar decomposition of su$(n+1)$ for any $n+1$. The su$(n+1)$ modules thus obtained are useful in understanding contractions of su$(n+1)$ and su$(n+1)$-phase states of quantum optics. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1479301]

I. INTRODUCTION

In this paper, we wish to present a new kind of coherent state[1] construction for the groups SU$(n+1)$. The construction is applicable to unitary irreducible representations (unirreps) of SU$(n+1)$ characterized by integral highest weights of the type $(\lambda, 0,...)$ for which there is no weight multiplicity, described by Young tableaux having a single row.

Our coherent states differ from the usual coherent states in that our basis functions are functions over a subgroup $\mathfrak{k}$ of SU$(n+1)$ rather than polynomials in holomorphic variables. Because there is no multiplicity of weights in SU$(n+1)$ unirreps of the type $(\lambda, 0,...)$, we can choose $\mathfrak{k}$ to be the Cartan subgroup of SU$(n+1)$. Basis functions for our modules are simple products of n exponential factors, and are closely related to the SU(3)$\supset$SO(3) construction of Ref. 2.

The realization of su$(n+1)$ that we obtain is particularly well-suited for a discussion of polar decompositions of su$(n+1)$ generators. We consider as an application a study of phase states,[3–6] and, in particular, of SU(2) and SU(3) phase states. The general case can be inferred from the discussion of the SU(3) case and from the results of Sec. III.

Coherent states are also useful in understanding the "semiclassical" behavior of systems.[7] Our construction can also be used to understand some of the possible asymptotic limits of quantum systems. For SU$(n+1)$ unirreps of the type $(\lambda, 0,...)$, which are applicable to $(n+1)$-channel interferometry,[6] the asymptotic limit corresponds to taking the number of (unpolarized) photons λ to be arbitrarily large. The parameters which enter in the explicit realization of the su$(n+1)$ generators will be related to the partition of λ photons between $n+1$ channels.

The construction is presented first for SU(2) in Sec. II. The general construction, valid for the irreps $(\lambda, 0,...)$ of SU$(n+1)$ is presented in Sec. III. Section IV contains an application to SU(3) of the general formalism. Our paper ends with a discussion containing further results and a short conclusion.

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II. SU(2)

A. Coherent state representation of the su(2) algebra

A basis for A_1, the complex extension of the su(2) algebra, is given in the usual way, by the three operators $\{\hat{h}_1, \hat{e}_+, \hat{e}_-\}$ with nonzero commutation relations

$$[\hat{h}_1, \hat{e}_\pm] = \pm 2\hat{e}_\pm, \quad [\hat{e}_+, \hat{e}_-] = \hat{h}_1. \tag{1}$$

For λ any positive integer, a highest weight $|\chi_\lambda\rangle$ for an irrep of dimension $\lambda+1$ (the number λ is just twice the spin of the representation) is defined by

$$\hat{h}_1 |\chi_\lambda\rangle = \lambda |\chi_\lambda\rangle, \quad \lambda \in \mathbb{Z}^+, \quad \hat{e}_+ |\chi_\lambda\rangle = 0. \tag{2}$$

Now it can be verified explicitly that the map Γ:

$$\begin{aligned}
\hat{h}_1 &\mapsto \Gamma(\hat{h}_1) = -i \frac{\mathrm{d}}{\mathrm{d}\varphi}, \\
\hat{e}_+ &\mapsto \Gamma(\hat{e}_+) = -\frac{1}{2} e^{2i\varphi} (\tan\beta)^{-1} \left(\lambda + i \frac{\mathrm{d}}{\mathrm{d}\varphi} \right), \\
\hat{e}_- &\mapsto \Gamma(\hat{e}_-) = -\frac{1}{2} e^{-2i\varphi} (\tan\beta) \left(\lambda - i \frac{\mathrm{d}}{\mathrm{d}\varphi} \right),
\end{aligned} \tag{3}$$

preserves the commutation relations of su(2) and is therefore a realization of A_1. A carrier space for this representation is the span of exponential functions $\{e^{i\nu\theta}, \nu = -\lambda, -\lambda+2, ..., \lambda-2, \lambda\}$. The highest and lowest weight state proportional to $e^{i\lambda\theta}$ and $e^{-i\lambda\theta}$, respectively.

To obtain Eq. (3), one first chooses some fixed but otherwise arbitrary (generic) angle β in the range $0 < \beta < 2\pi$. With β fixed, the state $R_y(\beta)|\chi_\lambda\rangle$, where $R_y(\beta) = e^{(\hat{e}_+ - \hat{e}_-)\beta}$, $R_z(\varphi) = e^{\varphi \hat{h}_1}$, is cyclic under the action of $R_z^{-1}(\varphi)$. $R_y(\beta)|\chi_\lambda\rangle$ then acts as a fiducial vector "translated" by $R_z^{-1}(\varphi)$.

Let $|\psi\rangle$ be an arbitrary state in the irrep with highest weight λ, and define the coherent state wave function for $|\psi\rangle$ by

$$|\psi\rangle \mapsto \psi_\beta(\varphi) \equiv \langle \chi_\lambda | R_y(\beta) R_z(\varphi) | \psi \rangle. \tag{4}$$

Since $\langle \chi_\lambda | \hat{e}_- = 0$, it is convenient to write $R_y(\beta)$ in antinormal-ordered form, so that, ignoring a normalization and a phase factor,

$$\psi_\beta(\varphi) = \langle \chi_\lambda | R_y(\beta) R_z(\varphi) | \psi \rangle \propto \langle \chi_\lambda | e^{\tan\beta \hat{e}_+} R_z(\varphi) | \psi \rangle. \tag{5}$$

The coherent state realization $\Gamma(\hat{X})$ of an operator $\hat{X}$ in su(2) is defined by

$$\hat{X} |\psi\rangle \to [\Gamma(\hat{X}) \psi]_\beta(\varphi) \equiv \langle \chi_\lambda | e^{\tan\beta \hat{e}_+} R_z(\varphi) \hat{X} | \psi \rangle. \tag{6}$$

Using $R_z(\varphi) = \exp(i\varphi \hat{h}_1)$, it follows immediately from this that

$$\hat{h}_1 \mapsto \Gamma(\hat{h}_1) = -i \frac{\mathrm{d}}{\mathrm{d}\varphi}, \tag{7}$$

since

$$\Gamma(\hat{h}_1) \psi_\beta(\varphi) = \langle \chi_\lambda | e^{\tan\beta \hat{e}_+} R_z(\varphi) \hat{h}_1 | \psi \rangle. \tag{8}$$

If $\hat{X}=\hat{e}_{\pm}$, we then have

$$[\Gamma(\hat{e}_{\pm})\psi_{\beta}](\varphi)=\langle\chi_{\lambda}|e^{\tan\beta\hat{e}_{+}}R_{z}(\varphi)\hat{e}_{\pm}|\psi\rangle=e^{\pm 2i\varphi}\langle\chi_{\lambda}|e^{\tan\beta\hat{e}_{+}}\hat{e}_{\pm}R_{z}(\varphi)|\psi\rangle. \tag{9}$$

The step which differentiates ours from the usual construction is to expand $\hat{e}_{\pm}$ as

$$\hat{e}_{\pm}=x_{\pm}e^{-\tan\beta\hat{e}_{+}}\hat{e}_{-}e^{\tan\beta\hat{e}_{+}}+y_{\pm}e^{-\tan\beta\hat{e}_{+}}\hat{h}_{1}e^{\tan\beta\hat{e}_{+}}+z_{\pm}\hat{h}_{1}, \tag{10}$$

where $x_{\pm}$, $y_{\pm}$, and $z_{\pm}$ are coefficients to be determined. This expansion is always possible since $\hat{e}_{\pm}$ is a traceless su(2) matrix and can therefore always be expanded in terms three linearly independent traceless matrices in A_1.

Before solving for the coefficients in Eq. (10), it is worth observing that, once substituted in Eq. (9), one obtains the simpler expression

$$\begin{aligned}[\Gamma(\hat{e}_{\pm})\psi_{\beta}](\varphi)&=e^{\pm 2i\varphi}(y_{\pm}\langle\chi_{\lambda}|\hat{h}_{1}e^{\tan\beta\hat{e}_{+}}R_{z}(\varphi)|\psi\rangle+z_{\pm}\langle\chi_{\lambda}|e^{\tan\beta\hat{e}_{+}}R_{z}(\varphi)\hat{h}_{1}|\psi\rangle)\\&=e^{\pm 2i\varphi}\left(\lambda y_{\pm}-iz_{\pm}\frac{\mathrm{d}}{\mathrm{d}\varphi}\right)\psi_{\beta}(\varphi),\end{aligned} \tag{11}$$

where Eq. (7), $\langle\chi_{\lambda}|\hat{h}_{1}=\lambda\langle\chi_{\lambda}|$ and $\langle\chi_{\lambda}|\hat{e}_{-}=0$ have been used.

Although they will depend on the parameter β, the coefficients $y_{\pm}$ and $z_{\pm}$ cannot depend on the particular choice of representation used to compute them, as long as the representation is faithful: if they did, commutation relations which would hold in a representation would not necessarily hold in another. Thus, one can compute these coefficients in the defining 2×2 representation, where

$$\hat{e}_{+}\mapsto\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \hat{e}_{-}\mapsto\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \hat{h}_{1}\mapsto\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad e^{\tan\beta\hat{e}_{+}}\mapsto\begin{pmatrix}1&\tan\beta\\0&1\end{pmatrix}. \tag{12}$$

For $\hat{X}=\hat{e}_{+}$, Eq. (10) yields the matrix system

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}=x_{+}\begin{pmatrix}-\tan\beta&-\tan^{2}\beta\\1&\tan\beta\end{pmatrix}+y_{+}\begin{pmatrix}1&2\tan\beta\\0&-1\end{pmatrix}+z_{+}\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{13}$$

It is immediately possible to solve for x_{+}, as it multiplies the only matrix with a nonzero entry below the diagonal. Knowing x_{+}, it is then easy to solve for y_{+} and z_{+}. The solution is simply $y_{+}=-z_{+}=\frac{1}{2}(\tan\beta)^{-1}$ so that the final expression for $\Gamma(\hat{e}_{+})$ corresponds to that given in Eq. (3). Repeating the steps for $\hat{e}_{-}$ yields $x_{-}=1$ and $y_{-}=-z_{-}=\frac{1}{2}\tan\beta$ so that $\Gamma(\hat{e}_{-})$ has the form given in Eq. (3).

