JOURNAL OF MATHEMATICAL PHYSICS VOLUME 41, NUMBER 7 JULY 2000

SL(3,$\mathbb{C}$) generator matrix elements in a Pauli subgroup basis

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(Received 21 December 1999; accepted for publication 21 January 2000)

We construct SL(3,$\mathbb{C}$) basis states reduced according to its finite subgroup $\wp_3$. Matrix elements of sl(3,$\mathbb{C}$) generators are calculated between $\wp_3$ basis states.
 [S0022-2488(00)01207-X]

I. INTRODUCTION

The theory of finite-dimensional representations of semisimple Lie algebras over the complex number field is one of the major achievements of mathematics of the 20th century. Its many applications lead to its use outside mathematics proper. To a mathematician, the theory is fairly complete, i.e., all the main problems have been solved. What remains is on the fringes of the theory. The situation is very different when it comes to actually applying the theory.

Many applications to physics more or less impose on the user a particular basis of generators for a Lie algebra $\mathcal{L}$; this basis for $\mathcal{L}$ is not always the easiest choice from a mathematics point of view. Finding matrix elements of generators in this basis is often the main problem without which applications cannot proceed. Typically one requires ''good'' transformation properties of the generators with respect to a Lie subalgebra $\mathcal{L}'\subset\mathcal{L}$ or a subgroup $\mathcal{G}'\subset\mathcal{G}$ of the corresponding Lie group. In particular, the subgroup may be discrete or finite. (We say that a basis is ''good,'' if its elements can be split to subsets generating subspaces of $\mathcal{L}$ that are irreducible with respect to $\mathcal{L}'$ or $\mathcal{G}'$.)

It is standard in the general theory to work with the root (or Cartan) decomposition of the Lie algebra and the corresponding weight decomposition of the representation spaces. More precisely, a Cartan subalgebra $\mathcal{H}$ is first chosen among the equivalent Cartan subalgebras and one decomposes the Lie algebra and its representation spaces into eigenspaces of $\mathcal{H}$. The appealing aspect of this prescription is its uniformity. Indeed, it applies to all semisimple Lie algebras of over $\mathbb{C}$. In applications, an eigenspace decomposition of this kind is far from satisfactory: the eigenspaces are, more often than not, of dimension greater than 1, and the theory does not provide a canonical way of constructing a basis inside each eigenspace.

In this paper, we consider the cases where $\mathcal{G}'$ is the finite subgroup $\wp_2\subset$SL(2,$\mathbb{C}$) or $\wp_3\subset$SL(3,$\mathbb{C}$). The matrices of $\wp_2$ are closely related to the familiar Pauli matrices, and the matrices of $\wp_3$ are closely related to a 3×3 generalization[1] of the Pauli matrices.

The $\wp_3$ matrices induce a $\mathbb{Z}_3\times\mathbb{Z}_3$ grading[1] of sl(3,$\mathbb{C}$), which is one of four gradings[2] providing sets of additive quantum numbers for SL(3,$\mathbb{C}$) representations. Of the other three cases, one is the standard Cartan grading by weights, and two are related to the continuous subgroups SL(2,$\mathbb{C}$) and $O(3)$ of GL(3,$\mathbb{C}$) (combined with a nontrivial outer automorphism of SL(3,$\mathbb{C}$)), and have been investigated in Ref. 3. These subgroups of SL(3,$\mathbb{C}$) are well known and have been extensively studied in physics and mathematics: the famous Gel'fand–Tsetlin states[4] reduce the SL(3,$\mathbb{C}$)$\supset$SL(2,$\mathbb{C}$) subgroup chain, while the SL(3,$\mathbb{C}$)$\supset O(3)$ chain was (originally, at least) studied in relation to nuclear physics problems.[5]

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By contrast, the subgroup $\wp_3$ is a finite subgroup of $SL(3,\mathbb{C})$. The associated grading property is closely related to the fact that there exists in the defining 3×3 representation, a basis for $sl(3,\mathbb{C})$ defined by eight linearly independent traceless matrices of $\wp_3$ with determinant one, so that these basis elements are simultaneously elements of $sl(3,\mathbb{C})$ algebra and of the group $SL(3,\mathbb{C})$. [This holds in general for $SL(n,\mathbb{C})$, where there exists, in the defining $n\times n$ representation, a basis of $sl(n,\mathbb{C})$ defined in terms of n-1 linearly independent matrices of determinant in $\wp_n$.] As a subgroup of $SL(3,\mathbb{C})$, $\wp_3$ has received little attention, although $\wp_n$ matrices were found to be useful in the study of $\mathbb{Z}_n$ symmetric one-dimensional quantum chains,[6] but without any reference to the $sl(n,\mathbb{C})$ algebra associated with them. They also appear in relation to two-dimensional hydrodynamical problems,[7] where the full $\mathbb{Z}_n\otimes\mathbb{Z}_n$ structure is exploited, but again without reference to the associated $sl(n,\mathbb{C})$ algebra. A $\mathbb{Z}_3$-graded generalization of supersymmetry has been proposed in Ref. 8, and a $\mathbb{Z}_n$-graded exterior calculus has also been developed in Ref. 9.

Our objective is to construct basis states that reduce the $SL(2,\mathbb{C})\supset\wp_2$ and $SL(3,\mathbb{C})\supset\wp_3$ subgroup chains, and compute matrix elements of $sl(2,\mathbb{C})$ and $sl(3,\mathbb{C})$ generators in their respective Pauli subgroup bases.

For $\wp_2\subset SL(2,\mathbb{C})$, the task is straightforward, and the results are included to illustrate how we intend to proceed with the more difficult problem of $\wp_3\subset SL(3,\mathbb{C})$.

To construct a basis, we choose elements of $\wp_3$ that can be simultaneously diagonalized, and consider any representation space of $sl(3,\mathbb{C})$ as decomposed into their eigenspaces. Since generalized Pauli matrices have a dual interpretation as elements of the $\wp_3$ subgroup or as generators of the $sl(3,\mathbb{C})$ Lie algebra, the ''diagonal'' elements of $\wp_3$ can also be interpreted as generators of a Cartan subalgebra of $sl(3,\mathbb{C})$, so that the decomposition into eigenspaces turns out to be the familiar weight decomposition in disguise.

The action of $\wp_3$ elements on weight subspaces is to ''permute,'' in a cyclic fashion, subspaces labeled by weights belonging to the same Weyl group orbit. Thus, we can regroup weight subspaces into orbits of a representative $sl(3,\mathbb{C})$ state under the action of $\wp_3$. It will be shown that these orbits generically comprise alternate weights in the Weyl orbits of the weight diagram of $SL(3,\mathbb{C})$, thereby naturally dividing a weight diagram into three sectors. Once a complete basis in one sector has been constructed, it remains to transfer, by the action of $\wp_3$, that basis to other subspaces of the orbit. This is usually the most difficult task associated with the construction of a subgroup basis but, in the case of $\wp_3$, the actual expressions remain relatively easy to obtain because of the simple orbit structure. Finally, it will be shown that the computation of generator matrix elements can almost always be done within a single sector; the only times where one explicitly needs to compute the action of $\wp_3$ elements is when states lie near or on the border of two or three sectors. This means that we can almost always dispense with the actual computation of the action of $\wp_3$ on basis states.

II. THE SUBGROUPS $\wp_2\subset SL(2,\mathbb{C})$ AND $\wp_3\subset SL(3,\mathbb{C})$

A. $\wp_2$ and its related $sl(2,\mathbb{C})$ basis

The $\wp_2$ subgroup of $SL(2,\mathbb{C})$ is constructed by first considering the three Pauli matrices,

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_z=\begin{pmatrix}i&0\\0&-i\end{pmatrix},\tag{1}$$

plus the unit matrix $\sigma_0=\mathbf{1}$. One can verify that the set of eight matrices $\{\pm i\sigma_x,\pm i\sigma_y,\pm\sigma_z,\pm\sigma_0\}$ is a subgroup of $SL(2,\mathbb{C})$. This subgroup is $\wp_2$. It turns out that $\wp_2$ is isomorphic to the double dihedral subgroup ${}^{(2)}D_2\subset SL(2,\mathbb{C})$. The subgroup is generated by the elements

$$A_0=i\sigma_y=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad D_0=i\sigma_z=\begin{pmatrix}i&0\\0&-i\end{pmatrix},\tag{2}$$

and $-\mathbf{1}$. The character table for $\wp_2$ is shown in Table I.

TABLE I. $\wp_2$ character table. The first row gives the conjugacy class in SL$(2,\mathbb{C})$ as elements of finite order (EFO) in the notation of Ref. 10. The second row is the number of elements in the class. The third row shows a representative element; for the last three classes, the second element is obtained by multiplying by the duality operator A_0^2. The subscript on the class symbol in the fourth row is the order of a class element.

	[10]	[01]	[11]	[11]	[11]
	1	1	2	2	2
	$\mathbb{1}$	$A_0^2=D_0^2$	A_0	D_0	A_0D_0
IR	C_1	C_2	C_4	C_4'	C_4''
Γ_1	1	1	1	1	1
Γ_2	1	1	-1	1	-1
Γ_3	1	1	1	-1	-1
Γ_4	1	1	-1	-1	1
Γ_5	2	-2	0	0	0

Elements in $\wp_2$ are $\mathbb{Z}_2\times\mathbb{Z}_2$ graded; if we make the explicit identification $\sigma_i\rightarrow\sigma_{(\nu,\mu)}$, where

$$\sigma_x\mapsto\sigma_{(1,1)}, \quad \sigma_y\mapsto\sigma_{(0,1)}, \quad \sigma_z\mapsto\sigma_{(1,0)}, \quad \sigma_0\mapsto\sigma_{(0,0)}, \tag{3}$$

then the product of any two $\wp_2$ elements is, up to a sign, given by

$$\sigma_{(\nu_i,\mu_i)}\cdot\sigma_{(\nu_k,\mu_k)}\propto\sigma_{(\nu_i+\nu_k,\mu_i+\mu_k)}, \tag{4}$$

where all indices are read modulo 2.

Clearly, the 2×2 Pauli matrices form a basis for the Lie algebra sl$(2,\mathbb{C})$. The generating elements of the sl$(2,\mathbb{C})$ algebra are any two Pauli matrices, for instance

$$\hat{A}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{D}=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{5}$$

(The overcarets denote elements of the algebra.) The third element of the basis can be obtained by commuting the previous two.

B. $\wp_3$ and its related sl$(3,\mathbb{C})$ basis

Let $\omega=e^{2\pi i/3}$. The finite subgroup $\wp_3$ of SL$(3,\mathbb{C})$ is the subgroup comprising the 27 elements,

$$A_k=\omega^k\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_k^-=\omega^{-k}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$B_k=\omega^k\begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_k^-=\omega^{-k}\begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix},$$

$$C_k=\omega^k\begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix}, \quad C_k^-=\omega^{-k}\begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, \tag{6}$$

TABLE II. Character table for $\wp_3$. The first row gives the SL(3,$\mathbb{C}$) conjugacy class for each $\wp_3$ class as elements of finite order in the notation of Ref. 10. The second row is the number of elements in each class. The third row shows a representative element of each class; for the last eight classes, the other two elements are obtained by multiplying by T and T^2. The subscript on a class symbol is the order of its elements.

