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su(1,1) intelligent states

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Abstract
We construct all the intelligent states of the non-compact generators of $su(1,1)$ for every positive discrete representation of this Lie algebra, and discuss some of the properties of these states.

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1. Introduction

The objective of this paper is to construct intelligent states for the operators $\hat{K}_x$ and $\hat{K}_y$ of $su(1,1)$ when the Hilbert space carries a representation of the positive discrete series of $su(1,1)$. As an application, we discuss squeezing properties of some of the resulting states.

Our work is motivated in part by application to quantum optics [1, 2], where the $su(1,1)$ generators are realized as operators involving either a single type of boson

$$\hat{K}_x = \tfrac{1}{4}(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}), \qquad \hat{K}_y = \tfrac{1}{4i}(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}), \qquad \hat{K}_0 = \tfrac{1}{4}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) \tag{1}$$

or two separate distinct bosons:

$$\hat{K}_x = \tfrac{1}{2}(\hat{c}^\dagger\hat{d}^\dagger + \hat{c}\hat{d}), \qquad \hat{K}_y = \tfrac{1}{2i}(\hat{c}^\dagger\hat{d}^\dagger - \hat{c}\hat{d}), \qquad \hat{K}_z = \tfrac{1}{2}(\hat{c}^\dagger\hat{c} + \hat{d}^\dagger\hat{d} + 1). \tag{2}$$

The first realization is usually taken to describe degenerate parametric down-conversion, whereas the second describes the non-degenerate case. Numerous other realizations and applications of $su(1,1)$ have been described in the literature [3, 4].

It is easy to verify that operators in equations (1) or (2) satisfy the abstract $su(1,1)$ commutation relations:

$$[\hat{K}_x, \hat{K}_y] = -\mathrm{i}\hat{K}_z, \qquad [\hat{K}_y, \hat{K}_z] = \mathrm{i}\hat{K}_x, \qquad [\hat{K}_z, \hat{K}_x] = \mathrm{i}\hat{K}_y. \tag{3}$$

By definition, intelligent states for $\hat{K}_x$ and $\hat{K}_y$ are the states $|\psi(\alpha)\rangle$ for which

$$\Delta K_x \Delta K_y = \tfrac{1}{2}|\langle\hat{K}_z\rangle| \tag{4}$$

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holds. They are solutions [5] to the eigenvalue problem

$$(\hat{K}_x - i\alpha\hat{K}_y)|\psi(\alpha)\rangle = \lambda|\psi(\alpha)\rangle, \qquad \alpha \in \mathbb{R}. \tag{5}$$

From a mathematical perspective, the infinite-dimensional but discrete nature of the positive discrete representations of $su(1,1)$ makes the intelligent states fall 'in between' the familiar case of momentum and position—where the spectra of $\hat{x}$ and $\hat{p}$ are continuous and their eigenstates non-normalizable—and the case of angular momentum intelligent states [2, 6–8]—where the spectra of $\hat{J}_x$ and $\hat{J}_y$ are discrete and their eigenstates normalizable. Unlike the $\hat{x} - \hat{p}$ case, the right-hand side of equation (4) is state dependent. The strict minimum for the right-hand side is 0 but, unlike the $\hat{J}_x - \hat{J}_y$ case, this minimum cannot be reached as the eigenstates of $\hat{K}_x$ and $\hat{K}_y$ are not normalizable.

Let us assume for the moment that the parameter $\alpha \geqslant 1$. The Perelomov $SU(1,1)$ coherent state

$$e^{-i\tau\hat{K}_y}|k,k\rangle \tag{6}$$

is then intelligent provided $\alpha = \cosh(\tau)$ and $|k,k\rangle$ is the lowest weight state of the irreducible representation (irrep) k (see section 2 for notational details). The eigenvalue λ is $-k\sinh\tau$. However, for any unitary positive discrete series representation, there are in addition to equation (6) an infinite number of other solutions to equation (5).

In this paper we describe these sets of solutions, reviewing in part and going beyond the work of [9–11] to include those solutions realized by the boson construction of equations (1) or (2) but belonging to a *single* $su(1,1)$ irrep of the positive discrete series. Our analysis, however, is restricted to intelligent states of the two non-compact generators and so does not overlap with the work of Puri and Agarwal, who constructed in [11] intelligent states for one compact and one non-compact generator.

Our starting point will be the mapping of equation (5) to a Schrödinger-like differential equation in 'dummy space', equation (20). This mapping is valid only for the realization of equation (1). Our differential equation is to be contrasted with that of [10], which exploited solutions to a second-order differential equation in the 'coherent variable' α.

The solutions of equation (20) belong to two infinite families: those containing only even and those containing only odd harmonic oscillator kets. They correspond, respectively, to states of the $k = \frac{1}{4}$ and $k = \frac{3}{4}$ series described in section 2. They were dubbed 'Hermite polynomial states' in [9]. The construction of [9] hinges on recognizing a recursion as the defining recursion for Hermite polynomials; these polynomials appear naturally in solving equation (20).

Starting from these, we can proceed to arbitrary k by using the powerful observation [6] that the direct product of two intelligent states is also intelligent. Since the tensor products

$$\tfrac{1}{4} \otimes \tfrac{1}{4} = \tfrac{1}{2} \oplus \tfrac{3}{2} \oplus \tfrac{5}{2} \oplus \cdots \qquad \tfrac{1}{4} \otimes \tfrac{3}{4} = 1 \oplus 2 \oplus 3 \oplus \cdots \tag{7}$$

decompose into an infinite sum, intelligent states in any representation contained in the decomposition of $\frac{1}{4} \otimes \frac{1}{4}$ or $\frac{1}{4} \otimes \frac{3}{4}$ can be constructed by projection from the tensor product of two $k = \frac{1}{4}$ solutions, or of one $k = \frac{1}{4}$ and one $k = \frac{3}{4}$ solution.

Intelligent states in this paper emphasize the role of $SU(1,1)$ transformations on carefully constructed finite linear combinations of $su(1,1)$ basis states. All moments of the $su(1,1)$ generators can be obtained analytically as finite sums. Further properties of the $su(1,1)$ intelligent states can be obtained using the $SU(1,1)$ d-function found in [12]. Examples of the final forms of some intelligent states are given in tables 1, 2 and B1. They can be contrasted with the recent construction of [13], where a recursion relation for expansion coefficients is identified with those of the Pollaczek polynomials.