B. Basis functions

First, we claim that the set of states $\{R_{z}^{-1}(\varphi)R_{y}^{-1}(\beta)|\chi_{\lambda}\rangle,\ R_{z}^{-1}(\varphi)\in\mathrm{U}(1),\ \beta\ \text{fixed}\}$, obtained by U(1) rotation of the state $R_{y}^{-1}(\beta)|\chi_{\lambda}\rangle$ through all possible angle φ, spans the carrier space V_{λ} for an irrep of SU(2) with highest weight λ. To show this, recall that V_{λ} is generated from $|\chi_{\lambda}\rangle$ by repeated action of the lowering operator e_{-}. Now,

$$R_{z}^{-1}(\varphi)R_{y}^{-1}(\beta)|\chi_{\lambda}\rangle\propto R_{z}^{-1}(\varphi)e^{\tan\beta\hat{e}_{-}}|\chi_{\lambda}\rangle\propto e^{\tan\beta e^{-2i\varphi}\hat{e}_{-}}|\chi_{\lambda}\rangle e^{i\lambda\varphi} \tag{14}$$

by using the normal form of $e^{-\beta(\hat{e}_{+}-\hat{e}_{-})}$. This can be seen to indeed generate the whole of V_{λ} (provided that $\tan\beta\neq 0$, which is our assumption about β being generic).

Thus, to any state $|\psi\rangle$ in V_{λ} there corresponds a unique coherent state wave function

$$|\psi\rangle\mapsto\psi_{\beta}(\varphi)\equiv\langle\chi_{\lambda}|R_{y}(\beta)R_{z}(\varphi)|\psi\rangle, \tag{15}$$

which belongs to the set of U(1) square-integrable functions. In particular, the basis functions $|\lambda\nu\rangle$ are given by $\psi_{\beta;\lambda\nu}(\varphi)=\langle\chi_\lambda|R_y(\beta)|\lambda\nu\rangle e^{i\nu\varphi}$ and must be proportional to the only normalized function on the half-circle with weight ν:

$$\psi_{\beta;\lambda\nu}(\varphi)\propto\frac{1}{\sqrt{\pi}}e^{i\nu\varphi}. \tag{16}$$

Note that we can restrict to the half circle because the difference of two weights in an invariant subspace is always an even integer.

C. Making the representation Hermitian

The representation of su(2) given in Eq. (3) is not Hermitian with respect to the natural U(1) inner product. If, as usual, the adjoint of $\hat{e}_+$ is taken as $\hat{e}_-$, i.e., $\hat{e}_+^\dagger=\hat{e}_-$, then

$$\langle\lambda\nu'|\Gamma(\hat{e}_+)|\lambda\nu\rangle\neq\langle\lambda\nu|\Gamma(\hat{e}_-)|\lambda\nu'\rangle^* \tag{17}$$

if

$$\langle\psi_\beta|\psi'_\beta\rangle=\int_0^\pi d\varphi\ \psi^*_\beta(\varphi)\psi'_\beta(\varphi). \tag{18}$$

However, since λ is integral, the representation Γ must be equivalent to a Hermitian representation γ, i.e., there must exist an intertwining operator $\mathcal{K}$ such that

$$\mathcal{K}^{-1}\Gamma\mathcal{K}=\gamma,\quad \text{with}\quad \langle\lambda\nu'|\gamma(\hat{e}_+)|\lambda\nu\rangle=\langle\lambda\nu|\gamma(\hat{e}_-)|\lambda\nu'\rangle^*. \tag{19}$$

To construct the operator $\mathcal{K}$, note that $\Gamma(\hat{h}_1)$ is actually Hermitian in the representation of Eq. (3), so that $\mathcal{K}^{-1}\Gamma(\hat{h}_1)\mathcal{K}=\Gamma(\hat{h}_1)=\gamma(\hat{h}_1)$. Thus, $\mathcal{K}$ commutes with $\Gamma(\hat{h}_1)$ and weight eigenstates of $\Gamma(\hat{h}_1)$ are also weight eigenstates of $\mathcal{K}$. Let

$$\hat{h}_1|\lambda\nu\rangle=\nu|\lambda\nu\rangle,\quad \mathcal{K}|\lambda\nu\rangle=\mathcal{K}_\nu|\lambda\nu\rangle. \tag{20}$$

Using Eq. (19), the Hermiticity condition reads

$$\langle\lambda,\nu+2|\gamma(\hat{e}_+)|\lambda\nu\rangle=\frac{1}{2\tan\beta}(\lambda-\nu)\frac{\mathcal{K}_\nu}{\mathcal{K}_{\nu+2}}=\langle\lambda\nu|\gamma(\hat{e}_-)|\lambda,\nu+2\rangle^*=\frac{1}{2}\tan\beta(\lambda+\nu+2)\frac{\mathcal{K}^*_{\nu+2}}{\mathcal{K}^*_\nu} \tag{21}$$

from which we conclude that the ratio of $\mathcal{K}_{\nu+2}/\mathcal{K}_\nu$ must satisfy, up to a phase that we choose to be $+1$,

$$\frac{\mathcal{K}_{\nu+2}}{\mathcal{K}_\nu}=\frac{1}{\tan\beta}\sqrt{\frac{\lambda-\nu}{\lambda+\nu+2}}, \tag{22}$$

so that γ is indeed Hermitian and given explicitly by

$$\langle\lambda,\nu+2|\gamma(\hat{e}_+)|\lambda\nu\rangle=\frac{1}{2}\sqrt{(\lambda+\nu+2)(\lambda-\nu)}=\langle\lambda\nu|\gamma(\hat{e}_-)|\lambda,\nu+2\rangle. \tag{23}$$

D. Application: Phase operators and phase states

Any matrix M can be factorized in polar form $U\cdot D$, with U a unitary matrix and D a semipositive definite diagonal matrix.[8] The operator D is always well-defined. The unitary matrix U is the exponential of a Hermitian ''phase'' operator associated with the phase of the observable described by the matrix M. The problems of constructing a phase operator in a finite or semi-infinite dimensional space are related to the lack of uniqueness in the definition of U which occurs when the rank of M is smaller than its dimension.

Our realization Γ acts in a natural way in the infinite dimensional irreducible space spanned by the set of U(1) functions (phase functions) $V_{\sigma_2}=\{e^{i(2p+\sigma_2)\varphi}/\sqrt{\pi},p\in\mathbb{Z}\}$, where σ_2 is the "duality" of the representation: $\sigma_2=0$ for bosons and $\sigma_2=1$ for fermions. Furthermore, the realization Γ of $\hat{e}_+$ or $\hat{e}_-$ can obviously be factored as a product of two operators. One may easily show that the "diagonal" part of the decomposition of $\Gamma(\hat{e}_\pm)$ obtained, to within a sign, from $\sqrt{\Gamma^\dagger(\hat{e}_\pm)\Gamma(\hat{e}_\pm)}$.

We are primarily interested in the matrix representation of the operator $\hat{E}_\varphi\equiv e^{2i\varphi}$. In the infinite-dimensional space V_{σ_2}, the matrix representation of $\hat{E}_\varphi$ contains zeroes everywhere, except immediately above the diagonal. $\hat{E}_\varphi$ is unitary with respect to the inner product of Eq. (18). Since $[\Gamma(\hat{h}_1),\hat{E}_\varphi]=\hat{E}_\varphi,\hat{E}_\varphi$ is the exponential of an Hermitian "phase" operator that is conjugate to $\hat{h}_1$.

The eigenstates of $\hat{E}_\varphi$, known as phase states,[3] are labeled by the continuous variable $\theta_\circ$ and given by

$$|\theta_\circ\rangle=\sum_{p=-\infty}^{\infty} e^{i(2p+\sigma_2)(\theta_\circ+\varphi)},\quad \hat{E}_\varphi|\theta_\circ\rangle=e^{-2i\theta_\circ}|\theta_\circ\rangle. \tag{24}$$

To obtain a finite dimensional Hermitian representation of su(2), we project from V_{σ_2} a finite dimensional subspace V^λ spanned by an appropriate subset of exponential functions. Rowe[9] has already observed that the appropriate projection operator is the intertwining operator $\mathcal{K}$ of Eq. (19). Since $\mathcal{K}_\nu=0$ for $|\nu|>\lambda$, $\mathcal{K}$ isolates from the set of all U(1) functions $\{e^{ip\varphi},p=-\infty,\ldots,\infty\}$ a subset of pertinent functions which form a basis for the *physical* $SU(2)$ *subspace* for the representation. $\mathcal{K}$ also adjusts the matrix elements of the various generators of the algebra so as to make γ Hermitian. Thus, the expression of γ in terms of an intertwining operator which acts as a projector ties in nicely with the work by Popov and collaborators[10] on phase operators in a finite dimensional subspace.

The restriction of $\hat{E}_\varphi$ to the finite-dimensional space V^λ is no longer unitary: the highest weight is annihilated by $\hat{E}_\varphi$ so that $\hat{E}_\varphi$ is now nilpotent, with the last line of its matrix representation containing only zeroes:

$$\hat{E}_\varphi=\begin{pmatrix} 0 & 1 & 0 & \ldots & \\ & 0 & 1 & 0 & \\ \vdots & & \ddots & 1 & \ddots \\ & & & & 1 \\ 0 & 0 & \ldots & & 0 \end{pmatrix}. \tag{25}$$

We would like to transform $\hat{E}_\varphi$ into a unitary matrix, but that transformation is not unique, as the rank of the matrix representation of $\gamma(\hat{e}_\pm)$ in V^λ is less than the dimension of this matrix. It is nevertheless possible to obtain a unitary operator closely related to $\hat{E}_\varphi$. The choice

$$E_{\hat{\varphi}}(\xi)=\begin{pmatrix} 0 & 1 & 0 & \ldots & \\ & 0 & 1 & 0 & \\ \vdots & & \ddots & 1 & \ddots \\ & & & & 1 \\ e^{i\xi} & 0 & \ldots & & 0 \end{pmatrix} \tag{26}$$

will produce a unitary matrix with determinant $e^{i\xi}$. The factor ξ is related to the phase of the vacuum state, which cannot be determined.

Vourdas[4] has done an extensive analysis of the case where $\xi=0$, which amounts to imposing a cyclic boundary condition by identifying $|\lambda,\lambda+2\rangle\sim|\lambda,-\lambda\rangle$. The case of general ξ does not differ significantly from this particular case where $\xi=0$: the eigenvalues and eigenvectors of the matrix of Eq. (26) are simply shifted by inessential phase factors. Thus, we set $\xi=0$ and define

$$E_{\hat{\varphi}}\equiv E_{\hat{\varphi}}(0), \tag{27}$$

so that $\det(E_{\hat{\varphi}})=1$. The notation indicates that $E_{\hat{\varphi}}$ is the exponential of a Hermitian "phase" operator $\hat{\varphi}$.

