

$su(3)$ intelligent states as coupled $SU(3)$ coherent states

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Received 12 September 2011

Published 30 May 2012

Online at stacks.iop.org/JPhysA/45/244013

Abstract

We extend previous work on intelligent states and show how to construct $su(3)$ intelligent states by coupling $SU(3)$ coherent states. We also discuss some properties of the resulting states.

This article is part of a special issue of *Journal of Physics A: Mathematical and Theoretical* devoted to ‘Coherent states: mathematical and physical aspects’.

PACS numbers: 03.65.Fd, 02.20.Sv, 03.65.—w

1. Introduction

The objective of this paper is to show how intelligent states for some observables in the algebra $su(3)$ can be explicitly constructed from $SU(3)$ coherent states. Intelligent states for the observables $\hat{\Omega}$ and $\hat{\Lambda}$ are states for which the strict equality

$$\Delta\Omega\Delta\Lambda = \frac{1}{2} |\langle [\hat{\Omega}, \hat{\Lambda}] \rangle| \quad (1)$$

holds [1]. A state $|\psi\rangle$ that satisfies equation (1) also satisfies

$$(\hat{\Omega} - i\alpha\hat{\Lambda})|\psi\rangle = \kappa|\psi\rangle, \quad \alpha \in \mathbb{R}, \quad (2)$$

i.e. intelligent states are eigenstates of the (non-Hermitian) operator $\hat{\Omega} - i\alpha\hat{\Lambda}$.

Coherent states are examples of intelligent states for appropriately chosen observables. It is well known for instance that the harmonic oscillator coherent states satisfy $\Delta x \Delta p = \frac{1}{2}\hbar$, with a similar property also holding for suitably chosen observables evaluated in an angular momentum coherent state. Coherent states are a special case of intelligent states because, in addition to saturating the uncertainty relation of equation (1) or its angular momentum analogue, the observables also satisfy $\Delta x = \Delta p$ (in appropriate units). Intelligent states generalize coherent states in the sense that equation (1) still holds but in general $\Delta\Omega \neq \Delta\Lambda$.

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Intelligent states, like coherent states, are not orthogonal. They were originally introduced for $SU(2)$ by Aragone and collaborators [1]. They were constructed for $SU(2)$ using a non-unitary transformation by Rashid [2], and using polynomial states in [3]. For $SU(1, 1)$ coherent states, the construction can involve solving recursion relations [4]. Properties of $SU(2)$ and $SU(1, 1)$ intelligent states have been used in the context of interferometry [5].

The construction of $su(3)$ intelligent states is motivated in part by the resurgence of interest in systems with higher symmetries, such as the three-well (or n -well) BEC, multiple-path interferometers, etc. Concurrent with this is the considerable attention paid by the quantum information community to the study of uncertainty relations and their connection with entanglement [6] and other non-classical properties of the system. Intelligent states for observables in higher Lie algebras thus appear to be a ‘natural’ family of states to consider in order to explore such properties.

In addition, the time evolution of angular wave packets associated with diatomic rigid molecules or with a quantum axially symmetric rigid body has been studied using $su(2)$ intelligent states [7]. One could thus envisage that $su(3)$ and $su(n)$ intelligent states could be used to study the time evolution of more general wave packets.

We will show in this paper how $su(3)$ intelligent states can be constructed by coupling together three $SU(3)$ coherent states, in direct generalization of the method proposed in [8] for $SU(2)$ and extended to $SU(1, 1)$ in [9]. We illustrate our method by choosing observables that do not transform by an $su(2)$ subalgebra of $su(3)$, thus bypassing the restrictions found in [10]. Although $SU(3)$ Clebsch–Gordan (CG) technology can be quite formidable, the couplings we will require are of the simplest kind as we will restrict our discussion to states in representations of the type $(\lambda, 0)$. The coupling method offers some distinct advantages over other possible constructions as it relies only on known special functions but not on recursion relations [4], and does not hinge on finding a suitable nonlinear transformation [2]. The coupling method is also immediately generalizable to higher groups once the appropriate coherent states have been found.

Finally, we note that the literature sometimes refers to intelligent states as ‘minimum uncertainty states’ [11]. We will stay away from this qualification: strictly speaking, the minimum of $\Delta\Omega$ is 0 and reached by choosing a normalized eigenstate of $\hat{\Omega}$, something always possible in a finite-dimensional representation.