	[100]	[111]	[111]	[111]	[111]	[111]	[111]	[111]	[111]	[111]	[111]
	1	1	1	3	3	3	3	3	3	3	3
	$\mathbb{1}$	T	T^2	A_0	D_0	A_0^2	D_0^2	A_0D_0	$A_0D_0^2$	$A_0^2D_0$	$A_0^2D_0^2$
IR	C_1	C_3	C_3'	C_3''	C_3'''	C_3^{iv}	C_3^{v}	C_3^{vi}	C_3^{vii}	C_3^{viii}	C_3^{ix}
Γ_1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	1	1	1	ω	1	ω^2	1	ω	ω	ω^2	ω^2
Γ_3	1	1	1	ω^2	1	ω	1	ω^2	ω^2	ω	ω
Γ_4	1	1	1	1	ω	1	ω^2	ω	ω^2	ω	ω^2
Γ_5	1	1	1	ω	ω	ω^2	ω^2	ω^2	1	1	ω
Γ_6	1	1	1	ω^2	ω	ω	ω^2	1	ω	ω^2	1
Γ_7	1	1	1	1	ω^2	1	ω	ω^2	ω	ω^2	ω
Γ_8	1	1	1	ω	ω^2	ω^2	ω	1	ω^2	ω	1
Γ_9	1	1	1	ω^2	ω^2	ω	ω	ω	1	1	ω^2
Γ_{10}	3	3ω	$3\omega^2$	0	0	0	0	0	0	0	0
Γ_{11}	3	$3\omega^2$	3ω	0	0	0	0	0	0	0	0

$$D_k=\omega^k\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},\quad D_k^-=\omega^{-k}\begin{pmatrix}\omega&0&0\\0&\omega^2&0\\0&0&\omega\end{pmatrix},$$

$$I_k=\omega^k\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad k=0,1,2.$$

The group $\wp_3$ is generated by the elements A_0, D_0, and I_1. The character table for $\wp_3$ is shown in Table II. Elements of $\wp_3$ are $\mathbb{Z}_3\times\mathbb{Z}_3$ graded, meaning that, if we make the assignment

$$\hat{A}\mapsto\hat{\Omega}_{(1,0)},\quad \hat{D}\mapsto\hat{\Omega}_{(0,1)},\tag{7}$$

the product of two general elements $\hat{\Omega}_{(k,m)},\hat{\Omega}_{(i,j)}\in\wp_3$ is proportional to

$$\hat{\Omega}_{(k,m)}\cdot\hat{\Omega}_{(i,j)}\propto\hat{\Omega}_{(i+k,j+m)},\tag{8}$$

where all indices are read modulo 3.

Much like $\wp_2\subset\mathrm{SL}(2,\mathbb{C})$, the $\wp_3\subset\mathrm{SL}(3,\mathbb{C})$ subgroup is distinguished in that any maximal set of linearly independent combination of its matrices can serve as a basis for the Lie algebra sl$(3,\mathbb{C})$. Thus, we choose the basis $\{\hat{A},\hat{A}^-,\hat{B},\hat{B}^-,\hat{C},\hat{C}^-,\hat{D},\hat{D}^-\}$ for sl$(3,\mathbb{C})$ by selecting the $\wp_3$ elements $\{A_0,A_0^-,B_0,B_0^-,C_0,C_0^-,D_0,D_0^-\}$.

Among the other interesting properties of this sl$(3,\mathbb{C})$ basis, one notes that it is possible to choose two elements, say $\hat{A}$ and $\hat{D}$, given explicitly by

$$\hat{A}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad \hat{D}=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},\tag{9}$$

and express any other element as a multiple commutator of $\hat{A}$ and $\hat{D}$.

TABLE III. $SL(2,\mathbb{C})\supset\wp_2$ branching rules for orbit $[jm]$.

$2j$	$2m$	IR
Odd	Odd	Γ_5
0 mod 4	0	Γ_1
	4, 8, 12, etc.	$\Gamma_1\oplus\Gamma_2$
	2, 6, 10, etc.	$\Gamma_3\oplus\Gamma_4$
2 mod 4	0	Γ_2
	4, 8, 12, etc.	$\Gamma_1\oplus\Gamma_2$
	2, 6, 10, etc.	$\Gamma_3\oplus\Gamma_4$

III. SL(2,$\mathbb{C}$) IN A $\wp_2$ BASIS

We start by presenting two solutions to the problem of constructing SL(2,$\mathbb{C}$) states in a $\wp_2$ basis. Our (later) results on SL(3,$\mathbb{C}$) will be modeled after the SL(2,$\mathbb{C}$) solutions.

A. A U(1) basis

We will use a realization of the sl(2,$\mathbb{C}$) Lie algebra in terms of creation and destruction operators with

$$L_+=a_1^\dagger a_2,\quad L_-=a_2^\dagger a_1,\quad L_0=\tfrac{1}{2}(a_1^\dagger a_1-a_2^\dagger a_2), \tag{10}$$

where, as usual, $[a_i,a_j^\dagger]=\delta_{ij}$.

Basis states for the representation of angular momentum j and dimension $2j+1$ are given by the familiar oscillator states:

$$|\nu_1\nu_2\rangle=\frac{(a_1^\dagger)^{\nu_1}(a_2^\dagger)^{\nu_2}}{\sqrt{\nu_1!\,\nu_2!}}|0\rangle,\quad j=\frac{1}{2}(\nu_1+\nu_2), \tag{11}$$

where $|0\rangle$ is the harmonic oscillator vacuum with $a_i|0\rangle=0$, and where the sl(2,$\mathbb{C}$) weight label m is given by $m=\frac{1}{2}(\nu_1-\nu_2)$.

B. Branching rules

The next step is to determine the $SL(2,\mathbb{C})\downarrow\wp_2$ branching rules (see Table III). Let

$$a_1^+|0\rangle=|10\rangle=\begin{pmatrix}1\\0\end{pmatrix},\quad a_2^+|0\rangle=|01\rangle=\begin{pmatrix}0\\1\end{pmatrix}.$$

One observes that the generating elements A_0, D_0, and $-\mathbf{1}$ on basis states of the fundamental two-dimensional representation must be, by definition,

$$A_0|10\rangle=|01\rangle,\quad A_0|01\rangle=-|10\rangle,$$
$$D_0|10\rangle=i|10\rangle,\quad D_0|01\rangle=-i|01\rangle. \tag{12}$$

From this we deduce that

$$A_0a_1^\dagger A_0^{-1}=a_2^\dagger,\quad A_0a_2^\dagger A_0^{-1}=-a_1^\dagger,$$
$$D_0a_1^\dagger D_0^{-1}=ia_1^\dagger,\quad D_0a_2^\dagger D_0^{-1}=-ia_2^\dagger, \tag{13}$$

and that

$$A_0|\nu_1\nu_2\rangle=(-1)^{\nu_2}|\nu_2\nu_1\rangle,\quad D_0|\nu_1\nu_2\rangle=(-i)^{\nu_2-\nu_1}|\nu_1\nu_2\rangle. \tag{14}$$

Thus, the generating elements of $\wp_2$ simply permute, up to a phase, the SL(2,ℂ) basis states $|\nu_1\nu_2\rangle$ and $|\nu_2\nu_1\rangle$. The state $|\nu_2\nu_1\rangle$ lies in the same $\wp_3$ orbit as $|\nu_1\nu_2\rangle$, and the decomposition of an SL(2,ℂ) module can be done orbit by orbit.

Consider a generic representation of SL(2,ℂ) of dimension $2j+1$, with basis states labeled as $|\nu_1\nu_2\rangle$. The action of A_0 transforms $|\nu_1\nu_2\rangle$ as per Eq. (14). Let $[\nu_1\nu_2]$, $\nu_1>\nu_2$ denote the orbit of a generic state $|\nu_1\nu_2\rangle$, $\nu_1>\nu_2$, under the action of $\wp_3$. In this subspace, the trace of A_0 vanishes whereas the trace of A_0^2 is ± 2 according to whether $\nu_1-\nu_2$ is even or odd. Thus, the $\wp_3$ orbit $[\nu_1\nu_2]$ with $\nu_1>\nu_2$ contains Γ_5 when $\nu_1-\nu_2$ is odd, and a sum of irreps,

$$x_1\Gamma_1+x_2\Gamma_2+x_3\Gamma_3+x_4\Gamma_4, \tag{15}$$

when $\nu_1-\nu_2$ is even; Γ_5 occurs in even-dimensional representations, and the one-dimensional reps occur in the odd-dimensional representations.

The coefficients x_i can be determined from the action of the $\wp_2$ element D_0. Since we have $D_0|\nu_1\nu_2\rangle=e^{i(\nu_1-\nu_2)\pi/2}|\nu_1\nu_2\rangle$, the trace of D_0 is 2 for $\nu_1-\nu_2=0 \bmod 4$, -2 for $\nu_1-\nu_2=2 \bmod 4$, and vanishes for $\nu_1-\nu_2$ odd. From Table I, it follows that a generic orbit contains Γ_5 when $\nu_1-\nu_2$ is odd, contains $\Gamma_1\oplus\Gamma_2$ when $\nu_1-\nu_2=0 \bmod 4$, and contains $\Gamma_3\oplus\Gamma_4$ when $\nu_1-\nu_2=2 \bmod 4$.

When $\nu_1=\nu_2$, $\mathrm{Tr}\,A_0=\pm 1$ according to whether $\nu_1+\nu_2=0 \bmod 4$ or $2 \bmod 4$, $\mathrm{Tr}\,D_0=1$, $\mathrm{Tr}\,A_0^2=1$. It follows that, in that case, we have Γ_1 when $\nu_1+\nu_2=0 \bmod 4$ and Γ_2 when $\nu_1+\nu_2=2 \bmod 4$.

C. $\wp_2$ basis states

We will use $|\nu_1\nu_2\rangle$ with $\nu_1\geqslant\nu_2$ as a $\wp_3$ orbit representative of $[\nu_1\nu_2]$.

We can obtain SL(2,ℂ)$\supset\wp_2$ states by projection from the orbit representative. We will denote by $|\nu_1\nu_2;\Gamma_\sigma;k\rangle,\nu_1\geqslant\nu_2$, the kth basis state of the representation Γ_σ of $\wp_2$ containing the state $|\nu_1\nu_2\rangle$.

Let Γ_σ be an irrep of $\wp_2$. The (character) projection operator for this representation is given by

$$P^\sigma=\frac{\dim_\sigma}{8}\sum_G \chi^\sigma(G)^*G, \tag{16}$$

where the sum is over the group elements, $\dim_\sigma$ is the dimension of the representation Γ_σ, and $\chi^\sigma(G)$ is the character of the element G in the irrep Γ_σ.