Phase states in the finite dimensional subspace V^{λ} are eigenstates of $E_{\hat{\varphi}}$. They are obtained by restricting the sum in Eq. (24) to those values of ν that correspond to states occurring in the su(2) irrep with highest weight λ:

$$|\lambda;\theta_\lambda\rangle=\sum_{\nu=-\lambda,-\lambda+2,\ldots,\lambda} e^{i\nu\theta_\lambda}|\lambda\,\nu\rangle=\frac{\sin((\lambda+1)(\varphi+\theta_\lambda))}{\sin(\varphi+\theta_\lambda)}, \quad \theta_\lambda=2\pi/(\lambda+1), \tag{28}$$

using $|\lambda\,\nu\rangle\mapsto e^{i\nu\varphi}$. The state $|\lambda;\theta_\lambda\rangle$ behaves like a periodic δ function as $\lambda\to\infty$, in accordance with the requirement of Ref. 5.

E. Application: Asymptotic SU(2) Wigner function

Let $\lambda\to\infty$ and set $\nu_0=\lambda\cos 2\beta$, i.e., set $\cos 2\beta=\nu_0/\lambda$ to its "classical value." Then

$$\lim_{\lambda\to\infty}\frac{\mathcal{K}_{\nu_0+p+2}}{\mathcal{K}_{\nu_0+p}}=\lim_{\lambda\to\infty}\tan\beta\sqrt{\frac{(\lambda+\nu_0+p+2)}{(\lambda-\nu_0-p)}}=\tan\beta\sqrt{\frac{1+\cos 2\beta}{1-\cos 2\beta}}=1+O(p/\lambda). \tag{29}$$

For finite values of p, we can therefore solve for $\mathcal{K}_{\nu_0+p}$ as $\mathcal{K}_{\nu_0+p}=1$. For finite p, the operators $\hat{e}_{\pm}$ are now represented by

$$\begin{aligned}\gamma(\hat{e}_+)&\to-\tfrac{1}{2}e^{2i\varphi}\lambda(1-\cos 2\beta)\operatorname{cotan}\beta=-\tfrac{1}{2}\lambda e^{2i\varphi}\sin 2\beta,\\ \gamma(\hat{e}_-)&\to-\tfrac{1}{2}e^{-2i\varphi}\lambda(1+\cos 2\beta)\tan\beta=-\tfrac{1}{2}\lambda e^{-2i\varphi}\sin 2\beta,\end{aligned} \tag{30}$$

and, in particular, we have $\gamma(\hat{L}_y)=i(\gamma(\hat{e}_+)-\gamma(\hat{e}_-))=\lambda\sin 2\beta\sin 2\varphi$. The reduced SU(2)-Wigner function, $\langle\lambda\,\nu|\exp(i\theta\hat{L}_y)|\lambda\,\nu'\rangle$, can therefore be written, in the limit, as

$$\begin{aligned}&\lim_{\lambda\to\infty}\langle\tfrac{1}{2}\lambda,\tfrac{1}{2}(\nu_0+p)|\exp(i\theta(\gamma(\hat{e}_+)-\gamma(\hat{e}_-)))|\tfrac{1}{2}\lambda,\tfrac{1}{2}(\nu_0+q)\rangle\\ &\quad\to\frac{1}{\pi}\int_0^{\pi}e^{i((1/2)(p-q))2\varphi}e^{i\theta\lambda\sin 2\beta\sin 2\varphi}\mathrm{d}\varphi\\ &\quad=J_{p-q}((-\lambda\sin 2\beta)\theta),\end{aligned} \tag{31}$$

where J_ν is a Bessel function and we have used an integral expression for J_ν found in Ref. 11. This result has been further investigated in Ref. 12.

III. GENERALIZATION TO SU$(N+1)$ IRREPS OF TYPE $(\lambda, 0,...)$

A. Algebraic formulation

In this section we generalize the above-mentioned construction to obtain a representation of su$(n+1)$ on the n-torus. We start by going to the complex extension of u$(n+1)$, spanned by the $(n+1)^2$ operators $\{\hat{C}_{ij}, i,j=1,...,n+1\}$ which satisfy the commutation relations

$$[\hat{C}_{ij}, \hat{C}_{kl}] = \delta_{jk}\hat{C}_{il} - \delta_{il}\hat{C}_{kj}. \tag{32}$$

The complex extension of su$(n+1)$ is obtained by selecting from the above set the operators $\hat{C}_{ij}$, $i \neq j$ and $\hat{h}_k$, where

$$\hat{h}_k = \hat{C}_{kk} - \hat{C}_{k+1,k+1}, \quad k=1,...,n. \tag{33}$$

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g} = sl(n+1,\mathbb{C})$ consisting of diagonal matrices. Let $(\lambda, 0, ...)$ be a dominant integral weight (with respect to $\mathfrak{h}$) and $|\chi_\lambda\rangle$ the highest weight vector of a representation on the space V which has only trivial weight multiplicities. Let $\mathfrak{s}$ be the stabilizer subalgebra of $\langle\chi_\lambda|$, i.e.,

$$\mathfrak{s} = \{s \in g \quad s.t. \ \langle\chi_\lambda|s = \alpha(s)\langle\chi_\lambda|\}, \tag{34}$$

where $\alpha(s) \in \mathbb{C}$. Note that the Cartan subalgebra $h \subset \mathfrak{s}$.

Choose and fix a generic element g in $SL(n+1,\mathbb{C})$ and construct another "twisted" copy of the Cartan subalgebra $g\mathfrak{h}g^{-1}$. The only condition on g must be that

$$\mathfrak{g} = \mathfrak{s} + g\mathfrak{h}g^{-1}, \tag{35}$$

i.e., it must be possible to expand an arbitrary element in $\mathfrak{g}$ as a sum of an element in $\mathfrak{s}$ and an element in $g\mathfrak{h}g^{-1}$. The coherent state representation of an operator $\hat{X} \in \mathfrak{g}$ is then defined by

$$\Gamma(\hat{X})\psi_g(k) \equiv \langle\chi_\lambda|g\, k\, \hat{X}|\psi\rangle, \quad k \in H. \tag{36}$$

Since H is just an n-dimensional torus, the group element $k \in H$ is parametrized by n angles $\varphi_1, \ldots, \varphi_n$ as in $k = \exp(i\Sigma_p \varphi_p \hat{h}_p)$, where p runs from $p=1,...,n$. We will abuse the notation and write the coherent state as a function of $\underline{\varphi} = [\varphi_1, \ldots, \varphi_n]$. With this notation we find that, for $\hat{X} = \hat{h}_k \in \mathfrak{h}$,

$$\Gamma(\hat{h}_k) = -i\frac{\partial}{\partial\varphi_k}. \tag{37}$$

If $\hat{X} = \hat{C}_{j\ell}$, $\ell \neq j$ so that $\hat{X} \notin \mathfrak{h}$, then we have

$$\begin{aligned}
\Gamma(\hat{C}_{j\ell})\psi_g(\varphi) &= \langle\chi_\lambda|g \exp(i\Sigma_k \varphi_k \hat{h}_k)\hat{C}_{j\ell}|\psi\rangle, \\
&= \exp(i\Sigma_k m_{j\ell}^k \varphi_k)\langle\chi_\lambda|g\hat{C}_{j\ell}\exp(i\Sigma\varphi_k\hat{h}_k)|\psi\rangle \\
&= \exp(i\Sigma_k m_{j\ell}^k \varphi_k)\langle\chi_\lambda|(g\hat{C}_{j\ell}g^{-1})g \exp(i\Sigma_k\varphi_k\hat{h}_k)|\psi\rangle,
\end{aligned} \tag{38}$$

where k runs from 1 to n and where

$$[\hat{h}_k, \hat{C}_{j\ell}] = m_{j\ell}^k \hat{C}_{j\ell} = \delta_{kj}\hat{C}_{k\ell} - \delta_{k+1,j}\hat{C}_{k+1,\ell} - \delta_{k\ell}\hat{C}_{jk} + \delta_{k+1,\ell}\hat{C}_{j,k+1} \tag{39}$$

for $j \neq k = 1,...,n+1$. With the understanding that $\varphi_0 = \varphi_{n+1} = 0$, the sum $\Sigma_{k=1}^n m_{j\ell}^k \varphi_k$ can be rewritten as

$$\sum_{k=1}^{n} m_{j\ell}^{k}\varphi_k=\varphi_j-\varphi_{j-1}-\varphi_\ell+\varphi_{\ell-1}. \tag{40}$$

In order to complete the description of our coherent state representation of $\mathfrak{g}$, we need to compute explicitly, for every $(j\ell)$, the decomposition

$$g\hat{C}_{j\ell}g^{-1}=\overbrace{\hat{s}_{j\ell}}^{\in\mathfrak{s}}+\overbrace{g\hat{d}_{j\ell}g^{-1}}^{\in g\mathfrak{h}\mathfrak{g}^{-1}}, \quad \hat{d}_{j\ell}=\sum_{k=1}^{n} d_{j\ell}^{k}\hat{h}_k, \tag{41}$$

as per Eq. (35). It is simpler (and equivalent) to compute

$$\hat{C}_{j\ell}=g^{-1}\hat{s}_{j\ell g}+\hat{d}_{j\ell}. \tag{42}$$

Substitution of (42) into Eq. (38) then yields

$$\begin{aligned}\Gamma(\hat{C}_{j\ell})\psi_g(\varphi)&=\exp(i\Sigma_k m_{j\ell}^{k}\varphi_k)\langle\chi_\lambda|(\hat{s}_{j\ell}g+\hat{d}_{j\ell})\exp(i\Sigma_k\varphi_k\hat{h}_k)|\psi\rangle,\\ &=\exp(i\Sigma_k m_{j\ell}^{k}\varphi_k)\langle\chi_\lambda|(\hat{s}_{j\ell}g\,\exp(i\Sigma_k\varphi_k\hat{h}_k)+g\,\exp(i\Sigma_k\varphi_k\hat{h}_k)\hat{d}_{j\ell})|\psi\rangle.\end{aligned} \tag{43}$$

It follows therefore that, in accordance with Eq. (37), $\hat{d}_{j\ell}$ will be a sum of differential operators in the variables φ_k, while the action on the left of $\hat{s}_{j\ell}$ will yield back $\langle\chi_\lambda|$ to within a normalization factor.