2. Review of $SU(3)$ coherent states

By restricting our discussion to $su(3)$ irreps of the type $(\lambda, 0)$, we may consider the $su(3)$ algebra to be spanned by the eight operators

$$\begin{aligned}\hat{C}_{ij} &= \hat{a}_i^\dagger \hat{a}_j, & i \neq j = 1, 2, 3, \\ \hat{h}_1 &= 2\hat{C}_{11} - \hat{C}_{22} - \hat{C}_{33}, \\ \hat{h}_2 &= \frac{1}{2}(\hat{C}_{22} - \hat{C}_3).\end{aligned}\tag{3}$$

constructed from harmonic oscillator creation and destruction operators. The elements in the algebra act in a natural way on three-dimensional harmonic oscillator states $|n_1 n_2 n_3\rangle$. For instance:

$$\hat{C}_{12}|n_1 n_2 n_3\rangle = \sqrt{(n_1 + 1)n_2}|n_1 + 1, n_2 - 1, n_3\rangle.\tag{4}$$

A set of basis states for the irrep $(\lambda, 0)$ of dimension $\frac{1}{2}(\lambda + 1)(\lambda + 2)$ is given by those three-dimensional harmonic oscillator states $\{|n_1 n_2 n_3\rangle, n_1 + n_2 + n_3 = \lambda\}$.

An element $R(\varpi)$ of the group $SU(3)$, with $\varpi \equiv (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma_1, \gamma_2)$, will be written in the form

$$\begin{aligned} R(\varpi) &= R_{12}(\alpha_1, \beta_1, -\alpha_1)R_{23}(\alpha_2, \beta_2, -\alpha_2)T(\alpha_3, \beta_3, \gamma_1, \gamma_2), \\ T(\alpha_3, \beta_3, \gamma_1, \gamma_2) &= R_{12}(\alpha_3, \beta_3, -\alpha_3) e^{i\gamma_1\hat{h}_1} e^{i\gamma_2\hat{h}_2}. \end{aligned} \tag{5}$$

Here, $R_{ab}(\vartheta, \varphi, \chi)$ is a $SU(2)$ subgroup transformation mixing modes a, b of the harmonic oscillator. The angles β_i range over $0 \leq \beta_i \leq \pi$. In the fundamental $(1, 0)$ representation, the subgroup transformations have the generic forms

$$R_{12}(\omega_{12}) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{23}(\omega_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}. \tag{6}$$

The $*$ are entries that make the blocks into $SU(2)$ transformations. The factored form of equation (5) can be verified by slightly modifying the factorization algorithm of [12].

The lowest weight state of the irrep $(\lambda, 0)$ is $|00\lambda\rangle$. It is isotropic under transformations of the type $T(\alpha_3, \beta_3, \gamma_1, \gamma_2)$ given in equation (5). Thus, we define an $SU(3)$ coherent state $|\omega\rangle$ as the ‘translated’ lowest weight state

$$|\omega\rangle \equiv R_{12}(\alpha_1, \beta_1, -\alpha_1)R_{23}(\alpha_2, \beta_2, -\alpha_2)|00\lambda\rangle. \tag{7}$$

3. $su(3)$ intelligent states

3.1. A choice of observables

Intelligent states are tied to the product of fluctuations of some specified observables. For $su(3)$, a completely general pair of observables would be overly complicated, would hide the simplicity of the procedure and would not allow for a comfortable discussion of some of the results. With this in mind, we consider the following pair:

$$\hat{A}' = \frac{2\pi}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \hat{B}' = \frac{2\pi i}{3\sqrt{3}} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \tag{8}$$

with the commutation relation

$$\hat{C}' = -i[\hat{A}', \hat{B}'] = \frac{4\pi^2}{9\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 0 \end{pmatrix}. \tag{9}$$

Besides being sufficiently simple yet not trivial, the physical motivation behind this choice is connected with properties of the eigenstates of \mathcal{A}' and \mathcal{B}' . If $|\Psi_i^{\mathcal{A}'}\rangle$ and $|\Phi_j^{\mathcal{B}'}\rangle$ denote any three-dimensional eigenvector of \hat{A}' and \hat{B}' , respectively, then these eigenvectors are said to be mutually unbiased [13]:

$$|\langle \Psi_i^{\mathcal{A}'} | \Phi_j^{\mathcal{B}'} \rangle|^2 = \frac{1}{3}. \tag{10}$$

In this perspective, \mathcal{A}' and \mathcal{B}' are the direct generalization of the Pauli matrices σ_x and σ_y , the eigenstates of which satisfy the overlap condition $|\langle \Psi_i^{\sigma_x} | \Phi_j^{\sigma_y} \rangle|^2 = \frac{1}{2}$.