Using this in conjuction with the character Table I, we find that, for the two-dimensional representation Γ_5, the basis states are just

$$|\nu_1\nu_2;\Gamma_5;1\rangle=|\nu_1\nu_2\rangle,\quad |\nu_1\nu_2;\Gamma_5;2\rangle=A_0|\nu_1\nu_2\rangle=(-1)^{\nu_2}|\nu_2\nu_1\rangle,\quad \nu_1>\nu_2. \tag{17}$$

For the one-dimensional representations, we find, using the branching rules and the explicit action of D_0 on our states, $D_0|\nu_1\nu_2\rangle=(-1)^{(1/2)(\nu_1-\nu_2)}|\nu_1\nu_2\rangle$, that the states have the generic form

$$|\nu_1\nu_2;\Gamma_\sigma\rangle=\frac{1}{\sqrt{2}}(\mathbf{1}+(-1)^{\sigma+1}A_0)|\nu_1\nu_2\rangle=\frac{1}{\sqrt{2}}(|\nu_1\nu_2\rangle+(-1)^{\sigma+1}\cdot(-1)^{\nu_2}|\nu_2\nu_1\rangle). \tag{18}$$

Since all weights of an SL(2,ℂ) irrep have multiplicity one, we find, for the case where $\nu_1=\nu_2$, that a basis state for this case is simply given by the state itself. This and the generic case can be handled together by defining

$$|\nu_1\nu_2;\Gamma_\sigma\rangle=\frac{1}{\sqrt{2(1+\delta_{\nu_1\nu_2})}}(|\nu_1\nu_2\rangle+(-1)^{\sigma+1+\nu_2}|\nu_2\nu_1\rangle). \tag{19}$$

D. $\wp_2$ generator matrix elements

It is clearly sufficient to compute matrix elements of $\hat{A}$ and $\hat{D}$ of Eq. (5), as the other element of the Lie algebra can be obtained by commuting these two. These operators can be expressed in terms of the usual sl(2,$\mathbb{C}$) operators as

$$\hat{A}=L_+-L_-=a_1^\dagger a_2-a_2^\dagger a_1, \quad \hat{D}=iL_0=\frac{i}{2}(a_1^\dagger a_1-a_2^\dagger a_2). \tag{20}$$

Acting on states of a two-dimensional representation of $\wp_2$, we find the nonzero matrix elements of $\hat{A}$ and $\hat{D}$ as

$$\begin{aligned}
\langle \nu_1+1,\nu_2-1;\Gamma_5;1|\hat{A}|\nu_1\nu_2;\Gamma_5;1\rangle&=\sqrt{(\nu+1)\nu_2},\\
\langle \nu_1-1,\nu_2+1;\Gamma_5;1|\hat{A}|\nu_1\nu_2;\Gamma_5;1\rangle&=-\sqrt{(\nu_1+1)\nu_2}, \quad \nu_2\neq\nu_1-1,\\
\langle \nu_1,\nu_1-1;\Gamma_5;2|\hat{A}|\nu_1,\nu_1-1;\Gamma_5;1\rangle&=(-1)^{\nu_2}\nu_1, \quad \nu_2=\nu_1-1,\\
\langle \nu_1,\nu_2;\Gamma_5;k|\hat{D}|\nu_1,\nu_2;\Gamma_5;k\rangle&=(-1)^{k+1}\frac{i}{2}(\nu_1-\nu_2).
\end{aligned} \tag{21}$$

The remaining matrix elements of $\hat{A}$ are found from the observation that

$$\hat{A}|\nu_1\nu_2;\Gamma_5;2\rangle=\hat{A}A_0|\nu_1\nu_2;\Gamma_5;1\rangle=A_0\hat{A}|\nu_1\nu_2\rangle, \tag{22}$$

since $\hat{A}$ and A_0 commute. Thus, we find

$$\begin{aligned}
\langle \nu_1'\nu_2';\Gamma_5;1|\hat{A}|\nu_1\nu_2;\Gamma_5;2\rangle&=(-1)^{\nu_2'}\langle \nu_1'\nu_2';\Gamma_5;2|\hat{A}|\nu_1\nu_2;\Gamma_5;1\rangle,\\
\langle \nu_1'\nu_2';\Gamma_5;2|\hat{A}|\nu_1\nu_2;\Gamma_5;2\rangle&=\langle \nu_1'\nu_2';\Gamma_5;1|\hat{A}|\nu_1\nu_2;\Gamma_5;1\rangle.
\end{aligned} \tag{23}$$

The matrix elements of $\hat{A}$ between states of one-dimensional representations of $\wp_2$ are straightforward:

$$\begin{aligned}
\langle \nu_1'\nu_2';\Gamma_{\sigma'}|\hat{A}|\nu_1\nu_2;\Gamma_\sigma\rangle=&-\sqrt{\frac{\nu_1(\nu_2+1)}{(1+\delta_{\nu_1-1,\nu_2+1})(1+\delta_{\nu_1\nu_2})}}[\delta_{\nu_1',\nu_1-1}\delta_{\nu_2',\nu_2+1}\\
&+(-1)^{\sigma+\nu_2}\delta_{\nu_2',\nu_1-1}\delta_{\nu_1',\nu_2+1}]\delta_{\sigma',\sigma+2}\\
&+\sqrt{\frac{(\nu_1+1)\nu_2}{(1+\delta_{\nu_1\nu_2})}}\delta_{\sigma',\sigma+2}\delta_{\nu_1',\nu_1+1}\delta_{\nu_2',\nu_2-1}, \quad \nu_1\geqslant\nu_2,
\end{aligned} \tag{24}$$

where, in $\delta_{\sigma',\sigma+2}$, the sum $\sigma+2$ is taken modulo 2.

IV. SL(3,$\mathbb{C}$) IN A $\wp_3$ BASIS

For SL(3,$\mathbb{C}$), we will choose a representative states of an orbit to be an SL(2,$\mathbb{C}$)$\times$U(1) basis state.

A. An SL(2,ℂ)×U(1) basis

It will be convenient to introduce basis states that reduce the subgroup chain,

$$\mathrm{SL}(3,\mathbb{C})\supset \mathrm{SL}(2,\mathbb{C})\times \mathrm{U}(1), \tag{25}$$

where SL(2,ℂ) is the subgroup whose Lie algebra sl(2,ℂ) is spanned by the I-spin operators,

$$\hat{I}_-=\hat{C}_{23}, \quad \hat{I}_+=\hat{C}_{32}, \quad \hat{I}_0=\tfrac{1}{2}(\hat{C}_{33}-\hat{C}_{22}), \tag{26}$$

U(1) the subgroup with Lie algebra spanned by $\hat{X}=2\hat{C}_{11}-\hat{C}_{22}-\hat{C}_{33}$, and where $\{\hat{C}_{ij},i,j=1,2,3\}$ are the familiar element of the gl(3,ℂ) Lie algebra:

$$\begin{aligned} &\hat{C}_{ij}, \quad i<j, \ \text{lowering operators},\\ &\hat{C}_{ij}, \quad i>j, \ \text{raising operators},\\ &\hat{h}_1=\hat{C}_{33}-\hat{C}_{22}, \quad \hat{h}_2=\hat{C}_{22}-\hat{C}_{11}, \quad \text{Cartan subalgebra operators,}\end{aligned} \tag{27}$$

with commutation relations

$$[\hat{C}_{ij},\hat{C}_{kl}]=\delta_{jk}\hat{C}_{il}-\delta_{il}\hat{C}_{kj}. \tag{28}$$

An explicit realization is given in terms of creation and destruction operators for two particles:

$$\hat{C}_{ij}=a^{\dagger}_{i1}a_{j1}+a^{\dagger}_{i2}a_{j2}, \tag{29}$$

where $[a_{is},a^{\dagger}_{jt}]=\delta_{ij}\delta_{st}$, as usual.

Basis states that reduce this subgroup chain are obtained, following Ref. 11, by the SU(2)-coupled product of harmonic oscillator states,

$$\begin{aligned}|\nu_1\nu_2\nu_3;I\rangle=&\sum_{m_1m_2m_3(N)}\left\langle \begin{matrix}\tfrac{1}{2}\nu_2 & \tfrac{1}{2}\nu_3 & I\\ m_2 & m_3 & N\end{matrix}\right\rangle\left\langle \begin{matrix}\tfrac{1}{2}\nu_1 & I & \tfrac{1}{2}p\\ m_1 & N & \tfrac{1}{2}p\end{matrix}\right\rangle\\ &\times\frac{(a^{\dagger}_{11})^{\nu_1/2+m_1}}{\sqrt{(\tfrac{1}{2}\nu_1+m_1)!}}\frac{(a^{\dagger}_{12})^{\nu_1/2-m_1}}{\sqrt{(\tfrac{1}{2}\nu_1-m_1)!}}\frac{(a^{\dagger}_{21})^{\nu_2/2+m_2}}{\sqrt{(\tfrac{1}{2}\nu_2+m_2)!}}\frac{(a^{\dagger}_{22})^{\nu_2/2-m_2}}{\sqrt{(\tfrac{1}{2}\nu_2-m_2)!}}\\ &\times\frac{(a^{\dagger}_{31})^{\nu_3/2+m_3}}{\sqrt{(\tfrac{1}{2}\nu_3+m_3)!}}\frac{(a^{\dagger}_{32})^{\nu_3/2-m_3}}{\sqrt{(\tfrac{1}{2}\nu_3-m_3)!}}|0\rangle,\end{aligned} \tag{30}$$

where $\nu_i\geqslant 0$ and $\nu_1+\nu_2+\nu_3=p+2q$. The angular momentum I can take either integer or half-odd integer values in the range

$$\max[\tfrac{1}{2}|\nu_3-\nu_2|,\tfrac{1}{2}|p-\nu_1|]\leqslant I\leqslant \mathrm{Min}[\tfrac{1}{2}(p+\nu_1),\tfrac{1}{2}(p+2q-\nu_1)]. \tag{31}$$

The states $\{|\nu_1\nu_2\nu_3;I\rangle\}$ are the familiar Gel'fand–Tsetlin basis

$$\left\{\left|\begin{matrix}p+q & & q & & 0\\ & w & & v & \\ & & r & & \end{matrix}\right\rangle;\ p+q\geqslant w\geqslant q\geqslant v\geqslant 0 \quad w\geqslant r\geqslant v\right\}, \tag{32}$$

once we make the identification

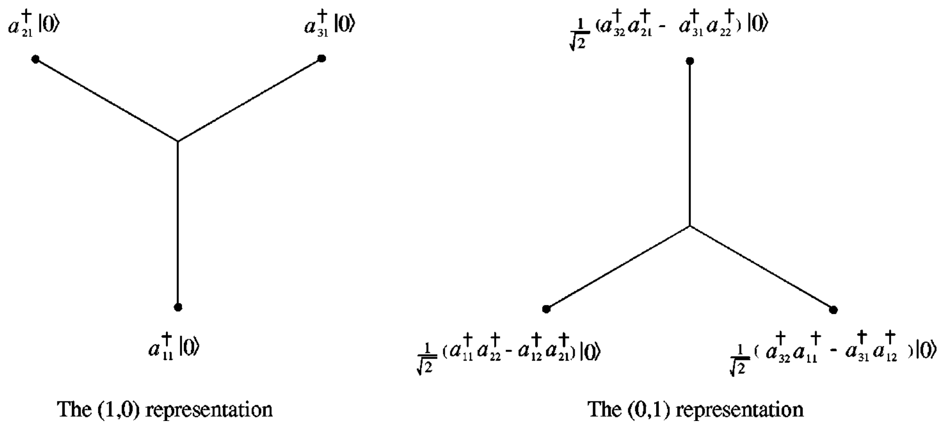

FIG. 1. The representations (1,0) and (0,1) of sl(3,ℂ).