Table 1. The first few normalized intelligent states of $k = \frac{1}{4}$ and $k = \frac{3}{4}$, expressed as combination of harmonic oscillator states. Here, $\alpha > 1$.

n	k	$\|\psi_n(\tau)\rangle_a$	$\|\psi_n(\tau)\rangle$
0	$\frac{1}{4}$	$e^{-i\tau\hat{K}_y}\|0\rangle_a$	$e^{-i\tau\hat{K}_y}\left\|\frac{1}{4},\frac{1}{4}\right\rangle$
1	$\frac{3}{4}$	$e^{-i\tau\hat{K}_y}\|1\rangle_a$	$e^{-i\tau\hat{K}_y}\left\|\frac{3}{4},\frac{3}{4}\right\rangle$
2	$\frac{1}{4}$	$e^{-i\tau\hat{K}_y}\frac{\left(-\sqrt{2}\cosh(\tau)\|0\rangle_a+2\sinh(\tau)\|2\rangle_a\right)}{\sqrt{3\cosh(2\tau)-1}}$	$e^{-i\tau\hat{K}_y}\frac{\left(-\sqrt{2}\cosh(\tau)\left\|\frac{1}{4},\frac{1}{4}\right\rangle+2\sinh(\tau)\left\|\frac{1}{4},\frac{5}{4}\right\rangle\right)}{\sqrt{3\cosh(2\tau)-1}}$
3	$\frac{3}{4}$	$e^{-i\tau\hat{K}_y}\frac{\left(-\sqrt{6}\cosh(\tau)\|1\rangle_a+2\sinh(\tau)\|3\rangle_a\right)}{\sqrt{5\cosh(2\tau)+1}}$	$e^{-i\tau\hat{K}_y}\frac{\left(-\sqrt{6}\cosh(\tau)\left\|\frac{3}{4},\frac{3}{4}\right\rangle+2\sinh(\tau)\left\|\frac{3}{4},\frac{7}{4}\right\rangle\right)}{\sqrt{5\cosh(2\tau)+1}}$

Table 2. The first few normalized intelligent states for $k = \frac{1}{2}, \frac{3}{2}$ and $\frac{5}{2}$. Here, $\alpha > 1$. The states $|m,n\rangle_{c,d}$ are those of equation (17).

(n_a, n_b)	k	$e^{-i\tau\hat{K}_y}\left(\sum_r g_r^k\|k,k+r\rangle\right)$	$e^{-i\tau\hat{K}_y}\left(\sum_{m,n} g_{m,n}^k\|m,n\rangle_{c,d}\right)$
(0, 0)	$\frac{1}{2}$	$e^{-i\tau\hat{K}_y}\left\|\frac{1}{2},\frac{1}{2}\right\rangle$	$e^{-i\tau\hat{K}_y}\|0,0\rangle_{c,d}$
(2, 0)	$\frac{3}{2}$	$e^{-i\tau\hat{K}_y}\left\|\frac{3}{2},\frac{3}{2}\right\rangle$	$e^{-i\tau\hat{K}_y}\|2,0\rangle_{c,d}$
	$\frac{1}{2}$	$e^{-i\tau\hat{K}_y}\frac{\left(\cosh\tau\left\|\frac{1}{2},\frac{1}{2}\right\rangle-\sinh\tau\left\|\frac{1}{2},\frac{3}{2}\right\rangle\right)}{\sqrt{\cosh 2\tau}}$	$e^{-i\tau\hat{K}_y}\frac{(\cosh\tau\|0,0\rangle_{c,d}-\sinh\tau\|1,1\rangle_{c,d})}{\sqrt{\cosh 2\tau}}$
(4, 0)	$\frac{5}{2}$	$e^{-i\tau\hat{K}_y}\left\|\frac{5}{2},\frac{5}{2}\right\rangle$	$e^{-i\tau\hat{K}_y}\|4,0\rangle_{c,d}$
	$\frac{3}{2}$	$e^{-i\tau\hat{K}_y}\frac{\left(\sqrt{3}\cosh\tau\left\|\frac{3}{2},\frac{3}{2}\right\rangle-\sinh\tau\left\|\frac{3}{2},\frac{5}{2}\right\rangle\right)}{\sqrt{1+2\cosh 2\tau}}$	$e^{-i\tau\hat{K}_y}\frac{(\sqrt{3}\cosh\tau\|2,0\rangle_{c,d}-\sinh\tau\|3,1\rangle_{c,d})}{\sqrt{1+2\cosh 2\tau}}$
	$\frac{1}{2}$	$\frac{e^{-i\tau\hat{K}_y}}{\sqrt{1+3\cosh 4\tau}}\left(2\cosh^2\tau\left\|\frac{1}{2},\frac{1}{2}\right\rangle -2\sinh 2\tau\left\|\frac{1}{2},\frac{3}{2}\right\rangle+2\sinh^2\tau\left\|\frac{1}{2},\frac{5}{2}\right\rangle\right)$	$\frac{e^{-i\tau\hat{K}_y}}{\sqrt{1+3\cosh 4\tau}}(2\cosh^2\tau\|0,0\rangle_{c,d} -2\sinh 2\tau\|1,1\rangle_{c,d}+2\sinh^2\tau\|2,2\rangle_{c,d})$

Because of equation (4), intelligent states have been investigated as input states to reduce the quantum noise in interferometers, with application to highly phase-sensitive measurements [14]. In this paper we discuss the $su(1,1)$ squeezing properties of some of our states. Our calculations show that, under reasonable definitions of squeezing, squeezing of $su(1,1)$ observables can occur even in 'single-particle' $su(1,1)$ states.

2. Review of the $su(1,1)$ algebra and positive discrete series representations

The major features of positive discrete series representations of $su(1,1)$ are excellently described in [4, 15]. They are briefly reviewed here to establish the notation used throughout the paper.

2.1. The generators and their action

The abstract $su(1,1)$ commutation relations are given in equation (3). We denote by $|k, k+r\rangle$ an eigenstate of $\hat{K}_z$:

$$\hat{K}_z|k, k+r\rangle = (k+r)|k, k+r\rangle. \tag{8}$$

The meaning of the labels k and r will be specified shortly. Introducing the usual raising and lowering operators $\hat{K}_\pm = \hat{K}_x \pm \mathrm{i}\hat{K}_y$ leads to

$$[\hat{K}_z, \hat{K}_+] = \hat{K}_+, \qquad [\hat{K}_z, \hat{K}_-] = \hat{K}_-, \qquad [\hat{K}_+, \hat{K}_-] = -2\hat{K}_z. \tag{9}$$

For positive discrete series representations, the values of k are restricted to $\frac{1}{2}, 1, \frac{3}{2}, \ldots$. In addition, there are two limits of discrete series representation labeled by $k = \frac{1}{4}$ or $k = \frac{3}{4}$. The label $r \in \mathbb{Z}^*$, and the state of equation (8) with $r = 0$ is such that

$$\hat{K}_-|k, k\rangle = 0. \tag{10}$$

Acting now on $|k, k\rangle$ with $(\hat{K}_+)^r$ produces a sequence of states proportional to

$$(\hat{K}_+)^r|k, k\rangle \sim |k, k + r\rangle. \tag{11}$$

Standard manipulations lead to

$$\begin{aligned} \hat{K}_+|k, k + r\rangle &= \sqrt{(2k + r)(r + 1)} \quad |k, k + r + 1\rangle, \\ \hat{K}_-|k, k + r\rangle &= \sqrt{r(2k + r - 1)} \quad |k, k + r - 1\rangle \end{aligned} \tag{12}$$

as a consequence of requiring Hermiticity. There is no upper bound on r. The states $\{|k, k + r\rangle, r = 0, 1, \ldots\}$ form a basis for the positive discrete series representation k.