For calculational simplicity, it is convenient to go to a basis where $\hat{C}' = -i[\hat{A}', \hat{B}']$ is diagonal. This is done through the transformation

$$\hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \frac{1-\sqrt{3}}{\sqrt{3-\sqrt{3}}} & \frac{1+\sqrt{3}}{\sqrt{3+\sqrt{3}}} \\ -1 & \frac{1}{\sqrt{3-\sqrt{3}}} & \frac{1}{\sqrt{3+\sqrt{3}}} \\ 1 & \frac{1}{\sqrt{3-\sqrt{3}}} & \frac{1}{\sqrt{3+\sqrt{3}}} \end{pmatrix}, \tag{11}$$

with the result that our final observables are

$$\begin{aligned}
 \hat{A} &= \hat{U}^{-1} \hat{A}' \hat{U} = -\frac{2\pi}{3} \begin{pmatrix} 0 & \frac{1}{\sqrt{3-\sqrt{3}}} & \frac{1}{\sqrt{3+\sqrt{3}}} \\ \frac{1}{\sqrt{3-\sqrt{3}}} & 0 & 0 \\ \frac{1}{\sqrt{3+\sqrt{3}}} & 0 & 0 \end{pmatrix}, \\
 \hat{B} &= \hat{U}^{-1} \hat{B}' \hat{U} = \frac{2\pi i}{3} \begin{pmatrix} 0 & \frac{1}{\sqrt{3-\sqrt{3}}} & \frac{-1}{\sqrt{3+\sqrt{3}}} \\ \frac{-1}{\sqrt{3-\sqrt{3}}} & 0 & 0 \\ \frac{1}{\sqrt{3+\sqrt{3}}} & 0 & 0 \end{pmatrix}, \\
 \hat{C} &= \hat{U}^{-1} \hat{C}' \hat{U} = \frac{4\pi^2}{9\sqrt{3}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 - \sqrt{3} & 0 \\ 0 & 0 & -1 + \sqrt{3} \end{pmatrix}.
 \end{aligned} \tag{12}$$

The equality that defines intelligence, equation (1), then reads

$$\Delta A \Delta B = \frac{1}{2} |\langle \hat{C} \rangle|. \tag{13}$$

3.2. Intelligent states in (1, 0)

In the three-dimensional space carried by the (fundamental) (1, 0) representation, the eigenvalue problem of equation (2) for the operator $\hat{A} - i\alpha\hat{B}$ can be solved to yield the eigenvectors $|\psi_k^1\rangle$ and eigenvalue κ_k given by

$$\begin{aligned}
 |\psi_1^1(\alpha)\rangle &= \mathcal{N}_1 \begin{pmatrix} 0 \\ \frac{1-\sqrt{3}}{\sqrt{2}}\mu^2 \\ 1 \end{pmatrix}, & \kappa_1 &= 0, \\
 |\psi_2^1(\alpha)\rangle &= \mathcal{N}_2 \begin{pmatrix} \sqrt{3+\sqrt{3}}\mu \\ \sqrt{2+\sqrt{3}}\mu^2 \\ 1 \end{pmatrix}, & \kappa_2 &= -\frac{2\pi}{3}\sqrt{1-\alpha^2}, \\
 |\psi_3^1(\alpha)\rangle &= \mathcal{N}_3 \begin{pmatrix} -\sqrt{3+\sqrt{3}}\mu \\ \sqrt{2+\sqrt{3}}\mu^2 \\ 1 \end{pmatrix}, & \kappa_3 &= \frac{2\pi}{3}\sqrt{1-\alpha^2},
 \end{aligned} \tag{14}$$