$$\nu_1=p+2q-w-v,\quad \nu_2=w+v-r,\quad \nu_3=r,\quad I=\tfrac{1}{2}(w-v). \tag{33}$$

B. The action of A_0 and D_0

The matrix representation of A_0 and D_0 must give the matrices of Eq. (6) on basis states of the (1,0) representation. The position of these basis states in weight space is presented in Fig. 1. From this, we conclude that

$$A_0a^\dagger_{11}A_0^{-1}=a^\dagger_{21},\quad A_0a^\dagger_{21}A_0^{-1}=a^\dagger_{31},\quad A_0a^\dagger_{31}A_0^{-1}=a^\dagger_{11}. \tag{34}$$

We then define

$$A_0a^\dagger_{12}A_0^{-1}=a^\dagger_{22},\quad A_0a^\dagger_{22}A_0^{-1}=a^\dagger_{32},\quad A_0a^\dagger_{32}A_0^{-1}=a^\dagger_{12}, \tag{35}$$

thereby fixing the phases in the (0,1) representation. From this, we find

$$D_0a^\dagger_{1i}D_0^{-1}=a^\dagger_{1i},\quad D_0a^\dagger_{2i}D_0^{-1}=\omega^2a^\dagger_{2i},\quad D_0a^\dagger_{3i}D_0^{-1}=\omega a^\dagger_{3i}, \tag{36}$$

where we have used the constraint that the triality operator $T=A_0D_0A^2D_0=\omega\mathbf{1}$ in the (1,0) representation and $T=\omega^2\mathbf{1}$ in (0,1).

The action of A_0 on any harmonic oscillator state $|\nu_1\nu_2\nu_3;I\rangle$ is then easy to compute:

$$\begin{aligned}A_0|\nu_1\nu_2\nu_3;I\rangle=&\sum_{m_1m_2m_3(\beta)}\left\langle\begin{matrix}\frac{1}{2}\nu_2 & \frac{1}{2}\nu_3 & I\\ m_2 & m_3 & \beta\end{matrix}\right\rangle\left\langle\begin{matrix}\frac{1}{2}\nu_1 & I & \frac{1}{2}p\\ m_1 & \beta & \frac{1}{2}p\end{matrix}\right\rangle\\ &\times\frac{(a^\dagger_{21})^{\nu_1/2+m_1}}{\sqrt{(\frac{1}{2}\nu_1+m_1)!}}\frac{(a^\dagger_{22})^{\nu_1/2-m_1}}{\sqrt{(\frac{1}{2}\nu_1-m_1)!}}\frac{(a^\dagger_{31})^{\nu_2/2+m_2}}{\sqrt{(\frac{1}{2}\nu_2+m_2)!}}\frac{(a^\dagger_{32})^{\nu_2/2-m_2}}{\sqrt{(\frac{1}{2}\nu_2-m_2)!}}\\ &\times\frac{(a^\dagger_{11})^{\nu_3/2+m_3}}{\sqrt{(\frac{1}{2}\nu_3+m_3)!}}\frac{(a^\dagger_{12})^{\nu_3/2-m_3}}{\sqrt{(\frac{1}{2}\nu_3-m_3)!}}|0\rangle,\end{aligned} \tag{37}$$

with the final result

TABLE IV. $SL(3,\mathbb{C})\supset\wp_3$ branching rules for the triangular orbit containing the weight $[\nu]\neq 0$ in the $SL(3,\mathbb{C})$ irrep (p,q).

$p+2q$	IR
0 mod 3	$\Gamma_{1+3(\nu_3-\nu_2)}\oplus\Gamma_{2+3(\nu_3-\nu_2)}\oplus\Gamma_{3+3(\nu_3-\nu_2)}$
1 mod 3	Γ_{10}
2 mod 3	Γ_{11}

$$A_0|\nu_1\nu_2\nu_3;I\rangle=\sum_{I'}|\nu_3\nu_1\nu_2;I'\rangle\langle\nu_3\nu_1\nu_2;I'|A_0|\nu_1\nu_2\nu_3;I\rangle,$$

$$\langle\nu_3\nu_1\nu_2;I'|A_0|\nu_1\nu_2\nu_3;I\rangle=(-1)^{(p+2q+\nu_3+2I')/2}\sqrt{(2I'+1)(2I+1)}\begin{Bmatrix}\frac{1}{2}\nu_1 & \frac{1}{2}\nu_2 & I'\\ \frac{1}{2}\nu_3 & \frac{1}{2}p & I\end{Bmatrix}, \tag{38}$$

where

$$\begin{Bmatrix}a & b & c\\ d & e & f\end{Bmatrix}$$

is an su(2) 6-j symbol. It is important to note that $(A_0)^3|\nu_1\nu_2\nu_3;I\rangle=|\nu_1\nu_2\nu_3;I\rangle$.

The action of D_0 is easily found to be

$$D_0|\nu_1\nu_2\nu_3;I\rangle=\omega^{\nu_3-\nu_2}|\nu_1\nu_2\nu_3;I\rangle. \tag{39}$$

C. $SL(3,\mathbb{C})\downarrow\wp_3$ branching rules

Denote by $[\nu]\equiv[\nu_1,\nu_2,\nu_3]$ a weight $(\nu_2-\nu_3,\nu_2-\nu_1)$ in the representation (p,q). A state with weight $[\nu]$ is an eigenstate of D_0 with eigenvalue $\omega^{\nu_3-\nu_2}$ and an eigenstate of the triality operator $T=A_0D_0A_0^2D_0^2$ with eigenvalue ω^{p+2q}. These operators are always diagonal.

Let $|\nu_1\nu_2\nu_3;I\rangle$ be a state in the weight subspace $[\nu]$ of (p,q). The action of A_0 reflects $|\nu_1\nu_2\nu_3;I\rangle$ according to Eq. (38). Since $A_0^3=\mathbf{1}$, a generic orbit of $\wp_3$ is therefore three dimensional if $[\nu]\neq 0$ and one dimensional if $[\nu]=0$. Thus, we can decompose an $SL(3,\mathbb{C})$ module triangle by triangle, regarding a zero-weight point orbit as a degenerate triangle.

Assume that $[\nu]\neq 0$ so that $|\nu_1\nu_2\nu_3;I\rangle$, $A_0|\nu_1\nu_2\nu_3;I\rangle$ and $A_0^2|\nu_1\nu_2\nu_3;I\rangle$ are distinct. On this three-dimensional subspace, the trace of A_0 vanishes. The trace of T is $3\omega^{p+2q}$, which implies that, when $p+2q=1$ or $2 \bmod 3$, the orbit contains either Γ_{10} or Γ_{11}, whereas it contains a sum of three one-dimensional irreps when $p+2q=0 \bmod 3$.

This sum can be determined from the action of D_0, whose trace is, in general given, by $\omega^{\nu_3-\nu_2}(1+\omega^{q+2p}+\omega^{2(q+2p)})$. When $q+2p=0 \bmod 3$, the trace of D_0 is therefore $3\omega^{\nu_3-\nu_2}$, and we see from the character table that the orbit contains $\Gamma_{1+3(\nu_3-\nu_2)}\oplus\Gamma_{2+3(\nu_3-\nu_2)}\oplus\Gamma_{3+3(\nu_3-\nu_2)}$, where the representation label is calculated modulo 9.

TABLE V. $SL(3,\mathbb{C})\supset\wp_3$ branching rules for point orbit, having $[\nu]=0$, in the $SL(3,\mathbb{C})$ irrep (p,q). The label n is $p_<+1$, where $p_<$ is the lesser of p and q.

$p+2q$	p	IR
0 mod 3	0 mod 3	$\frac{1}{3}(n+2)\Gamma_1+\frac{1}{3}(n-1)(\Gamma_2\oplus\Gamma_3)$
	1 mod 3	$\frac{1}{3}(n-2)\Gamma_1+\frac{1}{3}(n+1)(\Gamma_2\oplus\Gamma_3)$
	2 mod 3	$\frac{1}{3}n(\Gamma_1\oplus\Gamma_2\oplus\Gamma_3)$

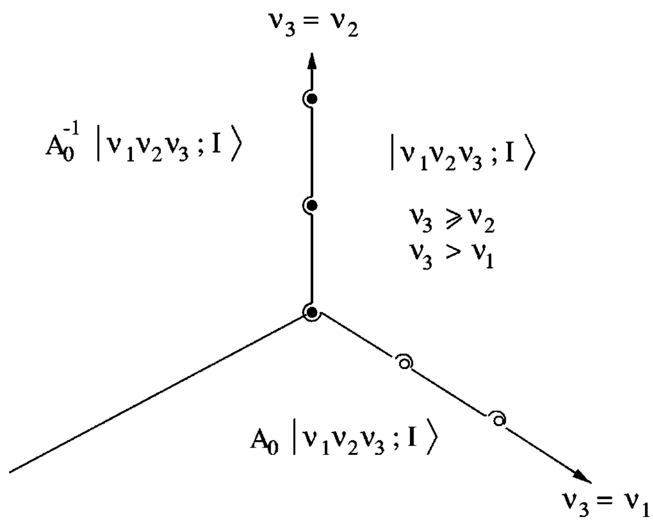

FIG. 2. The location of the orbit representatives, and of the reflection of these states.

Finally, consider the degenerate triangles with weight $[\nu]=0$ in the SL(3,ℂ) IR (p,q) whose triality is necessarily 0. The traces of D_0 and T are both n, the multiplicity of the point orbit; the value of n is $p_< +1$, where $p_<$ is the lesser of p,q. One can use the representations (3,0), (1,1), and (2,2) as prototypes to verify that the trace of A_0 is 1, -1, or 0 according to whether the triality of p (and of q) is 0, 1 or 2, respectively [the trace of A_0 for the zero weights is the same as for the whole SL(3,ℂ) IR]. It follows that the point orbits decompose into

$$\begin{gathered}\tfrac{1}{3}(n+2)\Gamma_1\oplus\tfrac{1}{3}(n-1)(\Gamma_2\oplus\Gamma_3),\quad \text{for } p=0 \bmod 3,\\ \tfrac{1}{3}(n-2)\Gamma_1\oplus\tfrac{1}{3}(n+1)(\Gamma_2\oplus\Gamma_3),\quad \text{for } p=1 \bmod 3,\\ \tfrac{1}{3}n(\Gamma_1\oplus\Gamma_2\oplus\Gamma_3),\quad \text{for } p=2 \bmod 3.\end{gathered}\tag{40}$$

We summarize the SL(3,ℂ)$\downarrow\wp_3$ branching rules in Tables IV and V.

D. $\wp_3$ basis states

The first step in constructing basis states is to identify one state from each occurrence of each $\wp_3$ orbit in (p,q). For triangular orbits, this state (henceforth called orbit representative) is taken to be the state $|\nu_1\nu_2\nu_3;I\rangle$ with $\nu_3\geqslant\nu_2$ and $\nu_3>\nu_1$. The location of orbit representatives in weight space is shown in Fig. 2, along with the location of the reflections of the representatives.