2.2. The degenerate case

For the case of the realization of equation (1), the operators $\hat{K}_\pm$ specialize to

$$\hat{K}_+ = \tfrac{1}{2}\hat{a}^\dagger\hat{a}^\dagger, \qquad \hat{K}_- = \tfrac{1}{2}\hat{a}\hat{a}. \tag{13}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ act in the familiar way on harmonic oscillator kets, two of which are killed by $\hat{K}_-$: $|0\rangle_a$ or $|1\rangle_a$. Since

$$\hat{K}_z|0\rangle_a = \tfrac{1}{4}|0\rangle_a, \qquad \hat{K}_z|1\rangle_a = \tfrac{3}{4}|1\rangle_a \tag{14}$$

and

$$(\hat{K}_+)^p|0\rangle_a \sim |2p\rangle_a, \qquad (\hat{K}_+)^p|1\rangle_a \sim |2p + 1\rangle_a, \tag{15}$$

we see that $\{|0\rangle_a, |2\rangle_a, |4\rangle_a, \ldots\}$ and $\{|1\rangle_a, |3\rangle_a, |5\rangle_a, \ldots\}$ span the $k = \frac{1}{4}$ and $k = \frac{3}{4}$ representations, respectively. (The subscript a has been added to explicitly indicate that these are boson kets.) These series correspond to even- and odd-parity harmonic oscillator states, respectively.

The harmonic oscillator kets can be identified with the $|k, k + p\rangle$ states as follows:

$$|2p\rangle_a \leftrightarrow \left|\tfrac{1}{4}, \tfrac{1}{4} + p\right\rangle, \qquad |2p + 1\rangle_a \leftrightarrow \left|\tfrac{3}{4}, \tfrac{3}{4} + p\right\rangle. \tag{16}$$

One rapidly verifies that the usual action of the harmonic oscillator raising and lowering operators is (of course) compatible with equation (12).

2.3. The non-degenerate case

The realization of equation (2) acts naturally on states of a two-dimensional harmonic oscillator. Note that, if

$$|m, n\rangle_{c,d} \equiv \frac{(\hat{c}^\dagger)^m(\hat{d}^\dagger)^n|0, 0\rangle_{c,d}}{\sqrt{m!n!}}, \tag{17}$$

then the states $\{x|\mu, 0\rangle_{c,d} + y|0, \mu\rangle_{c,d}, \mu = 0, 1, \ldots\}$, with $x, y \in \mathbb{C}$, are all killed by $\hat{K}_- = \hat{a}\hat{b}$ and thus lowest weight states for the irreps labeled by $k = \frac{1}{2}(\mu + 1)$. In particular, the two-boson vacuum $|0, 0\rangle_{c,d}$ is the lowest weight state for the irrep $k = \frac{1}{2}$. Since $\hat{K}_+ = \hat{c}^\dagger\hat{d}^\dagger$, we see that linear combinations of the form $x|n + \mu, n\rangle_{c,d} + y|n, n + \mu\rangle_{c,d}$ belong to the irrep $k = \frac{1}{2}(\mu + 1)$.

3. Intelligent states for the $k = 1/4$ and $k = 3/4$ series

We now solve equation (5). To this end write equation (5) as

$$\tfrac{1}{4}(1-\alpha)\hat{a}^\dagger\hat{a}^\dagger|\psi(\alpha)\rangle_a + \tfrac{1}{4}(1+\alpha)\hat{a}\hat{a}|\psi(\alpha)\rangle_a = \lambda|\psi(\alpha)\rangle_a. \tag{18}$$

Solutions to equation (18) were found in [9] using recursion relations closely related to Hermite polynomials.

3.1. The associated Schrödinger problem

We next introduce a *dummy* variable ξ and the identifications

$$\begin{aligned} \hat{a}^\dagger &\leftrightarrow \xi, & \hat{a} &\leftrightarrow \frac{\mathrm{d}}{\mathrm{d}\xi}, \\ \hat{K}_+ = \frac{1}{2}\hat{a}^\dagger\hat{a}^\dagger &\leftrightarrow \frac{1}{2}\xi^2, & \hat{K}_- = \frac{1}{2}\hat{a}\hat{a} &\leftrightarrow \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}, \\ |n\rangle_a &\leftrightarrow \frac{\xi^n}{\sqrt{n!}}, & {}_a\langle n| &\leftrightarrow \frac{1}{\sqrt{n!}}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}. \end{aligned} \tag{19}$$

The commutation relations are preserved under the identifications of equation (19), which transforms equation (5) into the differential equation

$$\frac{1}{4}(1+\alpha)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\psi(\xi) + \frac{1}{4}(1-\alpha)\xi^2\psi(\xi) = \lambda\psi(\xi). \tag{20}$$

This differential equation is formally similar to the Schrödinger equation for a harmonic oscillator in dummy space. The formal solutions can be expressed as

$$\psi_n(\xi) \sim \mathrm{e}^{-\varepsilon\xi^2/2}H_n(\sqrt{\varepsilon}\xi), \tag{21}$$

where H_n is the nth Hermite polynomial and

$$\varepsilon = \sqrt{\frac{\alpha-1}{\alpha+1}}. \tag{22}$$

We have made here a convenient choice of phase for ε.

It is important to realize that, in view of equation (19), $\psi_n(\xi)$ is really an operator acting in the space of polynomials in ξ, so the boundary conditions applicable to the usual harmonic oscillator problem must be correctly reinterpreted.

Convergence is imposed from

$$\mathrm{e}^{-\varepsilon\xi^2/2} \sim \mathrm{e}^{-\varepsilon\hat{K}_+} \tag{23}$$

through the requirement $|\varepsilon| < 1$; this implies $\alpha > 0$. In addition, we restrict the functions H_n to Hermite polynomials so that $H_n(\sqrt{\varepsilon}\xi)$ is a polynomial in integer powers of $\hat{a}^\dagger$. The formal solution

$$\psi_n(\xi) \sim \mathrm{e}^{-\varepsilon\xi^2/2}H_n(\sqrt{\varepsilon}\xi) \mapsto \mathrm{e}^{-\varepsilon\hat{a}^\dagger\hat{a}^\dagger/2}H_n(\sqrt{\varepsilon}\hat{a}^\dagger)|0\rangle_a \tag{24}$$

is thus a linear (but infinite) combination of the usual harmonic oscillator kets.