where

$$\mu = \frac{1+\alpha}{\sqrt{1-\alpha^2}} \in \mathbb{R} \tag{15}$$

and \mathcal{N}_k is a normalization constant. These eigenvectors can all be identified with coherent states. Indeed, a generic coherent state in the irrep (1, 0) is of the form

$$\begin{aligned}
 |\omega\rangle &= R_{12}(\alpha_1, \beta_1, -\alpha_1) R_{23}(\alpha_2, \beta_2, -\alpha_2) |001\rangle \\
 &= \begin{pmatrix} e^{-i(\alpha_1+\alpha_2)} \sin \frac{1}{2}\beta_1 \sin \frac{1}{2}\beta_2 \\ -e^{-i\alpha_2} \cos \frac{1}{2}\beta_1 \sin \frac{1}{2}\beta_2 \\ \cos \frac{1}{2}\beta_2 \end{pmatrix}.
 \end{aligned} \tag{16}$$

Thus, we will write

$$|\psi_k^1(\alpha)\rangle = R(\omega_k) |001\rangle, \tag{17}$$

where $\omega_k = (\alpha_{1k}, \beta_{1k}, \alpha_{2k}, \beta_{2k})$ are the angles defined so that equation (17) holds. A simple comparison with the eigenstates $|\psi_k^1\rangle$ produces the correct angles $\alpha_{1k}, \beta_{1k}, \alpha_{2k}, \beta_{2k}$. When $|\alpha| < 1$, we immediately see that $\alpha_{k1} = \alpha_{k2} = 0$ always. The other two angles are

$$\begin{aligned} \beta_{11} &= 0, & \tan \frac{1}{2}\beta_{21} &= -\frac{(1 - \sqrt{3})\mu^2}{\sqrt{2}}, \\ \tan \frac{1}{2}\beta_{12} &= -\frac{\sqrt{3 - \sqrt{3}}}{\mu}, & \tan \frac{1}{2}\beta_{22} &= -\mu\sqrt{3 + \sqrt{3} + (2 + \sqrt{3})\mu^2}, \\ \tan \frac{1}{2}\beta_{13} &= \frac{\sqrt{3 - \sqrt{3}}}{\mu}, & \tan \frac{1}{2}\beta_{23} &= -\mu\sqrt{3 + \sqrt{3} + (2 + \sqrt{3})\mu^2}. \end{aligned} \quad (18)$$

For $|\alpha| > 1$, the angles are easily determined from the magnitudes and phases of the appropriate ratios of entries. Thus we have, quite generally,

$$(\hat{\mathcal{A}} - i\alpha\hat{\mathcal{B}})R(\omega_k)|001\rangle = \kappa_k R(\omega_k)|001\rangle, \quad (19)$$

with κ_k the k th eigenvalue, which is real when $|\alpha| < 1$ but otherwise 0 or purely imaginary.

3.3. The coupling property

Now the eigenvalue problem is *linear* in the generators. Thus, if $|\psi_i^1(\alpha)\rangle$ and $|\psi_j^1(\alpha)\rangle$ are intelligent, so is the product $|\psi_i^1(\alpha)\rangle_1 |\psi_j^1(\alpha)\rangle_2$ of two ‘single-particle’ intelligent states. More formally, if we define the tensor product states

$$|\psi_i^1(\alpha)\rangle_1 |\psi_j^1(\alpha)\rangle_2 \equiv |\psi_i^1(\alpha)\rangle_1 \otimes |\psi_j^1(\alpha)\rangle_2 \quad (20)$$

and the (collective) operators

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \mathcal{A}_2, \quad \mathcal{B} = \mathcal{B}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \mathcal{B}_2, \quad (21)$$

with

$$\begin{aligned} \mathcal{A}|\psi_i^1(\alpha)\rangle_1 |\psi_j^1(\alpha)\rangle_2 &= (\mathcal{A}_1 \otimes \mathbb{1}_2)(|\psi_i^1(\alpha)\rangle_1 \otimes |\psi_j^1(\alpha)\rangle_2) \\ &\quad + (\mathbb{1}_1 \otimes \mathcal{A}_2)(|\psi_i^1(\alpha)\rangle_1 \otimes |\psi_j^1(\alpha)\rangle_2), \end{aligned} \quad (22)$$

$$= (\mathcal{A}_1|\psi_i^1(\alpha)\rangle_1) \otimes |\psi_j^1(\alpha)\rangle_2 + |\psi_i^1(\alpha)\rangle_1 \otimes (\mathcal{A}_2|\psi_j^1(\alpha)\rangle_2), \quad (23)$$

$$= (\kappa_i + \kappa_j)|\psi_i^1(\alpha)\rangle_1 |\psi_j^1(\alpha)\rangle_2, \quad (24)$$

we see that the product of intelligent states is also intelligent.