Again we observe that the expansion coefficients cannot depend on the choice of representation, so that we choose to work in the $(n+1)\times(n+1)$ representation where $|\chi_\lambda\rangle=(1,0,...,0)^t$. The computation is further facilitated if we observe that the dependence on g is actually only up to left multiplication of $\mathcal{S}$; hence we can write $g=S\cdot\overline{g}$, with $S\in\mathcal{S}$ in the stabilizer subgroup and $\overline{g}$ a conveniently chosen coset representative in $\mathcal{S}\backslash G$; a different choice of the representative $g'=s\cdot g$ will produce equivalent representations in which the coherent states are multiplied by a character $\chi(s)$. Then,

$$\hat{C}_{j\ell}=(\overline{g})^{-1}\hat{s}_{j\ell}\,\overline{g}+\hat{d}_{j\ell}. \tag{44}$$

If the highest weight vector $\langle\chi_\lambda|$ is the vector (1, 0,..., 0), then a general element $\hat{s}\in\mathfrak{s}$ and coset representative $\overline{g}$ have respective the matrix forms

$$\hat{s}\mapsto\begin{pmatrix} y & \mathbf{0}\\ \mathbf{x}^t & Y\end{pmatrix},\quad \overline{g}=\begin{pmatrix}1 & -\mathbf{v}\\ \mathbf{0}^t & \mathbb{1}\end{pmatrix}, \tag{45}$$

where Y is an $n\times n$ complex matrix, $\mathbf{x}=(x_2,x_3,\ldots,x_{n+1})$ is a complex $1\times n$ vector, $\mathbf{0}$ is the $\mathbf{1}\times n$ null vector, $y=-\mathrm{Tr}(Y)$, $\mathbf{v}=(v_2,v_3,\ldots,v_{n+1})$ is a complex vector, and $\mathbb{1}$ is the $n\times n$ unit matrix. [The matrix form of $\overline{g}$ can be compared with Eq. (12).]

The product $\overline{g}^{-1}\hat{s}\overline{g}$ is a matrix of the form

$$\begin{pmatrix} y+\mathbf{v}\cdot\mathbf{x} & -(y+\mathbf{v}\cdot\mathbf{x})\mathbf{v}+\mathbf{v}Y\\ \mathbf{x}^t & -\mathbf{x}\otimes\mathbf{v}+Y\end{pmatrix}, \tag{46}$$

where $\mathbf{v}\cdot\mathbf{x}$ is the usual scalar product and $\otimes$ denotes the outer product so that $\mathbf{x}\otimes\mathbf{v}$ is an $n\times n$ matrix.

We therefore seek to match the matrix expression of $\hat{C}_{j\ell}$ with the expansion

$$\begin{pmatrix} y+\mathbf{v}\cdot\mathbf{x} & -(y+\mathbf{v}\cdot\mathbf{x})\mathbf{v}+\mathbf{v}Y\\ \mathbf{x}^t & -\mathbf{x}\otimes\mathbf{v}+Y\end{pmatrix}+d, \tag{47}$$

where d is a diagonal matrix $d=\mathrm{diag}(d^1,...,d^{n+1})$ of zero trace.

For every different pair of indices (j,ℓ), $j\neq\ell$, and with $\mathbf{v}$ appearing as parameter (which does not depend on j, ℓ), we need to solve the above-given equation for y, Y, $\mathbf{x}$, d. These unknowns depend on j, ℓ but, to avoid overburdening the notation, we will keep writing y for $y_{j\ell}$, d^k for $d^k_{j\ell}$, etc., until we reach final formulas.

Using the form of $\hat{s}$, the highest weight state $(1,0,...)^t$ and Eq. (43), we find that the only coefficient in $\hat{s}$ that enters in the expression of $\Gamma(\hat{C}_{j\ell})$ is y. The only element in $\mathfrak{s}$ to have nonzero entry in position (1,1) is $\hat{h}_1$. As $\langle\chi_\lambda|\hat{h}_1=\langle\chi_\lambda|\lambda$, Eq. (43) simplies to

$$\Gamma(\hat{C}_{j\ell})=y_{j\ell}\exp(i\Sigma_k m_k^{j\ell}\varphi_k)\left(\lambda-i\sum_{k=1}^{n}z^k\frac{\partial}{\partial\varphi_k}\right),\quad y_{j\ell}z^k=d^k. \tag{48}$$

We divide the straightforward search for the solution into three subcases. It is also useful at this point to introduce an auxiliary set of $n+1$ vectors in the Cartan Lie-algebra $\mathfrak{h}$, given by

$$\hat{\rho}_k=-\sum_{j=1}^{k-1}j\hat{h}_j+\sum_{j=k}^{n}(n-j+1)\hat{h}_j=\mathrm{diag}(-1,...,-1,\overset{k\text{th-term}}{n},-1,...,-1),\quad k=1,...,n+1, \tag{49}$$

$$\Gamma(\hat{\rho}_k)=-i\left(\sum_{j=1}^{k-1}j\frac{\partial}{\partial\varphi_j}-\sum_{j=k}^{n}(n-j+1)\frac{\partial}{\partial\varphi_j}\right). \tag{50}$$

Note that, for $\hat{\rho}_{n+1}$, there is no contribution from the second sum in Eq. (49).

1. Case 1: $\hat{C}_{1\ell}$

Let $j=1$, $\mathbf{c}$ be the vector of components $(\mathbf{c})_k=\delta_{\ell k}$, $k=2,3,...,n+1$ and write

$$\hat{C}_{1\ell}=\begin{pmatrix}0&\mathbf{c}\\ \mathbf{0}^t&\mathbf{0}\end{pmatrix}, \tag{51}$$

From Eq. (47), $\mathbf{x}=\mathbf{0}$, $y+d^1=0$, and $Y+d=0$; Y is a diagonal matrix with entries $y=Y_{kk}=-d^k$.

From $\mathbf{v}(-y\mathbb{1}+Y)=\mathbf{c}$, we find first that $Y_{kk}=y$ for $k\neq\ell$, and, using the condition $y_{1\ell}+\mathrm{Tr}(Y)=0$, that $Y_{\ell\ell}=-ny$. Finally, from the nonzero component $\ell=k$ of $\mathbf{c}$, one obtains

$$y=\frac{-1}{(n+1)v_\ell}. \tag{52}$$

If Eq. (35) is to hold, then we must have $v_\ell\neq 0\ \forall\ell$. The matrix d is given by

$$d=\mathbf{diag}\left(-y,-y,\ldots,-y,\overbrace{ny}^{\ell\text{th}-\text{term}},-y\ldots,-y\right)=-y\hat{\rho}_\ell=\frac{-1}{(n+1)v_\ell}\hat{\rho}_\ell. \tag{53}$$

Therefore we finally have

$$\Gamma(\hat{C}_{1\ell})=y_{1\ell}e^{i(\varphi_1-\varphi_\ell+\varphi_{\ell-1})}(\lambda+\Gamma(\hat{\rho}_\ell)),\quad y_{1\ell}=\frac{-1}{(n+1)v_\ell}, \tag{54}$$

where we have found $\Sigma_{k=1}^n m_k^{1\ell}\varphi_k=\varphi_1-\varphi_\ell+\varphi_{\ell-1}$ using $\varphi_{n+1}=0$ and Eq. (40).

2. Case 2: $\hat{C}_{j\ell}$, $\ell \neq j$, ℓ, $j \geqslant 2$

We now write

$$\hat{C}_{j\ell} = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0}^t & c_{j\ell} \end{pmatrix}, \tag{55}$$

where $(c_{j\ell})_{mn} = \delta_{jm}\delta_{\ell n}$. Using Eq. (47), $\mathbf{x} = \mathbf{0}$ once more, so that $y + d^1 = 0$. Thus, the diagonal elements of Y are such that $Y_{kk} + d^k = 0$. The only off-diagonal entry in Y is a 1 in the jth row, ℓth column. (For consistency it is convenient to enumerate the entries of the $n\times n$ matrix Y_{jk} with j, $k = 2,...,n+1$.) From $\mathbf{v}(-y\mathbb{1} + Y) = 0$, we obtain the equations

$$v_k(-y + Y_{kk}) + v_j\delta_{\ell k} = 0, \tag{56}$$

from which we conclude that, if $k \neq \ell$, $Y_{kk} = y$. The coefficient $Y_{\ell\ell}$ is fixed by $y + \mathrm{tr}(Y) = 0$ to be $Y_{\ell\ell} = -ny$. Finally, from Eq. (56) with $\ell = k$, we conclude that

$$y = \frac{v_j}{(n+1)v_\ell}, \tag{57}$$

where $v_j \neq 0 \neq v_\ell$ by assumption. The diagonal matrix d is given by

$$d = \mathbf{diag}\left(-y, -y, \ldots, -y, \overbrace{ny}^{\ell\text{th}-\text{term}}, -y \ldots, -y \right) = -y\hat{\rho}_\ell = -\frac{v_j}{(n+1)v_\ell}\hat{\rho}_\ell \tag{58}$$

Summarizing, we find, using Eq. (40),

$$\Gamma(\hat{C}_{j\ell}) = y_{j\ell}e^{i(\varphi_j - \varphi_{j-1} - \varphi_\ell + \varphi_{\ell-1})}(\lambda + \Gamma(\hat{\rho}_\ell)), \quad y_{j\ell} = \frac{v_j}{(n+1)v_\ell}. \tag{59}$$

3. Case 3: $\hat{C}_{\ell 1}$, $\ell \neq 1$

In this case, the vector $\mathbf{x}^t$ has components $x_j = \delta_{\ell j}$. Thus, the scalar product $\mathbf{v}\cdot\mathbf{x} = v_\ell$, and the equation $y + v_\ell + d^1 = 0$ gives

$$d^1 = -(v_\ell + y). \tag{60}$$

From $\mathbf{v}(-y\mathbb{1} + v_\ell\mathbb{1} + \mathbf{Y}) = 0$, we conclude that

$$(-y - v_\ell)v_m + \sum_{k=1}^{n} v_k Y_{km} = 0, \quad m = 2,3,...,n+1. \tag{61}$$

Now, $(\mathbf{x}\otimes\mathbf{v})_{pm} = x_p v_m = \delta_{\ell p} v_m$. Thus, we have

$$0 = -(\mathbf{x}\otimes\mathbf{v})_{pm} + Y_{pm} + d^m\delta_{pm} = -\delta_{\ell p}v_m + Y_{pm} + d^m\delta_{pm}, \quad m = 2,3,...,n+1. \tag{62}$$