Although the phase of ε is immaterial, equation (22) divides the solutions in two regimes: $0 < \alpha < 1$, for which ε is purely imaginary, and $\alpha > 1$ for which ε is real.

We will assume henceforth that $\alpha > 1$: the case $0 \leqslant \alpha < 1$ is not essentially different from the latter case and so is sketched in appendix B.

3.2. Solutions for $\alpha > 1$

It is convenient to change the variable α to τ using $\alpha = \cosh(\tau)$, which in turn implies

$$\varepsilon = \tanh(\tau/2), \qquad \omega = \pm\tfrac{1}{2}\sinh\tau. \tag{25}$$

To choose the sign of ω, we note that the coherent states

$$\mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}|0\rangle_a, \qquad \mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}|1\rangle_a, \tag{26}$$

are both intelligent with eigenvalues $-\frac{1}{4}\sinh\tau$ and $-\frac{3}{4}\sinh\tau$, indicating that

$$\omega = -\tfrac{1}{2}\sinh\tau \quad\Rightarrow\quad \lambda_n = -\left(n+\tfrac{1}{2}\right)\omega = -\tfrac{1}{2}\left(n+\tfrac{1}{2}\right)\sinh\tau. \tag{27}$$

The form of equation (26), and the infinite sum explicit in equation (21), both suggest that we rewrite

$$\mathrm{e}^{-\varepsilon\hat{K}_+} = \mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}\,\mathrm{e}^{-\varepsilon\hat{K}_-}\,\mathrm{e}^{-\beta\hat{K}_z}, \tag{28}$$

where $\beta = -2\ln(\cosh(\tau/2))$. With this, one can write the intelligent state in dummy space as

$$\mathrm{e}^{-\varepsilon\xi^2/2}H_n(\sqrt{\varepsilon}\xi) = \mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}h_n(\xi;\tau) \tag{29}$$

where we have introduced the formal polynomial (of finite degree)

$$h_n(\xi;\tau) = \mathrm{e}^{-\varepsilon\hat{K}_-}\,\mathrm{e}^{-\beta\hat{K}_0}H_n(\sqrt{\varepsilon}\xi) = \mathrm{e}^{-\frac{\varepsilon}{2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}}H_n(\sqrt{\varepsilon}\cosh(\tau/2)\xi). \tag{30}$$

In terms of kets, we have

$$\mathrm{e}^{-\varepsilon\xi^2/2}H_n(\sqrt{\varepsilon}\xi) \mapsto |\psi_n(\tau)\rangle_a = \mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}|h_n(\tau)\rangle_a. \tag{31}$$

In most cases equation (31) is enough to compute to expectation values and other moments of any $su(1,1)$ operator.

With round kets denoting unnormalized states, we write

$$|h_n(\tau)\rangle_a = \mathcal{N}_n(\tau)|h_n(\tau))_a \tag{32}$$

where $\mathcal{N}_n(\tau)$ is a normalization factor. We show in appendix A that

$$|h_{2m}(\tau))_a \equiv \sum_{q=0}^{m}\frac{c_{2q}^{2m}(\tau)}{\sqrt{(2q)!}}|2q\rangle_a, \tag{33}$$

with

$$c_{2q}^{2m}(\tau) = \frac{(-1)^{m-q}}{(m-q)!}(2\sinh(\tau))^q(\cosh(\tau))^{m-q}, \tag{34}$$

for the even case. For the odd case, the result can be simplified to an expression similar to the even case:

$$|h_{2m+1}(\tau))_a \equiv \sum_{q=0}^{m}\frac{c_{2q+1}^{2m+1}(\tau)}{\sqrt{(2q+1)!}}|2q+1\rangle_a, \tag{35}$$

with

$$c_{2q+1}^{2m+1} = \frac{(-1)^{m-q}}{(m-q)!}(2\sinh(\tau))^q(\cosh(\tau))^{m-q}. \tag{36}$$

The first few solutions are presented explicitly in table 1.

4. The solution for arbitrary k and $\alpha > 1$

We now come to the main result of this paper: the construction of intelligent states for arbitrary k. We will use the powerful result of [6], which shows that, if $|\psi_{n_a}(\tau))_a$ and $|\psi_{m_b}(\tau))_b$ are intelligent, so is the product

$$|\psi_{n_a}(\tau); \psi_{m_b}(\tau))_{a,b} \equiv |\psi_{n_a}(\tau))_a|\psi_{m_b}(\tau))_b. \tag{37}$$

Formally, we work with two pairs of bosons: $\{\hat{a}^\dagger, a\}$ and $\{\hat{b}^\dagger, b\}$, defining the generators of the $su(1,1)$ algebra as

$$\begin{aligned} \hat{K}_+ &= \tfrac{1}{4}(\hat{a}^\dagger\hat{a}^\dagger + \hat{b}^\dagger\hat{b}^\dagger), \qquad \hat{K}_- = \tfrac{1}{4}(\hat{a}\hat{a} + \hat{b}\hat{b}), \\ \hat{K}_z &= \tfrac{1}{2}(\hat{a}^\dagger a + \hat{b}^\dagger b + 1). \end{aligned} \tag{38}$$

The realization of equation (2) is obtained from equation (38) by the unitary transformation

$$\hat{a}^\dagger \to \frac{1}{\sqrt{2}}(\hat{c}^\dagger + \hat{d}^\dagger), \qquad \hat{b}^\dagger \to \frac{\mathrm{i}}{\sqrt{2}}(\hat{c}^\dagger - \hat{d}^\dagger). \tag{39}$$

Suppose for simplicity we take $|\psi_{n_a}(\tau))_a$ and $|\psi_{m_b}(\tau))_b$ to be in the $k = \frac{1}{4}$ irrep. With $n_a = 2p$ and $m_b = 2q$ we can write

$$|\psi_{2p}(\tau); \psi_{2q}(\tau))_{a,b} = \mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}[|h_{2p}(\tau))_a|h_{2q}(\tau))_b]. \tag{40}$$

The product state $|h_{2p}(\tau))_a|h_{2q}(\tau))_b$ will generally belong to more than one $su(1,1)$ irrep since the tensor product of equation (7) is reducible. To construct an intelligent state in the $su(1,1)$ irrep k with k in equation (7), we project into the correct subspace:

$$\left|\psi^k_{2p,2q}(\tau)\right)_{a,b} = \mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}\sum_r |k, k+r\rangle\kappa_r^{k,2p,2q}(\tau). \tag{41}$$

The coefficient

$$\kappa_r^{k,2p,2q}(\tau) \equiv \langle k, k+r|h_{2p}(\tau); h_{2q}(\tau))_{a,b} \tag{42}$$

can be evaluated from the explicit expressions of $|h_{2p}(\tau))_a$ and $|h_{2q}(\tau))_b$:

$$\kappa_r^{k,2p,2q}(\tau) = \sum_{m,n} c^{2p}_{2m}(\tau)c^{2q}_{2n}(\tau)C^{k,k+r}_{\frac{1}{4},\frac{1}{4}+m;\frac{1}{4},\frac{1}{4}+n}, \tag{43}$$

with $c^{2p}_{2m}(\tau), c^{2q}_{2n}(\tau)$ given by equation (34) and $C^{k,k+r}_{\frac{1}{4},\frac{1}{4}+m;\frac{1}{4},\frac{1}{4}+n}$ an $su(1,1)$ Clebsch–Gordan coefficient [16].