4. All the intelligent states of $(\lambda, 0)$

We start by noting that the q -fold product

$$|00q\rangle = |001\rangle_1 \otimes |001\rangle_2 \otimes \cdots \otimes |001\rangle_q \quad (25)$$

is the lowest weight for $(q, 0)$. By the previous argument, it follows that

$$|\psi_k^q(\alpha)\rangle = R(\omega_k)|00q\rangle \quad (26)$$

$$= (R(\omega_k)_1|001\rangle_1) \otimes \cdots \otimes (R(\omega_k)_q|001\rangle_q) \quad (27)$$

is also coherent and simultaneously intelligent (we use the round ket to denote an unnormalized state).

Let us use this to construct the six intelligent states of \mathcal{A} and \mathcal{B} for the irrep $(2, 0)$. The eigenvalue problem takes the matrix form

$$-\frac{2\pi}{3}(\hat{\mathcal{A}} - i\alpha\hat{\mathcal{B}}) = \begin{pmatrix} 0 & (1-\alpha)\eta_- & 0 & (1+\alpha)\eta_+ & 0 & 0 \\ (1+\alpha)\eta_- & 0 & (1-\alpha)\eta_- & 0 & \frac{(1+\alpha)\eta_+}{\sqrt{2}} & 0 \\ 0 & (1+\alpha)\eta_- & 0 & 0 & 0 & 0 \\ (1-\alpha)\eta_+ & 0 & 0 & 0 & \frac{(1-\alpha)\eta_-}{\sqrt{2}} & (1+\alpha)\eta_+ \\ 0 & \frac{(1-\alpha)\eta_+}{\sqrt{2}} & 0 & \frac{(1+\alpha)\eta_-}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & (1-\alpha)\eta_+ & 0 & 0 \end{pmatrix},$$

with $\eta_+ = \sqrt{\frac{2}{3+\sqrt{3}}}$ and $\eta_- = \sqrt{\frac{2}{3-\sqrt{3}}}$.

Rather than direct diagonalization, we use the coupling property to verify that the product

$$|\psi_{k\ell}^2(\alpha)\rangle = |\psi_k^1(\alpha)\rangle_1 |\psi_\ell^1(\alpha)\rangle_2 + |\psi_\ell^1(\alpha)\rangle_1 |\psi_k^1(\alpha)\rangle_2 \quad (28)$$

is contained in the irrep $(2, 0)$ and is intelligent with the eigenvalue $\kappa_k + \kappa_\ell$. Clearly there are six such states, symmetric under permutation of the ‘particle index’; by construction they will be eigenstates of $\mathcal{A} - i\alpha\mathcal{B}$ and thus intelligent. They must therefore be all the intelligent states of $(2, 0)$.

More generally, we start with the product

$$|\psi_{\lambda_1\lambda_2\lambda_3}(\alpha)\rangle = [R(\omega_1)|00\lambda_1\rangle_1] \otimes [R(\omega_2)|00\lambda_2\rangle_2] \otimes [R(\omega_3)|00\lambda_3\rangle_3]. \quad (29)$$

This state is intelligent by construction but belongs to the decomposable representation $(\lambda_1, 0) \otimes (\lambda_2, 0) \otimes (\lambda_3, 0)$. We can project it to the irreducible space $(\lambda_1 + \lambda_2 + \lambda_3, 0)$ using Clebsch–Gordan technology.