Multiplying Eq. (62) by v_p and summing over p, we obtain

$$0 = \sum_p (-\delta_{\ell p}v_p v_m) + \sum_p v_p Y_{pm} + \sum_p v_p d^m \delta_{pm}, \quad m = 2,3,...,n+1. \tag{63}$$

Using Eq. (61) and the fact that $v_m \neq 0$, this can be simplified to

$$0 = -v_\ell + y + v_\ell + d^m = y + d^m, \quad m = 2,3,...,n+1. \tag{64}$$

Now using the fact that $\Sigma_{k=1}^{n+1} d^k = 0$, we immediately find

$$y=-\frac{v_\ell}{n+1}, \tag{65}$$

so that, using Eq. (60), we find the matrix d to be

$$d=\mathrm{diag}(ny,-y,\ldots,-y)=-y\hat{\rho}_1=-\frac{v_\ell}{n+1}\hat{\rho}_1. \tag{66}$$

Hence,

$$\Gamma(\hat{C}_{\ell 1})=y_{\ell 1}e^{i(\varphi_\ell-\varphi_{\ell-1}-\varphi_1)}(\lambda+\Gamma(\hat{\rho}_1)),\quad y_{\ell 1}=\frac{v_\ell}{n+1},\quad \ell=2,\ldots,n+1,\quad \varphi_{n+1}\equiv 0. \tag{67}$$

B. Evaluating the v_k coefficients

The coefficients v_k are related to SU$(n+1)$ Wigner functions as follows. If $\nu=(\nu_1,\nu_2,\ldots,\nu_{n+1})$ denotes an ordered partition of λ, i.e., $\nu_1+\nu_2+\ldots+\nu_{n+1}=\lambda$ with ν_i a non-negative integer, then the set of states $\{\psi_\nu\}$, labeled by different partitions ν of λ, can be chosen as basis states for the irrep (λ, 0,..., 0) of su(n). These states satisfy

$$\hat{h}_k\psi_\nu=(\nu_k-\nu_{k+1})\psi_\nu, \tag{68}$$

and are uniquely identified by the set of eigenvalues of the operators $\hat{h}_k$.

Next, we need $g=\overline{g}\cdot S$, or $\overline{g}=S^{-1}g=\overline{g}$, with $\mathcal{S}^{-1}$ and G matrices of the form

$$S^{-1}=\begin{pmatrix} w & \mathbf{0} \\ \mathbf{Q}^t & X\end{pmatrix},\quad g=\begin{pmatrix} a & \mathbf{b} \\ \mathbf{c}^t & U\end{pmatrix}, \tag{69}$$

where $a=\langle\chi_\lambda|g|\chi_\lambda\rangle$ and $b_k=\langle\chi_\lambda|g|\psi_{\nu_k}\rangle$, with ψ_{χ_λ} the highest weight state of the $(n+1)$-dimensional defining representation (1, 0,..., 0) and ψ_{ν_k}, $k=2,\ldots,n+1$, the remaining basis states of this irrep. Thus we have

$$w=\frac{1}{\langle\chi_\lambda|g|\chi_\lambda\rangle},\quad v_k=\frac{\langle\chi_\lambda|g|\psi_{\nu_k}\rangle}{\langle\chi_\lambda|g|\chi_\lambda\rangle}. \tag{70}$$

C. Basis functions

We have already observed that, with $\overline{g}$ of the form of Eq. (45) and $v_j\neq 0\ \forall j$, then $(\overline{g})^{-1}|\chi_\lambda\rangle$ generates the whole representation space. Thus, the state $(\overline{g})^{-1}|\chi_\lambda\rangle$ acts as a cyclic vector for the irrep with highest weight (λ, 0, 0,...) under the action of any fixed element $k\in H$ in the Cartan subgroup. Hence, to every vector ψ in the representation space, there corresponds a unique function on H:

$$\psi\mapsto\psi_g(k)=\langle\chi_\lambda|g\,k|\psi\rangle. \tag{71}$$

The basis state ψ_ν of (λ, 0,..., 0) is mapped to the normalized element on the n-torus

$$\psi_\nu\mapsto\langle\chi_\lambda|\overline{g}k|\psi_\nu\rangle\propto\langle\chi_\lambda|\overline{g}|\psi_\nu\rangle\exp(i\Sigma_k(\nu_k-\nu_{k+1})\varphi_k)\mapsto\frac{\exp(i\Sigma_k(\nu_k-\nu_{k+1})\varphi_k)}{(2\pi)^{n/2}}. \tag{72}$$

The highest weight state ψ_{χ_λ} of (λ, 0,..., 0) is represented by $\psi_{\chi_\lambda}\mapsto e^{i\varphi_1}/(2\pi)^{n/2}$.

D. Making the representation Hermitian

States on the torus are naturally normalized with respect to the inner product

$$\langle \psi_{\nu'} | \psi_\nu \rangle = \frac{1}{(2\pi)^n} \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \cdots \int_0^{2\pi} d\varphi_n \exp(-i\Sigma_k(\nu'_k - \nu'_{k+1})\varphi_k) \exp(i\Sigma_p(\nu_p - \nu_{p+1})\varphi_p)$$

$$= \prod \delta_{\nu'_k - \nu_k - \nu'_{k+1} + \nu_{k+1}}. \tag{73}$$

However, with this inner product, the action of the operators $\Gamma(\hat{C}_{ij})$, $i \neq j$, is not Hermitian:

$$\langle \psi_{\nu'} | \Gamma(\hat{C}_{ij}) | \psi_\nu \rangle \neq \langle \psi_{\nu'} | \Gamma^\dagger(\hat{C}_{ji}) | \psi_\nu \rangle = \langle \psi_\nu | \Gamma(\hat{C}_{ji}) | \psi_{\nu'} \rangle^*; \tag{74}$$

the resulting representation does not exponentiate to a unitary representation of the group. Since all representations of $\mathrm{su}(n+1)$ having integral dominant weight are equivalent to Hermitian representations, there must exist an intertwining operator $\mathcal{K}$ that will transform Γ into a Hermitian representation γ, i.e., there exists $\mathcal{K}$

$$\gamma(\hat{C}_{ij}) = \mathcal{K}^{-1}\Gamma(\hat{C}_{ij})\mathcal{K}, \quad \text{such that} \quad \langle \psi_{\nu'} | \gamma(\hat{C}_{ij}) | \psi_\nu \rangle = \langle \psi_\nu | \gamma(\hat{C}_{ji}) | \psi_{\nu'} \rangle^*. \tag{75}$$

We find $\mathcal{K}$ by combining Eq. (75) with its Hermitian conjugate $\gamma^\dagger(\hat{C}_{ji}) = \mathcal{K}^\dagger \Gamma^\dagger(\hat{C}_{ji})(\mathcal{K}^{-1})^\dagger$, so that

$$\gamma(\hat{C}_{ij}) = \gamma^\dagger(\hat{C}_{ji}) \Rightarrow \Gamma(\hat{C}_{ij})\mathcal{S} = \mathcal{S}\Gamma^\dagger(\hat{C}_{ji}), \tag{76}$$

where $\mathcal{S} = \mathcal{K}\mathcal{K}^\dagger$ is a Hermitian operator. Noting that the Cartan elements $\hat{h}_k$ are represented under the map Γ by operators Hermitian with respect to the inner product of Eq. (73), we may take $\mathcal{S}$ to be diagonal in the weight basis: $\mathcal{S}|\psi_\nu\rangle = \mathcal{S}_\nu |\psi_\nu\rangle$. Thus, using Eq. (76), we obtain the condition

$$\langle \psi_{\nu'} | \Gamma(\hat{C}_{ij})\mathcal{S} | \psi_\nu \rangle = \langle \psi_{\nu'} | \mathcal{S}\Gamma^\dagger(\hat{C}_{ji}) | \psi_\nu \rangle \Rightarrow \frac{\mathcal{S}_\nu}{\mathcal{S}_{\nu'}} = \frac{y^*_{ji}(\lambda + \rho_i(\nu'))}{y_{ij}(\lambda + \rho_j(\nu))} \equiv R_{\nu,\nu'}, \tag{77}$$

where $\nu_k = \nu'_k + \delta_{ik} - \delta_{jk}$ and

$$\rho_i(\nu) = \langle \psi_\nu | \hat{\rho}_i | \psi_\nu \rangle = \sum_{k=1}^{n+1} (\rho_i)_{kk} \nu_k, \tag{78}$$

where $(\rho_i)_{kk}$ is the kth entry in the diagonal matrix ρ_i defined in Eq. (49).

To construct the coefficients $\mathcal{S}_\nu$ from the ratios of Eq. (77), one starts by (arbitrarily) fixing to $+1$ the coefficient of the highest weight corresponding to the trivial partition (λ, 0, 0,...); changing this would just change $\mathcal{S}$ by an overall multiplicative factor. Noting now that $\mathcal{S} = \mathcal{K}\mathcal{K}^\dagger$ is a positive Hermitian matrix, the ratio $\mathcal{K}_\nu / \mathcal{K}_{\nu'}$ can therefore be obtained, up to a phase, as the square root of the right-hand side of Eq. (77). The phase of the ratio $\mathcal{K}_\nu / \mathcal{K}_{\nu'}$ should be chosen so that the matrix elements of $\gamma(\hat{C}_{ij})$ are real, something that it is always possible to do. In practice, one chooses, without loss of generality, the element $g \in G$, from which y_{ij} is obtained, so that y_{ij} is always real. Assuming therefore that g is chosen in this way, we have v_k always real and

$$\frac{\mathcal{K}_\nu}{\mathcal{K}_{\nu'}} = \sqrt{\frac{y_{ji}(\lambda + \rho_i(\nu'))}{y_{ij}(\lambda + \rho_j(\nu))}}. \tag{79}$$

IV. APPLICATION TO SU(3)

A. Representation on the 2-torus

The su(3) Lie algebra is spanned in the usual way by the subset of u(3) operators comprising the ladder operators $\{\hat{C}_{ij}, i\neq j\}$ together with the Cartan generators $\hat{h}_1=\hat{C}_{11}-\hat{C}_{22}$, $\hat{h}_2=\hat{C}_{22}-\hat{C}_{33}$. The $\{\hat{C}_{ij}\}$ operators satisfy the general commutation relations $[\hat{C}_{ij},\hat{C}_{kl}]=\delta_{jk}\hat{C}_{il}-\delta_{il}\hat{C}_{kj}$.