A similar analysis for the coupling of a state from the $k = 3/4$ irrep with a state from the $k = 1/4$ irrep produces

$$\kappa_r^{k,2p+1,2q}(\tau) = \sum_{m,n} c^{2p+1}_{2m+1}(\tau)c^{2q}_{2n}(\tau)C^{k,k+r}_{\frac{3}{4},\frac{3}{4}+m;\frac{1}{4},\frac{1}{4}+n}. \tag{44}$$

In practice, we have found it more convenient to work with the recursion relation

$$\kappa_{r+1}^{k,n_a,n_b}(\tau) = \frac{(k-r) - \frac{1}{2}(n_a + n_b + 1)}{\sqrt{(2k+r)(r+1)}}(\tanh\tau)\kappa_r^{k,n_a,n_b}(\tau), \tag{45}$$

which is obtained by a procedure given in appendix C. The major advantage of equation (45) is the bypass of $SU(1,1)$ Clebsch–Gordan technology. (The corresponding recursion for $0 \leqslant \alpha \leqslant 1$ is given in equation (C.9).)

Equation (45) makes it clear that, for a given $k \neq \frac{1}{4}$ or $\frac{3}{4}$, two intelligent states $\left|\psi^k_{n_a,n_b}(\tau)\right)$ and $\left|\psi^k_{n'_a,n'_b}(\tau)\right)$ such that $N = n_a + n_b = n'_a + n'_b$ will be (almost always) identical (up to an overall phase). Beautiful exceptions occur when $n_a = n_b$: in this case, the product state $\left|\psi_{n_a}(\tau)\right\rangle\left|\psi_{n_b}(\tau)\right\rangle$ is symmetrical under permutation of n_a and n_b. However, the irreps $\frac{3}{2}+t$, $t=0,1,2,\ldots$ in equation (7) are antisymmetric under permutation of n_a and n_b, so the calculation of $\kappa_r^{k,n_a,n_b}(\tau)$ for those values of k and $n_a = n_b$ using equation (43) yields 0 for every coefficient.

For labeling purposes, we henceforth set without loss of generality $n_b = 0$. For a given k, the recursion of equation (45) stops at $r_{\max}$ given by

$$r_{\max} = \tfrac{1}{2}(n_a+1) - k. \tag{46}$$

Because $\mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}$ is a unitary transformation, the norm of $|\psi^k_{n_a,0}(\tau))$ can be recovered from the (finite) sum of each $|\kappa_r^{k,n_a,0}(\tau)|^2$ to complete the calculation of the normalized state.

5. Application: squeezing of su(1,1) observables

The realization of $su(1,1)$ generators in terms of boson operators is the starting point for many applications in quantum optics: the creation and destruction operators correspond to creation and destruction of electromagnetic field quanta.

In [2] and [10], $su(1,1)$ squeezing was defined by the condition

$$(\Delta K_y)^2 \leqslant \tfrac{1}{2}|\langle\hat{K}_z\rangle|, \qquad \text{or} \qquad (\Delta K_x)^2 \leqslant \tfrac{1}{2}|\langle\hat{K}_z\rangle|. \tag{47}$$

This definition has also been used in [13].

However, as pointed out for spin squeezing by Kitawaga and Ueda in [17] and emphasized in [18], squeezing should be defined in a covariant manner so as not to depend on a specific choice of one amongst a family of $SU(1,1)$-translated bases.

In particular, Kitawaga and Ueda have provided for spin squeezing [17] a geometrical construction that relies first on finding a vector in the direction of the average angular momentum of a state. One then defines 'rotated' observables orthogonal to this direction vector. Squeezing occurs when the standard deviation of one of these rotated observables beats a reference standard.

The generalization to $SU(1,1)$ is simple. Suppose for simplicity that $\alpha > 1$. Given a state $|\psi\rangle$, we first construct a vector in the direction of the 'average $\vec{K}$':

$$\vec{n}_{\langle K\rangle} = \vec{n}_{K_{z'}} = (\langle K_x\rangle, \langle K_y\rangle, \langle K_z\rangle) = (\langle K_x\rangle, 0, \langle K_z\rangle). \tag{48}$$

Next we construct the orthogonal (in the $su(1,1)$ sense) vector

$$\vec{n}_{K_{x'}} = (\langle K_z\rangle, 0, \langle K_x\rangle). \tag{49}$$

Finally, we introduce the translated observables

$$\hat{K}_{z'} = \mathrm{e}^{-\mathrm{i}\beta\hat{K}_y}\hat{K}_z\,\mathrm{e}^{\mathrm{i}\beta\hat{K}_y} = \langle K_x\rangle\hat{K}_x + \langle K_z\rangle\hat{K}_z, \tag{50}$$

$$\hat{K}_{x'} = \mathrm{e}^{-\mathrm{i}\beta\hat{K}_y}\hat{K}_x\,\mathrm{e}^{\mathrm{i}\beta\hat{K}_y} = \langle K_z\rangle\hat{K}_x + \langle K_x\rangle\hat{K}_z. \tag{51}$$

Thus we see that the angle β is determined from

$$\langle K_x\rangle = \sinh\beta, \qquad \langle K_z\rangle = \cosh\beta. \tag{52}$$

With this we can generalize equation (47) to

$$(\Delta K_{y'})^2 \leqslant \tfrac{1}{2}|\langle\hat{K}_{z'}\rangle|, \qquad \text{or} \qquad (\Delta K_{x'})^2 \leqslant \tfrac{1}{2}|\langle\hat{K}_{z'}\rangle|. \tag{53}$$

This is not the only plausible definition; Luis and Korolkova propose in [18] to use the coherent state as a threshold to define squeezing. If this latter definition is adopted, the right-hand side of the inequalities becomes the constant $\frac{1}{2}k$.