We will denote by $|\nu_1\nu_2\nu_3;I;\Gamma_\sigma,\mu\rangle$ the μth basis state of the representation Γ_σ containing the orbit representative $|\nu_1\nu_2\nu_3;I\rangle$.

We will repeatedly use the character projection operator,

$$P^\sigma=\frac{\dim_\sigma}{27}\sum_G \chi^\sigma(G)^*G,\tag{41}$$

where the sum is over the group elements of $\wp_3$, $\dim_\sigma$ is the dimension of the irrep Γ_σ, and χ^σ is the character of the element G in irrep Γ_σ.

We have been unable to obtain closed form expressions for $\wp_3$ states in a triangular orbit: the projection method fails since it generates states that are not orthogonal. We will come back to this later.

1. The representations Γ_{10} and Γ_{11}

The representations Γ_{10} and Γ_{11} are three-dimensional. They occur in representations where $p-q=1$ or $2 \bmod 3$. From the character table, we see that the appropriate projection operators is a sum of diagonal elements, so that we can choose orthonormal $\wp_3$ basis states as

$$|\nu_1\nu_2\nu_3;I;\Gamma_\sigma,1\rangle=|\nu_1\nu_2\nu_3;I\rangle,$$

$$|\nu_1\nu_2\nu_3;I;\Gamma_\sigma,2\rangle=A_0|\nu_1\nu_2\nu_3;I\rangle=\sum_L |\nu_3\nu_1\nu_2;L\rangle\langle\nu_3\nu_1\nu_2;L|A_0|\nu_1\nu_2\nu_3;I\rangle,$$

$$|\nu_1\nu_2\nu_3;I;\Gamma_\sigma,3\rangle=A_0^2|\nu_1\nu_2\nu_3;I\rangle=\sum_L |\nu_2\nu_3\nu_1;L\rangle\langle\nu_2\nu_3\nu_1;L|A_0^2|\nu_1\nu_2\nu_3;I\rangle, \tag{42}$$

where $\sigma=10$ if $p-q=1 \bmod 3$ or $\sigma=11$ if $p-q=2 \bmod 3$, and

$$\langle\nu_2\nu_3\nu_1;L|A_0^2|\nu_1\nu_2\nu_3;I\rangle=\langle\nu_2\nu_3\nu_1;L|A_0^{-1}|\nu_1\nu_2\nu_3;I\rangle=\langle\nu_1\nu_2\nu_3;I|A_0|\nu_2\nu_3\nu_1;L\rangle, \tag{43}$$

and where we have used the fact that $A_0^3=\mathbf{1}$.

2. The representations Γ_7, Γ_8, and Γ_9

For $\nu_3-\nu_2=2 \bmod 3$ for $p-q=0$, we find that $T=\mathbf{1}$ and $D_0=\omega^2\mathbf{1}$, with the result that

$$P^\sigma=\tfrac{1}{3}(\mathbf{1}+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^2), \quad \sigma=7,8,9, \tag{44}$$

from which we obtain the orthonormal basis states for triangles as

$$|\nu_1\nu_2\nu_3;I;\Gamma_7\rangle=\frac{1}{\sqrt{3}}(|\nu_1\nu_2\nu_3;I\rangle+A_0|\nu_1\nu_2\nu_3;I\rangle+A_0^{-1}|\nu_1\nu_2\nu_3;I\rangle),$$

$$|\nu_1\nu_2\nu_3;I;\Gamma_8\rangle=\frac{1}{\sqrt{3}}(|\nu_1\nu_2\nu_3;I\rangle+\omega^2A_0|\nu_1\nu_2\nu_3;I\rangle+\omega A_0^{-1}|\nu_1\nu_2\nu_3;I\rangle), \tag{45}$$

$$|\nu_1\nu_2\nu_3;I;\Gamma_9\rangle=\frac{1}{\sqrt{3}}(|\nu_1\nu_2\nu_3;I\rangle+\omega A_0|\nu_1\nu_2\nu_3;I\rangle+\omega^2A_0^{-1}|\nu_1\nu_2\nu_3;I\rangle).$$

The μ has been omitted since it is not necessary.

3. The representations Γ_4, Γ_5, and Γ_6

These representations decompose orbits with $\nu_3-\nu_2=1 \bmod 3$ in SL(3,ℂ) irreps with $p-q=0 \bmod 3$, so that $T=\mathbf{1}$, $D_0=\omega\mathbf{1}$, and

$$P^\sigma=\tfrac{1}{3}(\mathbf{1}+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^2), \quad \sigma=4,5,6. \tag{46}$$

The resulting orthonormalized basis states are therefore given by

$$|\nu_1\nu_2\nu_3;I;\Gamma_4\rangle=\frac{1}{\sqrt{3}}(|\nu_1\nu_2\nu_3;I\rangle+A_0|\nu_1\nu_2\nu_3;I\rangle+A_0^{-1}|\nu_1\nu_2\nu_3;I\rangle),$$

$$|\nu_1\nu_2\nu_3;I;\Gamma_5\rangle=\frac{1}{\sqrt{3}}(|\nu_1\nu_2\nu_3;I\rangle+\omega^2A_0|\nu_1\nu_2\nu_3;I\rangle+\omega A_0^{-1}|\nu_1\nu_2\nu_3;I\rangle), \tag{47}$$

$$|\nu_1\nu_2\nu_3;I;\Gamma_6\rangle=\frac{1}{\sqrt{3}}(|\nu_1\nu_2\nu_3;I\rangle+\omega A_0|\nu_1\nu_2\nu_3;I\rangle+\omega^2A_0^{-1}|\nu_1\nu_2\nu_3;I\rangle).$$

Again, μ is unnecessary.

4. The representations Γ_1,Γ_2, and Γ_3 in triangular orbits

These representations are contained in orbits with $\nu_3=\nu_2 \bmod 3$, and $p-q=0 \bmod 3$, so that $D=T=\mathbf{1}$. Furthermore, using $0=p-q=p+2q=\nu_1+\nu_2+\nu_3 \bmod 3$ and $\nu_3-\nu_2=0 \bmod 3$, we find that the projection operator is again given by

$$P^{\sigma}=\tfrac{1}{3}(\mathbf{1}+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^2),\qquad \sigma=1,2,3, \tag{48}$$

so that we find

$$|\nu_1\nu_2\nu_3;I;\Gamma_1\rangle=\frac{1}{\sqrt{3}}(|\nu_1\nu_2\nu_3;I\rangle+A_0|\nu_1\nu_2\nu_3;I\rangle+A_0^{-1}|\nu_1\nu_2\nu_3;I\rangle),$$

$$|\nu_1\nu_2\nu_3;I;\Gamma_2\rangle=\frac{1}{\sqrt{3}}(|\nu_1\nu_2\nu_3;I\rangle+\omega^2A_0|\nu_1\nu_2\nu_3;I\rangle+\omega A_0^{-1}|\nu_1\nu_2\nu_3;I\rangle), \tag{49}$$

$$|\nu_1\nu_2\nu_3;I;\Gamma_3\rangle=\frac{1}{\sqrt{3}}(|\nu_1\nu_2\nu_3;I\rangle+\omega A_0|\nu_1\nu_2\nu_3;I\rangle+\omega^2A_0^{-1}|\nu_1\nu_2\nu_3;I\rangle).$$

We do not need the index μ since the representations are one-dimensional. These states are orthonormal when the orbit is triangular.

5. The representations Γ_1,Γ_2, and Γ_3 in point orbits

When the weight $[\nu]=0$ occurs more than thrice, at least one $\wp_3$ will occur more than once, and the projection procedure fails: the resulting states are not orthogonal. Explicilty, if $k=\frac{1}{3}(p+2q)$, then the inner product of two states based on different angular momentum I and I' is given by

$$\langle kkk;I'|P_{\sigma'}^{-1}P_{\sigma}|kkk;I\rangle=\delta_{\sigma\sigma'}\left(\delta_{II'}+\sqrt{(2I+1)(2I'+1)}\begin{Bmatrix}\frac{1}{2}k & \frac{1}{2}k & I\\ \frac{1}{2}k & \frac{1}{2}q & I'\end{Bmatrix}\right.$$
$$\left.\times((-1)^{I}\omega^{2\sigma+1}+(-1)^{I'}\omega^{\sigma-1})\right). \tag{50}$$

One way to construct basis states for point orbits is to diagonalize A_0, which acts diagonally on these states. Let τ label multiple occurrences of the eigenvalue $\omega^{\sigma-1}$ of A_0 in the weight subspace $[\nu]=0$, and let $|\Gamma_\sigma,\tau\rangle$ be an eigenstate of A_0 with eigenvalue $\omega^{\sigma-1}$. If

$$|\Gamma_\sigma,\tau\rangle=\sum_I c_I^{\sigma,\tau}|kkk,I\rangle \tag{51}$$

is the expansion of this eigenvector in terms of our harmonic oscillator states, then the eigenvalue equation for A_0 implies

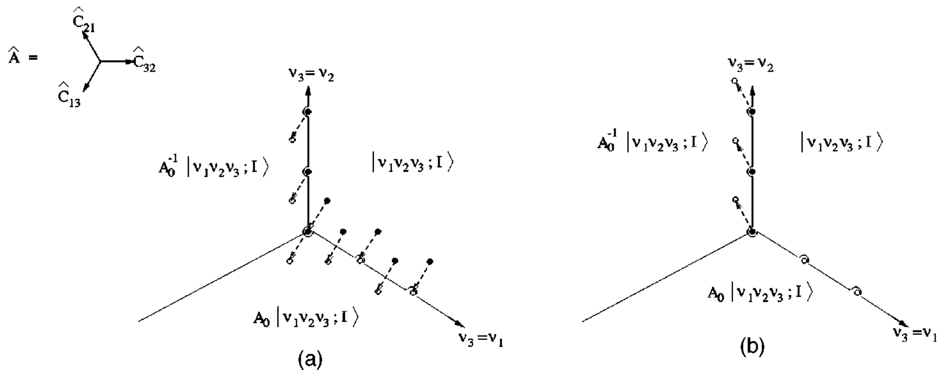

FIG. 3. The action of $\hat{C}_{13}$ and $\hat{C}_{21}$, respectively, on orbit representatives can sometimes produce states that need to be reflected back into representatives, as illustrated in (a) and (b).

$$\omega^{\sigma-1}c_{I'}^{\sigma,\tau}=\sum_{I}\ (-1)^{I}\sqrt{(2I+1)(2I'+1)}\begin{Bmatrix} \frac{1}{2}k & \frac{1}{2}k & I' \\ \frac{1}{2}k & \frac{1}{2}p & I \end{Bmatrix} c_I^{\sigma,\tau}, \tag{52}$$

while the eigenvalue equation for A_0^2 implies

$$\omega^{2\sigma+1}c_{I'}^{\sigma,\tau}=\sum_{I}\ (-1)^{I'}\sqrt{(2I+1)(2I'+1)}\begin{Bmatrix} \frac{1}{2}k & \frac{1}{2}k & I' \\ \frac{1}{2}k & \frac{1}{2}p & I \end{Bmatrix} c_I^{\sigma,\tau}. \tag{53}$$

Taking the sum of these two equations shows that the system of linear equations for the coefficients $c_I^{\sigma,\tau}$ can be separated in two systems containing, respectively, only even and only odd values of I.