First, we write

$$R(\omega_k)|00\lambda_k\rangle = \sum_{v_1 v_2} |v_1 v_2 v_3\rangle D_{v_1 v_2 v_3; 00\lambda_k}^{(\lambda_k, 0)}(\omega_k), \quad (30)$$

where v_3 is determined by $v_3 = \lambda_k - v_2 - v_1$, and $D_{v_1 v_2 v_3; 00\lambda_k}^{(\lambda_k, 0)}(\omega_k)$ is the $SU(3)$ D -function described in [12] and given by the overlap

$$\langle v_1 v_2 v_3 | R_{12}(\alpha_{1k}, \beta_{1k}, -\alpha_{1k}) R_{23}(\alpha_{2k}, \beta_{2k}, -\alpha_{2k}) | 00\lambda_k \rangle, \quad (31)$$

and ω_k is specified via equation (17). Next, we couple in sequence:

$$|v_1 v_2 v_3\rangle_1 |\mu_1 \mu_2 \mu_3\rangle_2 |\tau_1 \tau_2 \tau_3\rangle_3 = |v_1 v_2 v_3\rangle_1 |M_1 M_2 M_3\rangle_{23} \langle \mu_1 \mu_2 \mu_3; \tau_1 \tau_2 \tau_3 | M_1 M_2 M_3 \rangle, \quad (32)$$

$$= |N_1 N_2 N_3\rangle \langle v_1 v_2 v_3; M_1 M_2 M_3 | N_1 N_2 N_3 \rangle \langle \mu_1 \mu_2 \mu_3; \tau_1 \tau_2 \tau_3 | M_1 M_2 M_3 \rangle, \quad (33)$$

where $\langle a_1 a_2 a_3; b_1 b_2 b_3 | c_1 c_2 c_3 \rangle$ is the $SU(3)$ CG coefficient $C_{(q,0), a_1 a_2 a_3; (p,0), b_1 b_2 b_3}^{(p+q,0) c_1 c_2 c_3}$. Note that because all kets (including those occurring in the intermediate and final coupling) belong to an irrep of the type $(q, 0)$ that does not have any weight multiplicity, a simple listing of the triple $n_1 n_2 n_3$ is enough to uniquely identify the state and its weight. The product does not contain any sum because the weights of the intermediate and final states are completely specified by the weights of the initial states. Hence, we denote by $|\psi_{\lambda_1\lambda_2\lambda_3}^{\lambda_1+\lambda_2+\lambda_3}\rangle$ the intelligent state of the irrep $(\lambda_1 + \lambda_2 + \lambda_3, 0) \equiv (\lambda, 0)$ constructed from the coupling of $|\psi_1^{\lambda_1}\rangle |\psi_2^{\lambda_2}\rangle |\psi_3^{\lambda_3}\rangle$. Its explicit

expression is given by

$$\begin{aligned}
 |\psi_{\lambda_1\lambda_2\lambda_3}^\lambda(\alpha)\rangle &= \sum_{N_1N_2} |N_1N_2N_3\rangle F(N_1N_2N_3) \\
 F(N_1N_2N_3) &= \sum_{\nu_2} \langle 0\nu_2\nu_3; M_1M_2M_3|N_1N_2N_3\rangle D_{0\nu_2\nu_3;00\lambda_1}^{(\lambda_1,0)}(\omega_1) \\
 &\quad \times \sum_{\mu_1\mu_2} \langle \mu_1\mu_2\mu_3; \tau_1\tau_2\tau_3|M_1M_2M_3\rangle D_{\mu_1\mu_2\mu_3;00\lambda_2}^{(\lambda_2,0)}(\omega_2) D_{\tau_1\tau_2\tau_3;00\lambda_3}^{(\lambda_3,0)}(\omega_3), \quad (34)
 \end{aligned}$$

with $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, $\tau_k = M_k - \mu_k = N_k - \mu_k - \nu_k$.

The $SU(3)$ D -functions are given in terms of $SU(2)$ d -functions [12]:

$$\begin{aligned}
 D_{\nu_1\nu_2\nu_3;00\lambda_1}^{(\lambda_1,0)}(\omega_1) &= \delta_{\nu_10} d_{\frac{1}{2}(\nu_1+\nu_2-\nu_3),-\frac{1}{2}\lambda_1}^{\frac{1}{2}\lambda_1}(\beta_{21}) \\
 D_{\mu_1\mu_2\mu_3;00\lambda_2}^{(\lambda_2,0)}(\omega_2) &= d_{\frac{1}{2}(\mu_1-\mu_2),-\frac{1}{2}(\mu_1+\mu_2)}^{\frac{1}{2}(\mu_1+\mu_2)}(\beta_{12}) d_{\frac{1}{2}(\mu_1+\mu_2-\mu_3),-\frac{1}{2}\lambda_2}^{\frac{1}{2}\lambda_2}(\beta_{22}), \quad (35) \\
 D_{\tau_1\tau_2\tau_3;00\lambda_3}^{(\lambda_3,0)}(\omega_3) &= d_{\frac{1}{2}(\tau_1-\tau_2),-\frac{1}{2}(\tau_1+\tau_2)}^{\frac{1}{2}(\tau_1+\tau_2)}(\beta_{13}) d_{\frac{1}{2}(\tau_1+\tau_2-\tau_3),-\frac{1}{2}\lambda_3}^{\frac{1}{2}\lambda_3}(\beta_{23}),
 \end{aligned}$$