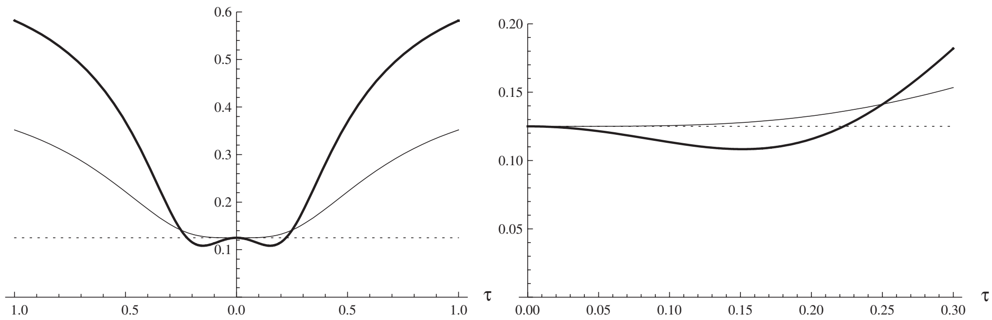

Figure 1. Thick line: the function $(\Delta K_{x'})^2$ as a function of τ for the single-particle state $|\psi_2(\tau)\rangle$. Dashed line: $\langle K_{z'}\rangle = 1/8$ obtained from a coherent state. Thin line: $\langle K_{z'}\rangle$ calculated using $|\psi_2(\tau)\rangle$.

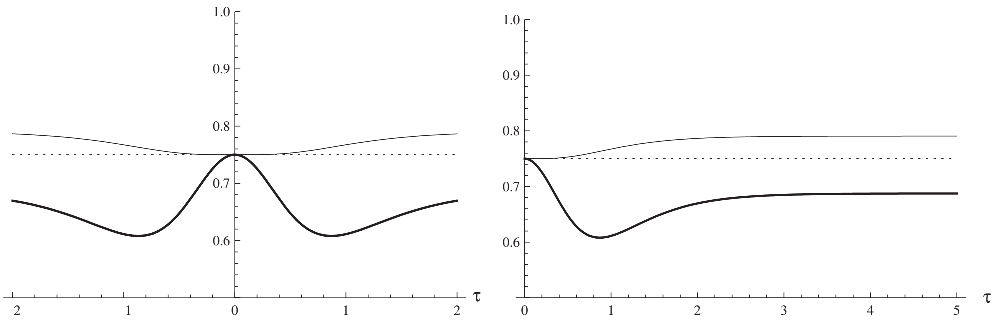

Figure 2. Thick line: the function $(\Delta K_{x'})^2$ as a function of τ for the state of equation (57). Dashed line: $\langle K_{z'}\rangle = 3/4$ obtained from a coherent state. Thin line: $\langle K_{z'}\rangle$ obtained from state (57).

As an illustration, consider squeezing in $\hat{K}_{x'}$ for the state

$$|\psi_2(\tau)\rangle = \mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}\,\frac{(-\sqrt{2}\cosh(\tau)|0\rangle + 2\sinh(\tau)|2\rangle)}{\sqrt{3\cosh(\tau)-1}}. \tag{54}$$

This state belongs to the $k = \frac{1}{4}$ series and is thus a 'single-particle' $su(1,1)$ state. Using this, we find

$$\langle K_x\rangle = -\frac{5}{4}\sinh(\tau), \qquad \langle K_y\rangle = 0, \qquad \langle K_z\rangle = \frac{1}{4}\cosh(\tau)\left(5 + \frac{8}{1-3\cosh(2\tau)}\right), \tag{55}$$

from which we extract the angle

$$\beta = -\mathrm{arctanh}\left(\frac{5(3\cosh(2\tau)-1)\tanh(\tau)}{15\cosh(2\tau)-13}\right). \tag{56}$$

The results for $(\Delta K_{x'})^2$ and $\frac{1}{2}\langle\hat{K}_{z'}\rangle$ as a function of τ are presented in figure 1. If one follows the suggestion of [18], the state-independent threshold $\frac{1}{2}k = \frac{1}{8}$ indicated by a dashed line should be considered the reference. The 'sombrero-hat' behavior of $(\Delta K_{x'})^2$ as a function of τ indicates squeezing for small enough $|\tau|$, i.e. whenever the sombrero-hat curve is below the dashed line or the thin black line (depending on the choice of criterion). The left graph gives a general view of the trend for $(\Delta K_{x'})^2$; the right graph focuses on the region of positive τ

where there is squeezing. Squeezing of $su(1,1)$ observables can occur even in 'single-particle' $su(1,1)$ states.

As a second example, we plot in figure 2 $(\Delta K_{y'})^2$ and $\frac{1}{2}\langle \hat{K}_{z'}\rangle$ as a function of τ for the state

$$\left|\psi^{3/2}_{(4,0)}(\tau)\right\rangle = \mathrm{e}^{-\mathrm{i}\tau\hat{K}_x}\frac{\left(\sqrt{3}\cosh(\tau)\left|\frac{3}{2},\frac{3}{2}\right\rangle - \mathrm{i}\sinh(\tau)\left|\frac{3}{2},\frac{5}{2}\right\rangle\right)}{\sqrt{1+2\cosh(2\tau)}}. \tag{57}$$

This state always exhibit squeezing.

6. Conclusion

In this paper we have shown how any $su(1,1)$ intelligent state belonging to an irrep of the positive discrete series can be constructed. The 'one-particle' intelligent states of the $\frac{1}{4}$ and $\frac{3}{4}$ series are connected to harmonic oscillator states in an abstract dummy space. Closed-form expressions for such states in terms of harmonic oscillator kets are provided in equations (33) and (34) for even states, and in equations (35) and (36) for odd states. The expansion coefficients are closely related to Hermite polynomials.

For higher representations, one can obtain the expansion coefficients either by using equations (43) or (44) and $su(1,1)$ Clebsch–Gordan technology, or more simply from the recursion relation of equation (45).

All our intelligent states are of the generic forms

$$|\psi(\tau)\rangle = \mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}|\phi(\tau)\rangle \qquad \text{or} \qquad |\psi(\tau)\rangle = \mathrm{e}^{-\mathrm{i}\tau\hat{K}_x}|\phi(\tau)\rangle \tag{58}$$

where $|\phi(\tau)\rangle$ is a finite linear combination of $su(1,1)$ states. This allows for efficient calculation of all moments of the generators $\hat{K}_x$, $\hat{K}_y$ or $\hat{K}_z$ as finite sums of explicit expressions.

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Appendix A. Closed-form expressions for $h_n(\xi;\tau)$ with $\alpha > 1$

In this section we analyze the formal polynomial

$$\langle\xi|h_n(\tau)\rangle \equiv h_n(\xi;\tau) \equiv \mathrm{e}^{-\frac{\varepsilon}{2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}} H_n(\sqrt{\varepsilon}\cosh(\tau/2)\xi) \tag{A.1}$$

of equation (30) for the case where $\alpha > 1$.