We have been unable to solve this system in closed form, so that, for the purpose of calculating matrix elements, we will use as a basis for the 0-weight subspace the $SU(2)\times U(1)$ states $|kkk,I\rangle$.

V. $\wp_3$ GENERATOR MATRIX ELEMENTS

We are interested in the matrix elements of the two generators $\hat{A}$ and $\hat{D}$ of Eq. (9). They are given by

$$\hat{A}=\hat{C}_{21}+\hat{C}_{32}+\hat{C}_{13}, \tag{54}$$

$$=\hat{C}_{32}+A_0^{-1}\hat{C}_{32}A_0+A_0\hat{C}_{32}A_0^{-1}, \tag{55}$$

$$\hat{D}=\hat{C}_{11}+\omega^2\hat{C}_{22}+\omega\hat{C}_{33}. \tag{56}$$

The matrix elements of $\hat{A}$ and $\hat{D}$ will be expressed in terms of the matrix elements of the $\hat{C}_{ij}$, which are given explicilty in Appendix A.

A. Geometrical considerations

We will repeatedly have to evaluate expressions of the kind

$$\langle \nu_1'\nu_2'\nu_3';I'|A_0^k\hat{C}_{i+1,i}|\nu_1\nu_2\nu_3;I\rangle, \tag{57}$$

where $k=0,1,2$, $i=0,1,2 \bmod 3$ and where $\langle \nu_1'\nu_2'\nu_3';I'|$ and $|\nu_1\nu_2\nu_3;I\rangle$ are orbit representatives with $\nu_3\geqslant\nu_2,\nu_3>\nu_1$ and $\nu_3'\geqslant\nu_2',\nu_3'>\nu_1'$.

As illustrated in Fig. 3, it follows from the location of such states in weight space that

$$\langle \nu_1'\nu_2'\nu_3';I'|A_0^{-1}\hat{C}_{32}|\nu_1\nu_2\nu_3;I\rangle=0,$$

$$\langle \nu_1'\nu_2'\nu_3';I'|A_0\hat{C}_{32}|\nu_1\nu_2\nu_3;I\rangle=0,$$

$$\langle \nu_1'\nu_2'\nu_3';I'|A_0^{-1}\hat{C}_{21}|\nu_1\nu_2\nu_3;I\rangle=0, \tag{58}$$

$$\langle \nu_1'\nu_2'\nu_3';I'|A_0\hat{C}_{21}|\nu_1\nu_2\nu_3;I\rangle\neq 0, \quad \text{only when} \quad \nu_2=\nu_3,$$

$$\langle \nu_1'\nu_2'\nu_3';I'|A_0^{-1}\hat{C}_{13}|\nu_1\nu_2\nu_3;I\rangle\neq 0, \quad \text{only when} \quad \nu_3=\nu_1+1 \quad \text{or} \quad \nu_1+2,$$

$$\langle \nu_1'\nu_2'\nu_3';I'|A_0\hat{C}_{13}|\nu_1\nu_2\nu_3;I\rangle\neq 0, \quad \text{only when} \quad \nu_3=\nu_2.$$

B. Three-dimensional representations

Since basis states are of the type $|\nu_1\nu_2\nu_3;I;\Gamma_\sigma,\mu\rangle=A_0^{\mu-1}|\nu_1\nu_2\nu_3;I\rangle$, we have, in general, expressions of the type

$$\langle \nu_1'\nu_2'\nu_3';I'|A_0^{2\mu'+1}\hat{A}A_0^{\mu-1}|\nu_1\nu_2\nu_3;I\rangle=\langle \nu_1'\nu_2'\nu_3';I'|A_0^{2\mu'+\mu}\hat{A}|\nu_1\nu_2\nu_3;I\rangle, \tag{59}$$

since $\hat{A}$ transforms into itself under conjugation by A_0. Using Eq. (58), we therefore have three cases.

1. $2\mu'+\mu=0 \bmod 3$

This simply yields

$$\begin{aligned}\langle \nu_1'\nu_2'\nu_3';I'|\hat{A}|\nu_1\nu_2\nu_3;I\rangle=&\langle \nu_1'\nu_2'\nu_3';I'|\hat{C}_{32}|\nu_1\nu_2\nu_3;I\rangle\\&+\langle \nu_1'\nu_2'\nu_3';I'|\hat{C}_{21}|\nu_1\nu_2\nu_3;I\rangle(1-\delta_{\nu_2\nu_3})\\&+\langle \nu_1'\nu_2'\nu_3';I'|\hat{C}_{13}|\nu_1\nu_2\nu_3;I\rangle\\&\times(1-\delta_{\nu_2\nu_3}-\delta_{\nu_3,\nu_1+1}-\delta_{\nu_3,\nu+2}).\end{aligned} \tag{60}$$

2. $2\mu'+\mu=1 \bmod 3$

This is nonzero only if $\nu_2=\nu_3$. We then have

$$\begin{aligned}&\langle \nu_1'\nu_2'\nu_3';I'|A_0(\hat{C}_{21}+\hat{C}_{13})|\nu_1\nu_2\nu_3;I\rangle\\&=\sum_{L'}\langle \nu_1'\nu_2'\nu_3';I'|A_0|\nu_2'\nu_3'\nu_1';L'\rangle\times[\langle \nu_2'\nu_3'\nu_1';L'|\hat{C}_{21}|\nu_1\nu_2\nu_3;I\rangle\\&\quad+\langle \nu_2'\nu_3'\nu_1';L'|\hat{C}_{13}|\nu_1\nu_2\nu_3;I\rangle]\delta_{\nu_3\nu_2}.\end{aligned} \tag{61}$$

3. $2\mu'+\mu=2 \bmod 3$

Here, we must have $\nu_3=\nu_1+1$ or $\nu_3=\nu_1+2$, with the result

$$\langle \nu_1'\nu_2'\nu_3';I'|A_0^{-1}\hat{C}_{13}|\nu_1\nu_2\nu_3;I\rangle = \sum_{L'} \langle \nu_1'\nu_2'\nu_3';I'|A_0^{-1}|\nu_3'\nu_1'\nu_2';L'\rangle \times \langle \nu_2'\nu_3'\nu_1';L'|\hat{C}_{13}|\nu_1\nu_2\nu_3;I\rangle \times (\delta_{\nu_3,\nu_1+1}+\delta_{\nu_3,\nu_1+2}). \tag{62}$$

C. One-dimensional representations with no point orbits

Consider now the case of a three-dimensional orbit which decomposes into a sum of three one-dimensional representations of $\wp_3$. To obtain the matrix elements of $\hat{A}$ as a sum of matrix elements of $\hat{C}_{ij}$, we write

$$|\nu_1\nu_2\nu_3;I;\Gamma_\sigma\rangle = \frac{1}{\sqrt{3}}(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1})|\nu_1\nu_2\nu_3;I\rangle, \tag{63}$$

$$\hat{A} = \hat{C}_{32} + A_0\hat{C}_{32}A_0^{-1} + A_0^{-1}\hat{C}_{32}A_0, \tag{64}$$

with the usual condition $\nu_3 \geqslant \nu_2, \nu_3 > \nu_1$, and note that

$$A_0(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1}) = \omega^{\sigma-1}(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1}),$$
$$A_0^{-1}(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1}) = \omega^{2\sigma+1}(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1}). \tag{65}$$

From this, we obtain

$$\langle \nu_1'\nu_2'\nu_3';I';\Gamma_{\sigma'}|\hat{A}|\nu_1\nu_2\nu_3;I;\Gamma_\sigma\rangle = \tfrac{1}{3}\langle \nu_1'\nu_2'\nu_3';I|(1+\omega^{2\sigma'+1}A_0+\omega^{\sigma'-1}A_0^{-1}) \times (1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1}) \times \hat{C}_{32}(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1})|\nu_1\nu_2\nu_3;I\rangle. \tag{66}$$

Now,

$$(1+\omega^{2\sigma'+1}A_0+\omega^{\sigma'-1}A_0^{-1})(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1}) = 3(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1})\delta_{\sigma'-\sigma,0}, \tag{67}$$

where the sum $\sigma'-\sigma$ is taken modulo 3, so that

$$\begin{aligned}\langle \nu_1'\nu_2'\nu_3';I';\Gamma_{\sigma'}|\hat{A}|\nu_1\nu_2\nu_3;I;\Gamma_\sigma\rangle &= \langle \nu_1'\nu_2'\nu_3';I'|(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1})\hat{C}_{13}|\nu_1\nu_2\nu_3;I\rangle \\ &+ \langle \nu_1'\nu_2'\nu_3';I'|(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1})\hat{C}_{21}|\nu_1\nu_2\nu_3;I\rangle \\ &+ \langle \nu_1'\nu_2'\nu_3';I'|(1+\omega^{2\sigma+1}A_0+\omega^{\sigma-1}A_0^{-1})\hat{C}_{32}|\nu_1\nu_2\nu_3;I\rangle,\end{aligned} \tag{68}$$

where we have assumed the $\sigma'-\sigma=0$ modulo 3, and where we have used equalities such as $A_0\hat{C}_{21}A_0^{-1}=\hat{C}_{32}$ to eliminate as many factors of A_0 as possible.

Assuming that no points orbits are involved, we can therefore simplify the matrix element into its final form:

$$\begin{aligned}\langle \nu_1'\nu_2'\nu_3';I';\Gamma_{\sigma'}|\hat{A}|\nu_1\nu_2\nu_3;I;\Gamma_\sigma\rangle &= \langle \nu_1'\nu_2'\nu_3';I'|\hat{C}_{13}|\nu_1\nu_2\nu_3;I\rangle(1-\delta_{\nu_2\nu_3}-\delta_{\nu_3,\nu_1+1}-\delta_{\nu_3,\nu_2+1})\delta_{\sigma-\sigma',0} \\ &+ \omega^{\sigma-1}\langle \nu_1'\nu_2'\nu_3';I'|A_0^{-1}\hat{C}_{13}|\nu_1\nu_2\nu_3;I\rangle(\delta_{\nu_3,\nu_1+1}+\delta_{\nu_3,\nu_1+2})\delta_{\sigma-\sigma',0}\end{aligned}$$

$$
\begin{aligned}
&+\omega^{2\sigma+1}\langle \nu_1'\nu_2'\nu_3';I'|A_0\hat{C}_{13}|\nu_1,\nu_2,\nu_3;I\rangle\,\delta_{\nu_2\nu_3}\delta_{\sigma-\sigma',0}\\
&+\langle \nu_1'\nu_2'\nu_3';I'|\hat{C}_{21}|\nu_1,\nu_2,\nu_3;I\rangle(1-\delta_{\nu_2\nu_3})\,\delta_{\sigma-\sigma',0}\\
&+\omega^{2\sigma+1}\langle \nu_1'\nu_2'\nu_3';I'|A_0\hat{C}_{21}|\nu_1,\nu_1,\nu_3;I\rangle\,\delta_{\nu_2\nu_3}\delta_{\sigma-\sigma',0}\\
&+\langle \nu_1'\nu_2'\nu_3';I'|\hat{C}_{32}|\nu_1,\nu_2,\nu_3;I\rangle\,\delta_{II'}\,\delta_{\sigma-\sigma',0}
\end{aligned}
\tag{69}
$$

where the sum $\sigma-\sigma'$ is taken modulo 3.