The highest weight state of the irrep $(\lambda,0)$ is mapped to the state $|\lambda,0,0\rangle \mapsto e^{i\lambda\varphi_1}/2\pi$. More generally, a state $|\nu_1,\nu_2,\nu_3\rangle$, with $\nu_1+\nu_2+\nu_3=\lambda$, is mapped to

$$|\nu_1,\nu_2,\nu_3\rangle \mapsto \frac{e^{i(\nu_1-\nu_2)\varphi_1+i(\nu_2-\nu_3)\varphi_2}}{2\pi}=\frac{e^{i(\nu_1-\nu_2)\varphi_1+i(2\nu_2+\nu_1-\lambda)\varphi_2}}{2\pi}, \tag{80}$$

where the condition $\nu_1+\nu_2+\nu_3=\lambda$ has been used.

Following the parametrization of Ref. 13 for SU(3) elements, and using the fact that we can choose g to be such that the matrix elements are real, we find

$$w=\frac{1}{\cos\frac{1}{2}\beta_2},\quad v_2=\cos\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2,\quad v_3=\sin\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2. \tag{81}$$

Simple application of Eqs. (37), (54), (59), and (67) then yields

$$\begin{aligned}
&\Gamma(\hat{h}_1)=-i\frac{\partial}{\partial\varphi_1},\quad \Gamma(\hat{h}_2)=-i\frac{\partial}{\partial\varphi_2},\\
&\Gamma(\hat{C}_{12})=\frac{-1}{3\cos\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2}e^{i(2\varphi_1-\varphi_2)}\left[\lambda+i\frac{\partial}{\partial\varphi_1}-i\frac{\partial}{\partial\varphi_2}\right],\\
&\Gamma(\hat{C}_{21})=\frac{-\cos\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2}{3}e^{-i(2\varphi_1-\varphi_2)}\left[\lambda-2i\frac{\partial}{\partial\varphi_1}-\frac{\partial}{\partial\varphi_2}\right],\\
&\Gamma(\hat{C}_{13})=\frac{-1}{3\sin\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2}e^{i(\varphi_1+\varphi_2)}\left[\lambda+i\frac{\partial}{\partial\varphi_1}+2i\frac{\partial}{\partial\varphi_2}\right],\\
&\Gamma(\hat{C}_{31})=\frac{-\sin\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2}{3}e^{-i(\varphi_1+\varphi_2)}\left[\lambda-2i\frac{\partial}{\partial\varphi_1}-i\frac{\partial}{\partial\varphi_2}\right],\\
&\Gamma(\hat{C}_{23})=\frac{1}{3\tan\frac{1}{2}\beta_3}e^{i(-\varphi_1+2\varphi_2)}\left[\lambda+i\frac{\partial}{\partial\varphi_1}+2i\frac{\partial}{\partial\varphi_2}\right],\\
&\Gamma(\hat{C}_{32})=\frac{\tan\frac{1}{2}\beta_3}{3}e^{i(\varphi_1-2\varphi_2)}\left[\lambda+i\frac{\partial}{\partial\varphi_1}-i\frac{\partial}{\partial\varphi_2}\right].
\end{aligned} \tag{82}$$

The ratios of $\mathcal{K}$ matrix elements required to compute the matrix elements of $\gamma(\hat{C}_{ij})$ are $\mathcal{K}_\nu/\mathcal{K}_{\nu'}$, with $\nu'_k=\nu_k+\delta_{ik}-\delta_{jk}$. Using $\nu_1+\nu_2+\nu_3=\lambda$, they are given explicitly by

$$\frac{\mathcal{K}_{\nu_1\nu_2\nu_3}}{\mathcal{K}_{\nu_1+1,\nu_2,\nu_3-1}}=v_3\sqrt{\frac{\lambda+\rho_1(\nu')}{\lambda+\rho_3(\nu)}}=\sin\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2\sqrt{\frac{\lambda+2(\nu_1+1-\nu_2)+(\nu_2-\nu_3+1)}{\lambda-(\nu_1-\nu_2)-2(\nu_2-\nu_3)}}$$
$$=\sin\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2\sqrt{\frac{\nu_1+1}{\nu_3}}, \tag{83}$$

$$\frac{\mathcal{K}_{\nu_1\nu_2\nu_3}}{\mathcal{K}_{\nu_1\nu_2-1,\nu_3+1}}=\frac{v_2}{v_3}\sqrt{\frac{\lambda+\rho_3(\nu')}{\lambda+\rho_2(\nu)}}=\frac{1}{\tan\frac{1}{2}\beta_3}\sqrt{\frac{\nu_3+1}{\nu_2}}, \tag{84}$$

$$\frac{\mathcal{K}_{\nu_1\nu_2\nu_3}}{\mathcal{K}_{\nu_1+1,\nu_2-1,\nu_3}}=v_2\sqrt{\frac{\lambda+\rho_1(\nu')}{\lambda+\rho_2(\nu)}}=\cos\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2\sqrt{\frac{\nu_1+1}{\nu_2}}. \tag{85}$$

To obtain the matrix element $\gamma(\hat{C}_{13})$, for instance, one computes

$$\begin{aligned}
&\langle\psi_{\nu_1+1,\nu_2,\nu_3-1}|\gamma(\hat{C}_{13})|\psi_{\nu_1,\nu_2,\nu_3}\rangle\\
&=\int\frac{d\varphi_1}{2\pi}\int\frac{d\varphi_2}{2\pi}\exp[-i(\nu_1-\nu_2+1)\varphi_1-i(\nu_2-\nu_3-1)\varphi_2]\\
&\quad\times(\mathcal{K}^{-1}\Gamma(\hat{C}_{13})\mathcal{K})\exp[i(\nu_1-\nu_2)\varphi_1+i(\nu_2-\nu_3)\varphi_2],\\
&=\int\frac{d\varphi_1}{2\pi}\int\frac{d\varphi_2}{2\pi}\exp[-i(\nu_1-\nu_2+1)\varphi_1-i(\nu_2-\nu_3-1)\varphi_2]\\
&\quad\times\left(\frac{1}{\mathcal{K}_{\nu_1+1,\nu_2,\nu_3-1}}e^{i(\varphi_1+\varphi_2)}\frac{-1}{3\sin\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2}\left[\lambda+i\frac{\partial}{\partial\varphi_1}+2i\frac{\partial}{\partial\varphi_2}\right]\mathcal{K}_{\nu_1,\nu_2,\nu_3}\right)\\
&\quad\times\exp[-i(\nu_1-\nu_2)\varphi_1+i(\nu_2-\nu_3)\varphi_2]\\
&\quad\times\frac{-1}{3\sin\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2}[\lambda-(\nu_1-\nu_2)-2(\nu_2-\nu_3)]\sin\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2\sqrt{\frac{\nu_1+1}{\nu_3}},\\
&=-\sqrt{(\nu_1+1)\nu_3}.
\end{aligned} \tag{86}$$

B. Application: the SU(3)→$[\mathbb{R}^6]$U(1)^2 contraction

Consider the limit where $\lambda\to\infty$. Set

$$\overline{\nu_1}=\lambda\left(\cos\frac{1}{2}\beta_2\right)^2,\quad \overline{\nu_2}=\lambda\left(\sin\frac{1}{2}\beta_2\right)^2\left(\cos\frac{1}{2}\beta_3\right)^2,\quad \overline{\nu_3}=\lambda\left(\sin\frac{1}{2}\beta_2\right)^2\left(\sin\frac{1}{2}\beta_3\right)^2. \tag{87}$$

The angles β_2 and β_3 then provide a convenient way to parametrize the distribution of λ photons in three modes with mode i containing a large number ν_i of photons.

With this, it is readily seen that the values of the angles β_2, β_3 for which the representation Γ of Eq. (82) is singular correspond to a distribution such that at least one of the three fields contains no quanta. Provided that Γ is nonsingular, we then have, for values of ν and ν' sufficiently close to the average values $\overline{\nu}$,

$$\lim_{\lambda\to\infty}\frac{\mathcal{K}_{\nu}}{\mathcal{K}_{\nu'}}=1+O(1/\lambda). \tag{88}$$

The representation Γ is then Hermitian; the diagonal operators $\hat{h}_1$ and $\hat{h}_2$ remain unchanged, and the ladder generators become, in the limit where $\lambda\to\infty$,

$$\begin{aligned}
\Gamma(\hat{C}_{12})&\to-2\lambda e^{i(2\varphi_1-\varphi_2)}\sin\beta_2\cos\frac{1}{2}\beta_3,\\
\Gamma(\hat{C}_{21})&\to-2\lambda e^{-i(2\varphi_1-\varphi_2)}\sin\beta_2\cos\frac{1}{2}\beta_3,\\
\Gamma(\hat{C}_{13})&\to-2\lambda e^{i(\varphi_1+\varphi_2)}\sin\beta_2\sin\frac{1}{2}\beta_3,\\
\Gamma(\hat{C}_{31})&\to-2\lambda e^{-i(\varphi_1+\varphi_2)}\sin\beta_2\cos\frac{1}{2}\beta_3,\\
\Gamma(\hat{C}_{23})&\to2\lambda e^{i(-\varphi_1+2\varphi_2)}\left(\sin\frac{1}{2}\beta_2\right)^2\sin\beta_3,\\
\Gamma(\hat{C}_{32})&\to2\lambda e^{i(\varphi_1-2\varphi_2)}\left(\sin\frac{1}{2}\beta_2\right)^2\sin\beta_3.
\end{aligned} \tag{89}$$

All the ladder operators commute with one another, and the resultant algebra is $[\mathbb{R}^6]\mathrm{U}(1)^2$.

C. Application: Phase operators and SU(3) phase states

The realization Γ acts naturally in the irreducible infinite-dimensional space of functions V_{σ_3}, where a state $|n,m\rangle$, $n,m\in\mathbb{Z}$ is represented by the function over the 2-torus $|n,m\rangle\mapsto e^{i(2n-m)\varphi_1}e^{i(2m-n-\sigma_3)\varphi_2}/2\pi$. Here, $\sigma_3=0$, 1, or 2 is the "triality" of the representation.