Consider first the case where $n = 2m$. Then

$$\mathrm{e}^{-\frac{\varepsilon}{2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}} H_{2m}(\sqrt{\varepsilon}\cosh(\tau/2)\xi) = \sum_{p=0}^{m}\frac{(-1)^p}{p!}\left(\frac{\varepsilon}{2}\right)^p\frac{\mathrm{d}^{2p}}{\mathrm{d}\xi^{2p}}H_{2m}(\sqrt{\varepsilon}\cosh(\tau/2)\xi). \tag{A.2}$$

Under derivation [19]

$$\frac{\mathrm{d}^{2p}}{\mathrm{d}\xi^{2p}}H_{2m}(\sqrt{\varepsilon}\cosh(\tau/2)\xi) = 2^{2p}(\varepsilon\cosh^2(\tau/2))^p\frac{(2m)!}{(2m-2p)!}H_{2m-2p}(\sqrt{\varepsilon}\cosh(\tau/2)\xi), \tag{A.3}$$

so

$$h_{2m}(\xi;\tau) = \sum_{p=0}^{m}\frac{1}{p!}(-1)^p(2\sinh^2(\tau/2))^p\frac{(2m)!}{(2m-2p)!}H_{2m-2p}(\sqrt{\varepsilon}\cosh(\tau/2)\xi). \tag{A.4}$$

Let us express this as the series

$$h_{2m}(\xi;\tau) = \sum_{q=0}^{m} \frac{c_{2q}^{2m}(\tau)}{(2q)!}\xi^{2q} = c_0^{2p}(\tau) + \frac{c_2^{2p}(\tau)}{2!}\xi^2 + \frac{c_4^{2p}(\tau)}{4!}\xi^4 + \cdots. \tag{A.5}$$

In particular,

$$c_0^{2m}(\tau) = \sum_{p=0}^{m} \frac{1}{p!}(-1)^p(2\sinh^2(\tau/2))^p \frac{(2m)!}{(2m-2p)!} H_{2m-2p}(0)$$
$$= (-1)^m \frac{(2m)!}{m!}(\cosh(\tau))^m.$$

In a similar manner one obtains the general expression

$$c_{2q}^{2m}(\tau) = (-1)^{m-q}(2\sinh(\tau))^q \frac{(2m)!}{(m-q)!}(\cosh(\tau))^{m-q}. \tag{A.6}$$

The factor $(2m)!$ will disappear upon normalization of the state, resulting in the expression of equation (34).

For $n = 2m+1$, $H_{2m+1}(\sqrt{\varepsilon}\xi)$ is a polynomial containing only odd powers of the dummy ξ, which is mapped to a polynomial of degree m in $\hat{K}_+$ acting on $|1\rangle$. Following the general method of the even case, suppose $n = 2m+1$. Then

$$h_{2m+1}(\xi;\tau) = \mathrm{e}^{-\frac{\varepsilon}{2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}} H_{2m+1}(\sqrt{\varepsilon}\cosh(\tau/2)\xi)$$
$$= \sum_{p=0}^{m} \frac{1}{p!}(-1)^p(2\sinh^2(\tau/2))^p \frac{(2m+1)!}{(2m-2p+1)!} H_{2m-2p+1}(\sqrt{\varepsilon}\cosh(\tau/2)\xi). \tag{A.7}$$

Again writing this as a series, we find, for instance,

$$c_1^{2m+1} = \sum_{p=0}^{m} \frac{1}{p!}(-1)^p(2\sinh^2(\tau/2))^p \frac{(2m+1)!}{(2m-2p+1)!}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}H_{2m-2p+1}(\sqrt{\varepsilon}\cosh(\tau/2)\xi)\right)_{\xi=0}, \tag{A.8}$$

$$= \sum_{p=0}^{m} \frac{1}{p!}(-1)^p(2\sinh^2(\tau/2))^p \frac{(2m+1)!}{(2m-2p+1)!} 2(\sqrt{\varepsilon}\cosh(\tau/2)) \frac{(2m-2p+1)!}{(2m-2p)!} H_{2m-2p}(0), \tag{A.9}$$

and more generally

$$c_{2q+1}^{2m+1} = (-1)^{m-q}2^{q+1}\sqrt{\varepsilon}\cosh(\tau/2)(\sinh(\tau))^q \frac{(2m+1)!}{(m-q)!}(\cosh(\tau))^{m-q}. \tag{A.10}$$

Once we identify terms independent of q that will disappear upon normalization we obtain equation (36).

Appendix B. The case $0 \leqslant \alpha < 1$

In this appendix we give the major results for the case where $0 \leqslant \alpha \leqslant 1$. The Perelomov $SU(1,1)$ coherent state

$$\mathrm{e}^{-\mathrm{i}\tau\hat{K}_x}|k,k\rangle \tag{B.1}$$

Table B1. The first few normalized intelligent states of $k = \frac{1}{4}$ and $k = \frac{3}{4}$, expressed as a combination of harmonic oscillator states. Here, $0 < \alpha < 1$.

n	k	$\lvert\psi_n(\tau)\rangle_a$	$\lvert\psi_n(\tau)\rangle$
0	$\frac{1}{4}$	$e^{-i\tau\hat{K}_x}\lvert 0\rangle_a$	$e^{-i\tau\hat{K}_x}\left\lvert\frac{1}{4},\frac{1}{4}\right\rangle$
1	$\frac{3}{4}$	$e^{-i\tau\hat{K}_x}\lvert 1\rangle_a$	$e^{-i\tau\hat{K}_x}\left\lvert\frac{3}{4},\frac{3}{4}\right\rangle$
2	$\frac{1}{4}$	$e^{-i\tau\hat{K}_x}\frac{(-\sqrt{2}\cosh(\tau)\lvert 0\rangle_a+2i\sinh(\tau)\lvert 2\rangle_a)}{\sqrt{3\cosh(2\tau)-1}}$	$e^{-i\tau\hat{K}_x}\frac{\left(-\sqrt{2}\cosh(\tau)\left\lvert\frac{1}{4},\frac{1}{4}\right\rangle+2i\sinh(\tau)\left\lvert\frac{1}{4},\frac{5}{4}\right\rangle\right)}{\sqrt{3\cosh(2\tau)-1}}$
3	$\frac{3}{4}$	$e^{-i\tau\hat{K}_x}\frac{(-\sqrt{6}\cosh(\tau)\lvert 1\rangle_a+2i\sinh(\tau)\lvert 3\rangle_a)}{\sqrt{5\cosh(2\tau)+1}}$	$e^{-i\tau\hat{K}_x}\frac{\left(-\sqrt{6}\cosh(\tau)\left\lvert\frac{3}{4},\frac{3}{4}\right\rangle+2\sinh(\tau)\left\lvert\frac{3}{4},\frac{7}{4}\right\rangle\right)}{\sqrt{5\cosh(2\tau)+1}}$

is intelligent provided $\alpha = 1/\cosh(\tau)$. This does not change equations (20) or (21) except that the auxiliary quantities now become

$$\varepsilon = i\tanh(\tau/2), \qquad \omega = \frac{i}{2}\tanh\tau, \qquad \lambda_n = -\frac{i}{2}\left(n+\frac{1}{2}\right)\tanh\tau. \tag{B.2}$$