D. One-dimensional representations with point orbits

When there are point orbits, we must be satisfied with the computation of $\langle kkk,I'|\hat{A}|k-1,k,k+1;I;\Gamma_\sigma\rangle$ and $\langle k,k-1,k+1;I';\Gamma_\sigma|\hat{A}|kkk,I\rangle$, where $k=\frac{1}{3}(p+2q)$, as we have no explicit expressions for $\wp_3$ basis states in these cases.

To compute $\langle kkk,I'|\hat{A}|k-1,k,k+1;I;\Gamma_\sigma\rangle$, note that

$$
\begin{aligned}
&\langle kkk,I'|\hat{A}|k-1,k,k+1;I;\Gamma_\sigma\rangle\\
&=\frac{1}{\sqrt{3}}\langle kkk,I'|(\hat{C}_{13}+\hat{C}_{21}\omega^{2\sigma+1}A_0+\hat{C}_{32}\omega^{\sigma-1}A_0^2)|k-1,k,k+1;I\rangle,\\
&=\frac{1}{\sqrt{3}}\langle kkk,I'|(\hat{C}_{13}+\omega^{2\sigma+1}A_0\hat{C}_{13}+\omega^{\sigma-1}A_0^2\hat{C}_{13})|k-1,k,k+1;I\rangle,\\
&=\frac{1}{\sqrt{3}}\langle kkk,I'|\hat{C}_{13}|k-1,k,k+1;I\rangle+\frac{1}{\sqrt{3}}\sum_L\,[\omega^{2\sigma+1}(-1)^L+\omega^{\sigma-1}(-1)^{I'}]\\
&\quad\times\sqrt{(2L+1)(2I'+1)}\begin{Bmatrix}\frac{1}{2}k & \frac{1}{2}k & L\\ \frac{1}{2}k & \frac{1}{2}p & I'\end{Bmatrix}\langle kkk,L|\hat{C}_{13}|k-1,k,k+1;I\rangle.
\end{aligned}
\tag{70}
$$

From the selection rules on $\hat{C}_{13}$, we note that we must have $L=I\pm\frac{1}{2}$.

In particular, for $\sigma=1 \bmod 3$, this shows that the sum extends over those values of L that have the same parity as I'. However, from the selection rules on the matrix elements of $\hat{C}_{13}$, the possible values of L are $I\pm\frac{1}{2}$, and the only value of L that is of the same parity as I' in $I\pm\frac{1}{2}$ is $L=I'$, further simplifying the final expression.

VI. CONCLUSION

In this paper we have presented an explicit construction of SL$(3,\mathbb{C})$ states in a $\wp_3$ subgroup basis. This $\wp_3$ subgroup has several interesting properties.

The group $\wp_3$ is generated by three elements, and induces a $\mathbb{Z}_3\otimes\mathbb{Z}_3$ grading of SL$(3,\mathbb{C})$. In the $(1,0)$ representation of sl$(3,\mathbb{C})$, linearly independent combinations of $\wp_3$ elements can be chosen as a basis for the sl$(3,\mathbb{C})$ algebra. The chosen $\wp_3$ matrices of this representation play a dual role as either subgroup elements or as elements of the sl$(3,\mathbb{C})$ algebra. Any basis element of sl$(3,\mathbb{C})$ can be obtained from the (possibly multiple) commutator of two generators for instance A_0 and D_0 corresponding to two generating elements of $\wp_3$.

The orbits of $\wp_3$ in the representation space of SL$(3,\mathbb{C})$ are simple, and make the construction of SL$(3,\mathbb{C})\supset\wp_3$ basis states devoid of many of the difficulties usually associated with the construction of a finite subgroup basis. This makes the action of a $\wp_3$ element on a basis state is relatively easy to compute. One needs to know only the matrix elements of A_0 and D_0. Moreover, only one of them acts in a nontrivial way on basis states.

The computation of sl$(3,\mathbb{C})$ generators can almost always be related to the computation of sl$(3,\mathbb{C})$ generators between states in a single sector covering one-third of the weight space.

Finally, it is quite clear that the method presented here can be generalized to the subgroup $\wp_n\subset$SL$(n,\mathbb{C})$ described in Ref. 1.

ACKNOWLEDGMENTS

Supported in part by N.S.E.R.C. of Canada and by the Fonds FCAR of the Government of Québec.

APPENDIX A: SU(2)×U(1) GENERATOR MATRIX ELEMENTS IN THE OSCILLATOR BASIS

It is simplest to compute the matrix elements of $\hat{C}_{31}$ and to extract from it those of $\hat{C}_{13}$ and $\hat{C}_{21}$.

In the harmonic oscillator basis:

$$
\begin{aligned}
&\langle \nu_1'\nu_2'\nu_3';I'|\hat{C}_{31}|\nu_1\nu_2\nu_3;I\rangle \\
&= \sum_{m_1m_2m_3(N)} \sum_{m_1'm_3'(N')} \left\langle \begin{matrix} \frac{1}{2}\nu_3 & \frac{1}{2}\nu_2 & I \\ m_3 & m_2 & N \end{matrix} \right\rangle \left\langle \begin{matrix} I & \frac{1}{2}\nu_1 & \frac{1}{2}p \\ N & m_1 & \frac{1}{2}p \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{1}{2}\nu_3' & \frac{1}{2}\nu_2 & I' \\ m_3' & m_2 & N' \end{matrix} \right\rangle \\
&\quad \times \left\langle \begin{matrix} I' & \frac{1}{2}\nu_1' & \frac{1}{2}p \\ N' & m_1' & \frac{1}{2}p \end{matrix} \right\rangle \frac{1}{\sqrt{(\frac{1}{2}\nu_1+m_1)!(\frac{1}{2}\nu_1-m_1)!(\frac{1}{2}\nu_3+m_3)!(\frac{1}{2}\nu_3-m_3)!}} \\
&\quad \times \frac{1}{\sqrt{(\frac{1}{2}\nu_1'+m_1')!(\frac{1}{2}\nu_1'-m_1')!(\frac{1}{2}\nu_3'+m_3')!(\frac{1}{2}\nu_3'-m_3')!}} \times [(\tfrac{1}{2}\nu_1+m_1)!(\tfrac{1}{2}\nu_1-m_1)! \\
&\quad \times (\tfrac{1}{2}\nu_3+m_3+1)!(\tfrac{1}{2}\nu_3-m_3)!\,\delta_{(1/2)\nu_1+m_1-1,(1/2)\nu_1'+m_1'}\delta_{(1/2)\nu_1-m_1,(1/2)\nu_1'-m_1'} \\
&\quad \times \delta_{(1/2)\nu_3+m_3+1,(1/2)\nu_3'+m_3'}\delta_{(1/2)\nu_3-m_3,(1/2)\nu_3'-m_3'} + (\tfrac{1}{2}\nu_1+m_1)!(\tfrac{1}{2}\nu_1-m_1)! \\
&\quad \times (\tfrac{1}{2}\nu_3+m_3)!(\tfrac{1}{2}\nu_3-m_3+1)!\,\delta_{(1/2)\nu_1+m_1,(1/2)\nu_1'+m_1'}\delta_{(1/2)\nu_1-m_1-1,(1/2)\nu_1'-m_1'} \\
&\quad \times \delta_{(1/2)\nu_3+m_3,(1/2)\nu_3'+m_3'}\delta_{(1/2)\nu_3-m_3+1,(1/2)\nu_3'-m_3'}],
\end{aligned}
\tag{A1}
$$

which, after simplification, yields

$$
\begin{aligned}
&\langle \nu_1-1,\nu_2,\nu_3+1;I'|\hat{C}_{31}|\nu_1\nu_2\nu_3;I\rangle \\
&= \sum_{m_1m_2m_3(N)} \left\langle \begin{matrix} \frac{1}{2}\nu_3 & \frac{1}{2}\nu_2 & I \\ m_3 & m_2 & N \end{matrix} \right\rangle \left\langle \begin{matrix} I & \frac{1}{2}\nu_1 & \frac{1}{2}p \\ N & m_1 & \frac{1}{2}p \end{matrix} \right\rangle \left[\left\langle \begin{matrix} \frac{1}{2}\nu_3+\frac{1}{2} & \frac{1}{2}\nu_2 & I' \\ m_3+\frac{1}{2} & m_2 & N+\frac{1}{2} \end{matrix} \right\rangle \right. \\
&\quad \times \left\langle \begin{matrix} I' & \frac{1}{2}\nu-\frac{1}{2} & \frac{1}{2}p \\ N+\frac{1}{2} & m_1-\frac{1}{2} & \frac{1}{2}p \end{matrix} \right\rangle \sqrt{(\tfrac{1}{2}\nu_1+m_1)(\tfrac{1}{2}\nu_3+m_3+1)} + \left\langle \begin{matrix} \frac{1}{2}\nu_3+\frac{1}{2} & \frac{1}{2}\nu_2 & I' \\ m_3-\frac{1}{2} & m_2 & N-\frac{1}{2} \end{matrix} \right\rangle \\
&\quad \left. \times \left\langle \begin{matrix} I' & \frac{1}{2}\nu-\frac{1}{2} & \frac{1}{2}p \\ N-\frac{1}{2} & m_1+\frac{1}{2} & \frac{1}{2}p \end{matrix} \right\rangle \sqrt{(\tfrac{1}{2}\nu_1-m_1)(\tfrac{1}{2}\nu_3-m_3+1)} \right].
\end{aligned}
\tag{A2}
$$