We introduce three "phase-like" operators:

$$\hat{E}_{\varphi_{12}}=e^{i(2\varphi_1-\varphi_2)},\quad \hat{E}_{\varphi_{23}}=e^{i(-\varphi_1+2\varphi_2)},\quad \hat{E}_{\varphi_{13}}=e^{i(\varphi_1+\varphi_2)}. \tag{90}$$

In V_{σ_3}, the operators $\hat{E}_{\varphi_{12}}$ and $\hat{E}_{\varphi_{23}}$ are unitary with respect to the natural inner product over the 2-torus; they are the exponential of phase operators conjugate to $\hat{h}_1$ and $\hat{h}_2$, respectively, since

$$\left[\frac{1}{2}\Gamma(\hat{h}_1),\hat{E}_{\varphi_{12}}\right]=\hat{E}_{\varphi_{12}},\quad \left[\frac{1}{2}\Gamma(\hat{h}_2),\hat{E}_{\varphi_{23}}\right]=\hat{E}_{\varphi_{23}}. \tag{91}$$

Note that $[\frac{1}{2}\Gamma(\hat{h}_1),\hat{E}_{\varphi_{23}}]\neq0$, $[\frac{1}{2}\Gamma(\hat{h}_2),\hat{E}_{\varphi_{12}}]\neq0$.

The realization of an element of A_2, say, $\Gamma(\hat{C}_{12})$ can be expressed as products of a unitary and a diagonal matrix: $\Gamma(\hat{C}_{12})=-\hat{E}_{\varphi_{12}}\hat{e}_{12}$, where $\hat{e}_{12}=\sqrt{\Gamma^{\dagger}(\hat{C}_{12})\Gamma(\hat{C}_{12})}$.

In the infinite-dimensional space V_{σ_3}, the unitary operators $\hat{E}_{\varphi_{12}}$ and $\hat{E}_{\varphi_{23}}$ commute, and it is possible to find their common set of eigenvectors. It can be verified that, for any θ_1, θ_2, states of the type

$$|\theta_1,\theta_2\rangle = \sum_{n,m\in\mathbb{Z}} e^{i(2n-m)\theta_1} e^{i(2m-n)\theta_2}|n,m\rangle = \sum_{n,m\in\mathbb{Z}} e^{i(2n-m)(\varphi_1+\theta_1)} e^{i(2m-n)(\varphi_2+\theta_2)}, \tag{92}$$

are the simultaneous eigenstate of $\hat{E}_{\varphi_{12}}$, $\hat{E}_{\varphi_{23}}$, and $\hat{E}_{\varphi_{13}}$:

$$\begin{aligned} &\hat{E}_{\varphi_{12}}|\theta_1,\theta_2\rangle = e^{-i(2\theta_1-\theta_2)}|\theta_1,\theta_2\rangle, \quad \hat{E}_{\varphi_{23}}|\theta_1,\theta_2\rangle = e^{-i(-\theta_1+2\theta_2)}|\theta_1,\theta_2\rangle, \\ &\hat{E}_{\varphi_{13}}|\theta_1,\theta_2\rangle = e^{-i(\theta_1+\theta_2)}|\theta_1,\theta_2\rangle. \end{aligned} \tag{93}$$

The states $|\theta_1,\theta_2\rangle$ are therefore phase states.

Consider now the finite-dimensional subspace $V^\lambda \subset V_{\sigma_3}$ such that V^λ is the carrier space for a unirrep of highest weight (λ, 0). This subspace is projected using the $\mathcal{K}$ operator from V_{σ_3}. In going from the infinite-dimensional space V_σ to the finite-dimensional V^λ, a number of problems arise in connection with the definition and properties of the phase operators.

We denote the states in V^λ by three non-negative integers as per Eq. (80). First, however, we note that it is not difficult to properly define the radial part of an operator. For instance, the radial part $\hat{J}_{12}$ of $\gamma(\hat{C}_{12})$ is found from $\hat{J}_{12} = \sqrt{\gamma^\dagger(\hat{C}_{12})\gamma(\hat{C}_{12})}$.

In V^λ, the restrictions of the operators $\hat{E}_{\varphi_{ij}}$ are nilpotent and therefore no longer unitary. The rank of $\hat{E}_{\varphi_{ij}}$ is equal to $\dim(V^\lambda)-(\lambda+1)$ as there are $(\lambda+1)$ states annihilated by $\hat{E}_{\varphi_{ij}}$ [one state in each $\mathrm{su}(2)_{ij}$ subrepresentation occurring in the su(3) irrep (λ, 0)].

In contrast with the SU(2) case, where a single entry of $\hat{E}_\varphi$ could be changed so as to obtain the unitary operator $E_{\hat{\varphi}}$, an arbitrary complex linear combination of the $(\lambda+1)$ states annihilated by $\hat{E}_{\varphi_{ij}}$ yields another state annihilated by $\hat{E}_{\varphi_{ij}}$. Thus, we are left with infinitely many ways of transforming $\hat{E}_{\varphi_{ij}}$ into a unitary phase operator $E_{\hat{\varphi}_{ij}}$, even if we insist that the determinant of $E_{\hat{\varphi}_{ij}}$ be 1. Furthermore, it can easily be verified that, for $\lambda \geqslant 2$, the restriction of $\hat{E}_{\varphi_{ij}}$ is an operator that does not necessarily commute with the other $\hat{E}_{\varphi_{kl}}$ operators: $[\hat{E}_{\varphi_{ij}},\hat{E}_{\varphi_{kl}}] \neq 0$.

We point out that, in the matrix representation of $[\hat{E}_{\varphi_{12}},\hat{E}_{\varphi_{23}}]$ there are precisely λ entries which are 1 rather than zero in this commutator. The "faulty" nonzero matrix elements appear in positions corresponding to matrix elements of the type $\langle \nu_1+1,0,\nu_3-1|\hat{E}_{\varphi_{13}}|\nu_1,0,\nu_3\rangle$, i.e., matrix elements involving vacuum states in mode 2: the familiar problems associated with the construction of unitary phase operators in the presence of vacuum states are still present.

Thus, the number of "faulty" nonzero matrix elements in commutators of the type $[\hat{E}_{\varphi_{ij}},\hat{E}_{\varphi_{jk}}]$ will grow like λ, since the number of states having the vacuum in one mode grows like λ. On the other hand, the number of states in the irrep (λ, 0) grows like λ^2. The classical limit where $\lambda \to \infty$ corresponds to the limit where the phases commute, provided that we ignore the relatively small number of "faulty" nonzero matrix elements compared to the number of "correct" zero matrix elements. This relative number grows like 1/λ.

In particular, in the interpretation of Eq. (89), the realization Γ becomes singular for states near the vacuum state when $\lambda \to \infty$ limit. Hence, provided that the distribution of photons in an input state is such that the vacuum can be safely ignored, phase operators can be considered as commuting.

A similar result on the lack of commutativity between the *total* and relative su(2) phase operators in systems containing few photons has been obtained in Ref. 14. These authors found that commutativity was recovered in the classical limit. Our results are similar to those found in Ref. 14, albeit applicable to the case of noncommuting *relative* phases in a three-beam system.

The three-dimensional representation (1, 0) merits special attention. Besides providing an illustrative example for our previous discussion, this representation is the only one that allows commuting *unitary* phase operators while preserving the polar decomposition.

More precisely, in a system containing a total of $\lambda=\nu_1+\nu_2+\nu_3=1$ quantum, the explicit matrix realization of (some of) $\hat{C}_{ij}$ in terms of $\hat{E}_{\varphi_{ij}}\cdot\hat{J}_{ij}$ can easily be found:

$$\begin{array}{cccc}
(ij) & \hat{E}_{\varphi_i} & \hat{J}_{ij} & \hat{C}_{ij}=\hat{E}_{\varphi_{ij}}\cdot\hat{J}_{ij} \\
(1,2) & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
(2,3) & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\
(1,3) & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{array} \tag{94}$$

To compute $\hat{E}_{\varphi_{12}}$, for instance, one uses $\hat{E}_{\varphi_{12}}=e^{i(2\varphi_1-\varphi_2)}$, i.e., the "phase" part of $\Gamma(\hat{C}_{12})$, the basis states $\{\frac{1}{2\pi}e^{i\varphi_1},\frac{1}{2\pi}e^{i(-\varphi_1+\varphi_2)},\frac{1}{2\pi}e^{i\varphi_2}\}$, and the inner product

$$\langle\Psi|\Phi\rangle=\int_0^{2\pi}\int_0^{2\pi}\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\Psi^*\Phi. \tag{95}$$

The operators $\hat{E}_{\varphi_{ij}}$ are explicitly not unitary. Whereas $\hat{E}_{\varphi_{12}}\cdot\hat{E}_{\varphi_{23}}=\hat{E}_{\varphi_{13}}$ it is not true that $[\hat{E}_{\varphi_{12}},\hat{E}_{\varphi_{23}}]=\hat{E}_{\varphi_{13}}$: the phase operators do not commute.

There are many ways of turning $\hat{E}_{\varphi_{ij}}$ into a unitary operator $E_{\hat{\varphi}_{ij}}$ while still preserving the decomposition of $\hat{C}_{ij}$ into a phase and a diagonal part. What is unique of the (1,0) representation is that it is also possible to find unitary operators $\hat{E}_{\hat{\varphi}_{ij}}$ such that $\hat{E}_{\hat{\varphi}_{ij}}$ preserves the polar decomposition of $\hat{C}_{ij}$ and simultaneously produces commuting phase operators: $[\hat{E}_{\hat{\varphi}_{ij}},\hat{E}_{\hat{\varphi}_{kl}}]=0$. This remarkable choice is

$$E_{\hat{\varphi}_{12}}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\quad E_{\hat{\varphi}_{23}}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\quad E_{\hat{\varphi}_{13}}=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{96}$$

It is not possible to convert $\hat{E}_{\varphi_{ij}}$ into a unitary operator $E_{\hat{\varphi}_{ij}}$ that will have all of the above enumerated properties when the total number of photons is greater than 1.

This result on the existence of commuting unitary phase operators is expected, as the representation (1,0) is pertinent to the classical description of a three-channel interferometer,[15] for which the phases are (of course) expected to commute.

V. DISCUSSION AND CONCLUSION

In deriving the representation Γ acting over the functions defined over the maximal torus H of $G = \mathrm{SU}(n+1)$, the main hinge is the decomposition of the Lie algebra as per Eq. (35). This is equivalent to requiring that the dimension of the subgroup H or, equivalently, the rank of the group G [i.e., n in the case of $\mathrm{SU}(n+1)$] is exactly of complementary dimension to the stabilizer of the highest weight state.

The question arises now as to whether there are other representations and/or groups G which allow a similar decomposition. The answer is—unfortunately—no, i.e., we need some extensions to the above-presented picture in order to accommodate other groups. We discuss why there is such an obstruction in the Appendix.