where

$$d_{M,-J}^J(\beta) = \sqrt{\frac{(2J)!}{(J+M)!(J-M)!}} (\cos \frac{1}{2}\beta)^{J-M} (-\sin \frac{1}{2}\beta)^{J+M}. \quad (36)$$

The $SU(3)$ Clebsch–Gordan coefficient $\langle n_1n_2n_3; m_1m_2m_3|n_1+m_1, n_2+m_2, n_3+m_3\rangle$ is easily evaluated as

$$\begin{aligned}
 &\langle n_1n_2n_3; m_1m_2m_3|n_1+m_1, n_2+m_2, n_3+m_3\rangle \\
 &= \sqrt{\frac{p!q!}{(p+q)!} \frac{(n_1+m_1)!}{n_1!m_1!} \frac{(n_2+m_2)!}{n_2!m_2!} \frac{(n_3+m_3)!}{n_3!m_3!}} \quad (37)
 \end{aligned}$$

subject to the constraints $n_1 + n_2 + n_3 = p$, $m_1 + m_2 + m_3 = q$.

By construction, $|\psi_{\lambda_1\lambda_2\lambda_3}^\lambda(\alpha)\rangle$ is intelligent and belongs to the irrep $(\lambda, 0)$ (albeit not correctly normalized). By simply going over all those (positive, integer) values of λ_k such that $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, we find $\frac{1}{2}(\lambda + 1)(\lambda + 2)$ different linearly independent intelligent states. As the dimension of $(\lambda, 0)$ is precisely $\frac{1}{2}(\lambda + 1)(\lambda + 2)$, it must be that all the intelligent states of this representation are of the form given in equation (34). If all but one of the λ_i are 0, then the state is an $SU(3)$ coherent state.

5. Selected results

The $su(3)$ intelligent states are the solutions to the eigenvalue equation

$$(\hat{A} - i\alpha\hat{B})|\psi_{\lambda_1\lambda_2\lambda_3}^\lambda(\alpha)\rangle = \lambda|\psi_{\lambda_1\lambda_2\lambda_3}^\lambda(\alpha)\rangle, \quad (38)$$

and they have the eigenvalues

$$\lambda = \frac{2\pi}{3}\sqrt{1-\alpha^2}(\lambda_3 - \lambda_2). \quad (39)$$

One also shows that

$$(\Delta\mathcal{A})^2 = -\frac{1}{2}\alpha\langle\hat{C}\rangle, \quad (\Delta\mathcal{B})^2 = -\frac{1}{2\alpha}\langle\hat{C}\rangle. \quad (40)$$

The uncertainty curves for the $su(3)$ intelligent states display expected behaviour: for $\alpha = 0$, the uncertainty is zero as the states are eigenstates of \mathcal{A} . For $\alpha = \pm\infty$, they are eigenstates of \mathcal{B} so the uncertainty goes to zero again. There are discontinuities at $\alpha = \pm 1$. Despite extensive

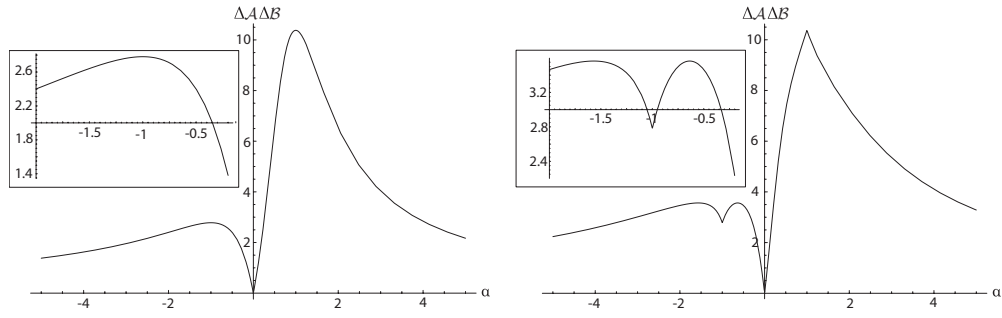


Figure 1. Two plots of $\Delta\mathcal{A}\Delta\mathcal{B}$ for $\lambda = 3$. The inset is an expanded view around $\alpha = -1$. Left: $\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = 0$. Right: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 0$.