Using now the two equalities between CG with arguments differing by $\frac{1}{2}$,

$$
\begin{aligned}
&\left\langle \begin{matrix} \frac{1}{2}\nu_3+\frac{1}{2} & \frac{1}{2}\nu_2 \\ m_3+\frac{1}{2} & m_2 \end{matrix} \middle| \begin{matrix} I' \\ N' \end{matrix} \right\rangle \\
&\quad = \sqrt{\frac{(I'+N')(\frac{1}{2}\nu_3-\frac{1}{2}\nu_2+\frac{1}{2}+I')(\frac{1}{2}\nu_2+\frac{1}{2}\nu_3+I'+\frac{3}{2})}{2I'(2I'+1)(\frac{1}{2}\nu_3+m_3+1)}} \left\langle \begin{matrix} \frac{1}{2}\nu_3 & \frac{1}{2}\nu_2 \\ m_3 & m_2 \end{matrix} \middle| \begin{matrix} I'-\frac{1}{2} \\ N'-\frac{1}{2} \end{matrix} \right\rangle \\
&\qquad + \sqrt{\frac{(I'-N'+1)(\frac{1}{2}\nu_2-\frac{1}{2}\nu_3+\frac{1}{2}+I')(\frac{1}{2}\nu_2+\frac{1}{2}\nu_3+\frac{1}{2}-I')}{2(I'+1)(2I'+1)(\frac{1}{2}\nu_3+m_3+1)}} \\
&\qquad \times \left\langle \begin{matrix} \frac{1}{2}\nu_3 & \frac{1}{2}\nu_2 \\ m_3 & m_2- \end{matrix} \middle| \begin{matrix} I'+\frac{1}{2} \\ N'-\frac{1}{2} \end{matrix} \right\rangle,
\end{aligned}
$$

$$
\begin{aligned}
&\left\langle \begin{matrix} \frac{1}{2}\nu_3+\frac{1}{2} & \frac{1}{2}\nu_2 & I' \\ m_3-\frac{1}{2} & m_2 & N' \end{matrix} \right\rangle \\
&\quad = \sqrt{\frac{(I'-N')(\frac{1}{2}\nu_3-\frac{1}{2}\nu_2+\frac{1}{2}+I')(\frac{1}{2}\nu_2+\frac{1}{2}\nu_3+I'+\frac{3}{2})}{2I'(2I'+1)(\frac{1}{2}\nu_3-m_3+1)}} \left\langle \begin{matrix} \frac{1}{2}\nu_3 & \frac{1}{2}\nu_2 \\ m_3 & m_2 \end{matrix} \middle| \begin{matrix} I'-\frac{1}{2} \\ N'+\frac{1}{2} \end{matrix} \right\rangle \\
&\qquad - \sqrt{\frac{(I'+N'+1)(\frac{1}{2}\nu_2-\frac{1}{2}\nu_3+\frac{1}{2}+I')(\frac{1}{2}\nu_2+\frac{1}{2}\nu_3+\frac{1}{2}-I')}{2(I'+1)(2I'+1)(\frac{1}{2}\nu_3-m_3+1)}} \left\langle \begin{matrix} \frac{1}{2}\nu_3 & \frac{1}{2}\nu_2 \\ m_3 & m_2 \end{matrix} \middle| \begin{matrix} I'+\frac{1}{2} \\ N'+\frac{1}{2} \end{matrix} \right\rangle,
\end{aligned}
\tag{A3}
$$

for $N'=N\pm\frac{1}{2}$, respectively, one can eliminate the sums over m_2 and m_3 to find

$$
\begin{aligned}
&\langle \nu_1-1,\nu_2,\nu_3+1;I'|\hat{C}_{31}|\nu_1\nu_2\nu_3;I\rangle \\
&\quad = \sum_{m_1 N} \left\langle \begin{matrix} I & \frac{1}{2}\nu_1 \\ N & m_1 \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle \sqrt{\frac{(I-\frac{1}{2}\nu_2+\frac{1}{2}\nu_3+1)(\frac{1}{2}\nu_2+\frac{1}{2}\nu_3+I+2)}{2(I+1)(2I+1)}} \\
&\qquad \times \Bigg[\left\langle \begin{matrix} I+\frac{1}{2} & \frac{1}{2}\nu_1-\frac{1}{2} \\ N+\frac{1}{2} & m_1-\frac{1}{2} \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle \sqrt{(\tfrac{1}{2}\nu_1+m_1)(I+N+1)} \\
&\qquad + \left\langle \begin{matrix} I+\frac{1}{2} & \frac{1}{2}\nu_1-\frac{1}{2} \\ N-\frac{1}{2} & m_1+\frac{1}{2} \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle \sqrt{(\tfrac{1}{2}\nu_1-m_1)(I-N+1)} \Bigg] \delta_{I',I+\frac{1}{2}} + \sum_{m_1 N} \left\langle \begin{matrix} I & \frac{1}{2}\nu_1 \\ N & m_1 \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle \\
&\qquad \times \sqrt{\frac{(I+\frac{1}{2}\nu_2-\frac{1}{2}\nu_3)(\frac{1}{2}\nu_2+\frac{1}{2}\nu_3-I+1)}{2I(2I+1)}} \Bigg[\left\langle \begin{matrix} I-\frac{1}{2} & \frac{1}{2}\nu_1-\frac{1}{2} \\ N+\frac{1}{2} & m_1-\frac{1}{2} \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle \sqrt{(\tfrac{1}{2}\nu_1+m_1)(I-N)} \\
&\qquad - \left\langle \begin{matrix} I-\frac{1}{2} & \frac{1}{2}\nu_1-\frac{1}{2} \\ N-\frac{1}{2} & m_1+\frac{1}{2} \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle \sqrt{(\tfrac{1}{2}\nu_1-m_1)(I+N)} \Bigg] \delta_{I',I-\frac{1}{2}}.
\end{aligned}
\tag{A4}
$$

Finally, using the explicit expression for CG of the type

$$
\left\langle \begin{matrix} J_1 & J_2 \\ M_1 & M_2 \end{matrix} \middle| \begin{matrix} J \\ J \end{matrix} \right\rangle,
$$

we find

$$\left\langle \begin{matrix} I+\frac{1}{2} & \frac{1}{2}\nu_1-\frac{1}{2} \\ N+\frac{1}{2} & m_1-\frac{1}{2} \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle = \sqrt{\frac{(I+N+1)(-I+\frac{1}{2}\nu_1+\frac{1}{2}p)}{(\frac{1}{2}\nu_1+m_1)(I-\frac{1}{2}\nu_1+1+\frac{1}{2}p)}} \left\langle \begin{matrix} I & \frac{1}{2}\nu_1 \\ N & m_1 \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle, \tag{A5}$$

$$\left\langle \begin{matrix} I+\frac{1}{2} & \frac{1}{2}\nu_1-\frac{1}{2} \\ N-\frac{1}{2} & m_1+\frac{1}{2} \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle = -\sqrt{\frac{(\frac{1}{2}\nu_1-m_1)}{(I-\frac{1}{2}\nu_1+1+\frac{1}{2}p)(I-N+1)}} \left\langle \begin{matrix} I & \frac{1}{2}\nu_1 \\ N & m_1 \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle \tag{A6}$$

$$\left\langle \begin{matrix} I-\frac{1}{2} & \frac{1}{2}\nu_1-\frac{1}{2} \\ N+\frac{1}{2} & m_1-\frac{1}{2} \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle = -\sqrt{\frac{(I-N)(I+\frac{1}{2}\nu_1+1+\frac{1}{2}p)}{(\frac{1}{2}\nu_1+m_1)(I+\frac{1}{2}\nu_1-\frac{1}{2}p)}} \left\langle \begin{matrix} I & \frac{1}{2}\nu_1 \\ N & m_1 \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle, \tag{A7}$$

$$\left\langle \begin{matrix} I-\frac{1}{2} & \frac{1}{2}\nu_1-\frac{1}{2} \\ N-\frac{1}{2} & m_1+\frac{1}{2} \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle = \sqrt{\frac{(\frac{1}{2}\nu_1-m_1)(I+\frac{1}{2}\nu_1+\frac{1}{2}p+1)}{(I+\frac{1}{2}\nu_1-\frac{1}{2}p)(I+N)}} \left\langle \begin{matrix} I & \frac{1}{2}\nu_1 \\ N & m_1 \end{matrix} \middle| \begin{matrix} \frac{1}{2}p \\ \frac{1}{2}p \end{matrix} \right\rangle, \tag{A8}$$

and we can eliminate the remaining sums (remembering that $N+m_1=\frac{1}{2}p$) to obtain the final result,

$$\begin{aligned}\langle \nu_1-1,\nu_2,\nu_3+1;I'|\hat{C}_{31}|\nu_1\nu_2\nu_3;I\rangle \\ = \sqrt{\frac{(I-\frac{1}{2}\nu_2+\frac{1}{2}\nu_3+1)(\frac{1}{2}\nu_2+\frac{1}{2}\nu_3+I+2)(I-\frac{1}{2}\nu_1+\frac{1}{2}p+1)(-I+\frac{1}{2}\nu_1+\frac{1}{2}p)}{2(I+1)(2I+1)}}\,\delta_{I',I+1/2} \\ -\sqrt{\frac{(I+\frac{1}{2}\nu_2-\frac{1}{2}\nu_3)(\frac{1}{2}\nu_2+\frac{1}{2}\nu_3-I+1)(I+\frac{1}{2}\nu_1+\frac{1}{2}p+1)(I+\frac{1}{2}\nu_1-\frac{1}{2}p)}{2I(2I+1)}}\,\delta_{I',I-1/2}.\end{aligned} \tag{A9}$$

Taking the adjoint of Eq. (A9), we find

$$\begin{aligned}\langle \nu_1+1,\nu_2,\nu_3-1;I'|\hat{C}_{13}|\nu_1\nu_2\nu_3;I\rangle \\ = \sqrt{\frac{(I-\frac{1}{2}\nu_2+\frac{1}{2}\nu_3)(I+\frac{1}{2}\nu_2+\frac{1}{2}\nu_3+1)(I-\frac{1}{2}\nu_1+\frac{1}{2}p)(-I+\frac{1}{2}\nu_1+\frac{1}{2}p+1)}{2I(2I+1)}}\,\delta_{I',I-1/2} \\ -\sqrt{\frac{(I+\frac{1}{2}\nu_2-\frac{1}{2}\nu_3+1)(-I+\frac{1}{2}\nu_2+\frac{1}{2}\nu_3)(I+\frac{1}{2}\nu_1+\frac{1}{2}p+2)(I+\frac{1}{2}\nu_1-\frac{1}{2}p+1)}{2(I+1)(2I+1)}}\,\delta_{I',I+1/2}.\end{aligned} \tag{A10}$$

The operator $\hat{C}_{21}$ is the $\frac{1}{2}$ component of a $\hat{f}^{1/2}$ tensor operator, whose $-\frac{1}{2}$ component is $\hat{C}_{21}$, so that, using the Wigner–Eckart theorem, we find

$$\begin{aligned}\langle \nu_1-1,\nu_2+1,\nu_3;I'|\hat{C}_{21}|\nu_1\nu_2\nu_3;I\rangle \\ = \sqrt{\frac{(I+\frac{1}{2}\nu_2-\frac{1}{2}\nu_3+1)(\frac{1}{2}\nu_2+\frac{1}{2}\nu_3+I+2)(I-\frac{1}{2}\nu_1+\frac{1}{2}p+1)(-I+\frac{1}{2}\nu_1+\frac{1}{2}p)}{2(I+1)(2I+1)}}\,\delta_{I',I+1/2} \\ +\sqrt{\frac{(I-\frac{1}{2}\nu_2+\frac{1}{2}\nu_3)(\frac{1}{2}\nu_2+\frac{1}{2}\nu_3-I+1)(I+\frac{1}{2}\nu_1+\frac{1}{2}p+1)(I+\frac{1}{2}\nu_1-\frac{1}{2}p)}{2I(2I+1)}}\,\delta_{I',I-1/2}.\end{aligned} \tag{A11}$$

Finally, the matrix element of $\hat{C}_{32}$ is simply given by

$$\langle \nu_1, \nu_2-1, \nu_3+1, I' | \hat{C}_{32} | \nu_1 \nu_2 \nu_3 ; I \rangle = \sqrt{(I-\tfrac{1}{2}\nu_3+\tfrac{1}{2}\nu_2)(I+\tfrac{1}{2}\nu_3-\tfrac{1}{2}\nu_2+1)}\, \delta_{II'} . \quad \text{(A12)}$$

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