The technique adopted in this paper has been limited in scope to unirreps of $\mathrm{SU}(n+1)$ with highest weights of the type (λ, 0,...). However, it is possible to extend the formalism presented here to general irreps by suitably enlarging the subgroup over which the coherent states are defined. Irreps of the type (λ, μ, 0,...) are particularly interesting as they can be expected to have applications to the description of polarized beams. For irreps with highest weight (λ, μ, 0,...), the appropriate subgroup $\mathfrak{k}$ of $\mathrm{SU}(n+1)$ is $S(U(2)\times U(1)\times \ldots \times U(1))$: the basis states and the representation Γ will then be expressed[16] in terms of Wigner functions over this subgroup.

The major result of this paper is a realization of the $\mathrm{su}(n+1)$ Lie algebra (or, more precisely, of the complex extension of this algebra), appropriate for irreps with integral highest weights of the form (λ, 0,...), for which basis functions and generators are expressed in terms of exponential functions and derivatives of phase angles. This would appear to be particularly suitable for applications to phase states, and for the study of the asymptotic limits of a representation and the appropriate limit of Wigner functions.

Although this has been done explicitly only for SU(2) and SU(3), the parameters which enter in the realization can be generally interpreted as projective coordinates, related to $\mathrm{SU}(n+1)$ Wigner functions, and understood physically as related to the distribution of λ photons between $n+1$ fields.

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APPENDIX A: HOW TO COUNT THE DIMENSION OF THE STABILIZER

In this section, we show why our construction only works for algebras which have A_n as their complexification, and for which representations of A_n our construction is possible.

Let $\mathfrak{h}$ be the Cartan subalgebra of a Lie algebra of one of the classical groups [i.e., $\mathrm{SU}(n)$, $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n)$ or their noncompact versions]. A good comprehensive review of Lie algebra structure is Refs. 17 and 18.

Let f_α be a lowering operator corresponding to the positive root α: in order that $f_\alpha \chi_\lambda \neq 0$ it is necessary and sufficient that $(\alpha,\lambda)\geqslant 0$, where $(\,,)$ is the Killing form. Notice that such lowering operators form a nilpotent subalgebra: we denote by $\mathfrak{t}_\lambda$ the subset of positive roots for which $(\lambda,\alpha)>0$.

Now, let $\{\alpha_i\}_{i=1\ldots l}$ be a set of simple (positive) roots, $\{\omega_i\}_{i=1\ldots l}$ the corresponding set of weights, i.e., $(\omega_i,\alpha_j)=\delta_{ij}$.

Any dominant weight and positive root can be written as

$$\lambda = \sum_{i=1}^{n} \lambda_i \omega_i , \quad \lambda_i \in \mathbb{N}_+ , \quad \alpha = \sum_{i=1}^{n} m_i \alpha_i , \quad m_i \in \mathbb{N}_+ . \tag{A1}$$

Let us fix a fundamental weight ω_k (which is equivalent to choosing a node on the Dynkin diagram).

For a given Lie algebra $\mathfrak{g}$ (i.e., for a given Dynkin diagram) one can compute the number of roots in $\mathfrak{t}_{\omega_k}$ and check to see, as we choose different nodes of the Dynkin diagram, if there are sufficiently many roots to allow a decomposition of the type proposed in Eq. (35).

The results are here below (we refer to the Planches in Ref. 18).

(1) SU$(n+1)$ (A_n): choosing the kth node there are $k(n+1-k)$ positive roots in $\mathfrak{t}_{\omega_k}$. The minimum for $\mathfrak{t}_{\omega_k}$ is n, which occurs when $k=1$ or $k=n$. These choices correspond, respectively, to the representations $(\lambda, 0,\ldots)$ or $(0,\ldots, 0,\lambda)$.

(2) SO$(2n+1)$ $(B_n, n>1)$: choosing the kth node there are $k+k(n-k)+k(n+1-k) =2k(n+1-k)$ positive roots in $\mathfrak{t}_{\omega_k}$. The minimum is $2n$, which is bigger than the rank n of the group: it is impossible to construct SO$(2n+1)$ on the torus.

(3) Sp$(2n)$ $(C_n, n>2)$: choosing the kth node there are $k(n-k)+k+k=k(n+2-k)$ positive roots. The minimum is $n+1$, and this is again greater than the rank n of Sp$(2n)$.

(4) SO$(2n)$ (D_n): choosing the kth node there are $k(n-k)+k+k=k(n+2-k)$ positive roots, the minimum is $n+1$, which is also greater than the rank of the group.

As we see, the minimal number is equal to the rank of the algebra only for A_n.

APPENDIX B: A COHOMOLOGICAL PERSPECTIVE ON $\mathcal{S}$-MATRIX THEORY

We wish to draw the attention of the reader to the following interesting "cohomological" interpretation of the solution for the operator $\mathcal{S}$.

In the construction of the coefficients $\mathcal{S}_\nu$ of the operator $\mathcal{S}$ from the ratios of Eq. (77) as described above, it is not *a priori* clear that the coefficient $\mathcal{S}_\nu$ corresponding to a nontrivial partition ν defined starting from the trivial partition does not depend on the particular "path" we have followed to reach the given partition ν. Indeed—in general—there are different ways of getting to a given partition ν starting from the trivial one; for instance, we have

$$(\lambda,0,0,0,\ldots)\overset{C_{21}}{\mapsto}(\lambda-1,1,0,0,\ldots)\overset{C_{31}}{\mapsto}(\lambda-2,1,1,0,\ldots) \quad \text{or} \tag{B1}$$

$$(\lambda,0,0,0,\ldots)\overset{C_{31}}{\mapsto}(\lambda-1,0,1,0,\ldots)\overset{C_{21}}{\mapsto}(\lambda-2,1,1,0,\ldots). \tag{B2}$$

We have to make sure that the coefficient $\mathcal{S}_{(\lambda-2,1,1,0,\ldots)}$ defined along these two different "paths" does not depend on the choice of path. We observe here that this in particular implies that the following *cocycle* condition holds

$$R_{\nu,\nu'}R_{\nu',\nu''}R_{\nu'',\nu}=1 \tag{B3}$$

for any partitions ν, ν', ν'' which are adjacent in the following sense: Two partitions ν, ν' of λ are said to be *adjacent* if there exist $i\neq j$ such that

$$\nu_k=\nu'_k+\delta_{ik}-\delta_{jk}, \quad k=1,\ldots,n+1. \tag{B4}$$

Let us verify this fact and consider the small loop

$$\nu\overset{C_{ij}}{\longrightarrow}\nu'\overset{C_{jk}}{\longrightarrow}\nu''\overset{C_{ki}}{\longrightarrow}\nu, \tag{B5}$$

$$\nu'_r=\nu_r+\delta_{ir}-\delta_{jr}, \quad \nu''_r=\nu'_r+\delta_{jr}-\delta_{kr}=\nu_k+\delta_{ir}-\delta_{kr}, \tag{B6}$$

and the associated cocycle condition

$$R_{\nu\nu'}R_{\nu\nu'}R_{\nu'\nu''}=\frac{y_{ji}^*(\lambda+\rho_i(\nu'))}{y_{ij}(\lambda+\rho_j(\nu))}\frac{y_{kj}^*(\lambda+\rho_j(\nu''))}{y_{jk}(\lambda+\rho_k(\nu'))}\frac{y_{ik}^*(\lambda+\rho_k(\nu))}{y_{ki}(\lambda+\rho_i(\nu''))}, \tag{B7}$$

which, according to Eq. (B3), should be 1.

Using the expressions for y_{ij}, it can be verified explicitly that, if we set $v_1\equiv 1$, then

$$\frac{(y_{ji})^*}{y_{ij}}=\left(\frac{v_j^*v_j}{v_i^*v_i}\right), \quad v_1\equiv 1. \tag{B8}$$

Using Eq. (B8), one sees at once that the y dependence drops out Eq. (B7). Moreover, since i, j, k are distinct indices, one checks also that

$$\rho_i(\nu'')=\rho_i(\nu'), \quad \rho_j(\nu'')=\rho_j(\nu), \quad \rho_k(\nu)=\rho_k(\nu'). \tag{B9}$$

Therefore Eq. (B7) is consistent with Eq. (B3). In a similar way, one can easily check that $R_{\nu,\nu'}=(R_{\nu',\nu})^{-1}$. This equation, together with Eq. (B7), define a *cocycle* over partitions.

[1] For reviews, see W.-M. Zhang, D. H. Feng, and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990); D. J. Rowe, Prog. Part. Nucl. Phys. **37**, 265 (1996).
[2] D. J. Rowe, R. LeBlanc, and J. Repka, J. Phys. A **22**, L309 (1989).
[3] R. Lynch, Phys. Rep. **256**, 367 (1995); J. Bergou and B. G. Englert, Ann. Phys. (N.Y.) **209**, 479 (1991); L. Susskind and J. Glogower, Physics (Long Island City, N.Y.) **1**, 49 (1964).
[4] A. Vourdas, Phys. Rev. A **41**, 1653 (1990).
[5] D. T. Pegg and S. M. Barnett, J. Mod. Opt. **44**, 225 (1997).
[6] B. C. Sanders, H. de Guise, D. J. Rowe, and A. Mann, J. Phys. A **32**, 7791 (1999).
[7] F. T. Arecchi *et al.*, Phys. Rev. A **6**, 2211 (1972); E. Lieb, Commun. Math. Phys. **31**, 327 (1973); R. Gilmore, J. Math. Phys. **20**, 891 (1979); B. Simon, Commun. Math. Phys. **71**, 247 (1980).
[8] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1985).
[9] D. J. Rowe, J. Math. Phys. **36**, 1520 (1995); D. J. Rowe and J. Repka, *ibid.* **32**, 2614 (1991).
[10] V. N. Popov and V. S. Yarunin, J. Mod. Opt. **39**, 1525 (1992).
[11] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer, Berlin, 1966); *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1965).
[12] D. J. Rowe, H. de Guise, and B. C. Sanders, J. Math. Phys. **42**, 2315 (2001).
[13] D. J. Rowe, B. C. Sanders, and H. de Guise, J. Math. Phys. **40**, 3604 (1997).
[14] A. Luis and L. L. Sanchez-Soto, Phys. Rev. A **48**, 4702 (1993).
[15] B. C. Sanders (private communication).
[16] H. de Guise and M. Bertola (unpublished).
[17] H. Samelson, *Notes on Lie algebras*, 2nd ed. (Springer, New York, 1990).
[18] N. Bourbaki, Groupes et Algèbres de Lie (1981), Vol. VI.