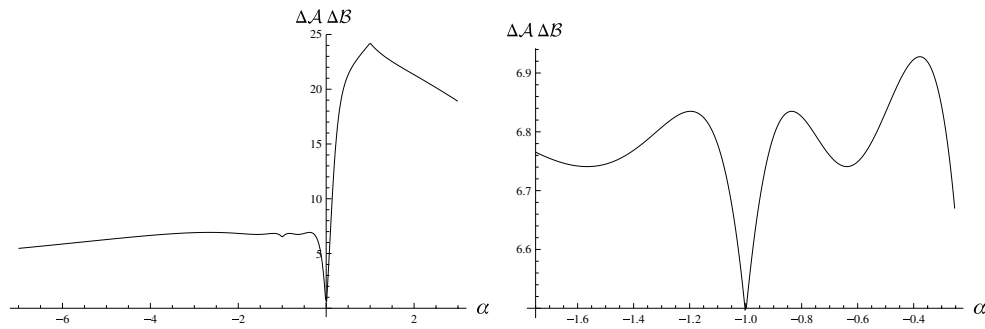


Figure 2. Two plots of $\Delta\mathcal{A}\Delta\mathcal{B}$ for $\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 1$. This state is one of the 36 intelligent states in the irrep $(7, 0)$. The rightmost figure gives details of the curve near $\alpha = -1$.

numerical experiments, we have not been able to determine a trend where the uncertainty, overall, is higher or lower for states of a given λ with different values of $\lambda_1, \lambda_2, \lambda_3$.

Figures 1 and 2 illustrate typical uncertainty curves for the $su(3)$ states. A striking feature of the $su(3)$ graphs is the difference in amplitude between positive and negative α . The overall uncertainty for negative α is significantly less than that for positive α for every graph produced up to this point. However, the height of the graph at $\alpha = -1$ can easily be determined. From the definition of μ , equation (15), it can be shown that

$$\lim_{\alpha \rightarrow -1} \frac{1 + \alpha}{\sqrt{1 - \alpha^2}} = 0. \tag{41}$$

From this, we deduce that the angles in the transformations ω_k go to 0 as $\alpha \rightarrow -1$. The uncertainty is then simply

$$\Delta\mathcal{A}\Delta\mathcal{B} = \frac{1}{2} |\langle \hat{C} \rangle| = \frac{2\pi^2(\sqrt{3} - 1)\lambda}{9\sqrt{3}}. \tag{42}$$

Thus, for any λ the uncertainty is easily determined for $\alpha = -1$. Due to the relative complexity of the general expression for the $su(3)$ intelligent states, it was not possible to establish other analytical results. It should be noted, however, that the curves presented above do not change if λ_2 and λ_3 are interchanged.

6. Conclusion

This paper shows how one can construct intelligent states for $su(3)$ observables using as ingredients $SU(3)$ coherent states. It is clear that the method, originally developed for $su(2)$ and $su(1, 1)$ and here applied to $su(3)$, can be generalized to $su(n)$ intelligent states. For the simplest symmetric (or one-row) representations, the coupling coefficients required are easy to calculate so the construction is immediate, and not limited as to the choice of observables. The CG coefficients and the simple form of the group functions given in equations (5) and (6) guarantee that the coherent states are properly normalized. One must, in the end, properly normalize the intelligent state but this (numerical) procedure and the extraction of relevant matrix elements remains simpler than the expressions obtained using the polynomial method of [3]. It is also applicable to the construction of intelligent states for any representation, something that becomes complicated using polynomial states.

Acknowledgments

This work was supported by NSERC of Canada. The work of BRL was supported by the Government of Ontario through its OGS award program.

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