Again using disentangling backward,

$$e^{-\varepsilon\hat{K}_+} = e^{-i\tau\hat{K}_x}\, e^{\varepsilon\hat{K}_-}\, e^{-\beta\hat{K}_0}, \qquad \beta = -2\ln(\cosh(\tau/2)), \tag{B.3}$$

so

$$e^{-\varepsilon\xi^2/2}H_n(\sqrt{\varepsilon}\xi) = e^{-i\tau\hat{K}_x}\, e^{\frac{\varepsilon}{2}\frac{d^2}{d\xi^2}}H_n(\sqrt{\varepsilon}\cosh(\tau/2)\xi) \tag{B.4}$$

and we can now analyze

$$\tilde{h}_n(\xi;\tau) = e^{\frac{\varepsilon}{2}\frac{d^2}{d\xi^2}}H_n(\sqrt{\varepsilon}\cosh(\tau/2)\xi). \tag{B.5}$$

For $n = 2m$, we write

$$\tilde{h}_{2m}(\xi;\tau) = \sum_{p=0}^{m}\frac{1}{p!}\left(\frac{\varepsilon}{2}\right)^p\frac{d^{2p}}{d\xi^{2p}}H_{2m}(\sqrt{\varepsilon}\cosh(\tau/2)\xi), \tag{B.6}$$

$$= \sum_{q=0}^{m}\frac{\tilde{c}_{2q}^{2m}}{(2q)!}\xi^{2q}. \tag{B.7}$$

Using the identification of equation (19), this becomes

$$\lvert\tilde{h}_{2m}(\tau)\rangle = \sum_{q=0}^{m}\frac{\tilde{c}_{2q}^{2m}}{(2q)!}\sqrt{(2q)!}\lvert 2q\rangle. \tag{B.8}$$

After manipulations similar to those of the even case, this yields the general expression

$$\tilde{c}_{2q}^{2m} = (-1)^{m-q}(2i\sinh(\tau))^q\frac{(2m)!}{(m-q)!}(\cosh(\tau))^{m-q}. \tag{B.9}$$

Likewise, for $n = 2m+1$,

$$\begin{aligned}\tilde{h}_{2m+1}(\xi;\tau) &= e^{\frac{\varepsilon}{2}\frac{d^2}{d\xi^2}}H_{2m+1}(\sqrt{\varepsilon}\cosh(\tau/2)\xi)\\ &= \sum_{p=0}^{m}\frac{1}{p!}(-1)^p(2\sinh^2(\tau/2))^p\frac{(2m+1)!}{(2m-2p+1)!}H_{2m-2p+1}(\sqrt{\varepsilon}\cosh(\tau/2)\xi).\end{aligned} \tag{B.10}$$

Writing again

$$\tilde{h}_{2m+1}(\xi;\tau) = \sum_{q=0}^{m} \frac{\tilde{c}_{2q+1}^{2m+1}}{(2q+1)!}\xi^{2q+1}, \tag{B.11}$$

$$|\tilde{h}_{2m+1}(\tau)\rangle = \sum_{q=0}^{m} \frac{\tilde{c}_{2q+1}^{2m+1}}{(2q+1)!}\sqrt{(2q+1)!}|2q+1\rangle, \tag{B.12}$$

we find, using once more the properties of the Hermite polynomials under differentiation,

$$\tilde{c}_1^{2m+1} = \sum_{p=0}^{m} \frac{1}{p!}(-1)^p(2\sinh^2(\tau/2))^p \frac{(2m+1)!}{(2m-2p+1)!} \times 2(\sqrt{\varepsilon}\cosh(\tau/2))\frac{(2m-2p+1)!}{(2m-2p)!}H_{2m-2p}(0), \tag{B.13}$$

$$= (-1)^m \frac{(2m+1)!}{m!}(2\sqrt{\varepsilon}\cosh(\tau/2))(\cosh(\tau))^m \tag{B.14}$$

and, quite generally,

$$\tilde{c}_{2q+1}^{2m+1} = (-1)^{m-q}(2\sqrt{\varepsilon}\cosh(\tau/2))(2\mathrm{i}\sinh(\tau))^q\frac{(2m+1)!}{(m-q)!}(\cosh(\tau))^{m-q}. \tag{B.15}$$

Appendix C. Recursion relations for $\langle k, k+r|h_{n_1}(\tau); h_{n_2}(\tau))$

First, assume that $\alpha > 1$. Start from

$$\langle k, k+r|\,\mathrm{e}^{\mathrm{i}\tau\hat{K}_y}(\hat{K}_x - \mathrm{i}\alpha\hat{K}_y)\,\mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}|h_{n_1}(\tau); h_{n_2}(\tau)) \tag{C.1}$$

and recall that $\mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}|h_{n_1}(\tau); h_{n_2}(\tau))$ is just $|\psi_{n_1}(\tau)\rangle|\psi_{n_2}(\tau)\rangle$ so that

$$\begin{aligned}&\langle k, k+r|\,\mathrm{e}^{\mathrm{i}\tau\hat{K}_y}(\hat{K}_x - \mathrm{i}\alpha\hat{K}_y)\,\mathrm{e}^{-\mathrm{i}\tau\hat{K}_y}|h_{n_1}(\tau); h_{n_2}(\tau))\\ &\quad= -\tfrac{1}{2}(n_1+n_2+1)(\sinh\tau)\langle k, k+r|h_{n_1}(\tau); h_{n_2}(\tau)),\\ &\quad= -\tfrac{1}{2}(n_1+n_2+1)(\sinh\tau)\kappa_r^{k,n_1,n_2}(\tau)\end{aligned} \tag{C.2}$$

where equation (25) has been used.

On the other hand, $\alpha = \cosh(\tau)$ so

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\tau\hat{K}_y}(\hat{K}_x - \mathrm{i}\alpha\hat{K}_y)\,\mathrm{e}^{-\mathrm{i}\tau\hat{K}_y} &= \hat{K}_x\cosh\tau - \hat{K}_z\sinh\tau - \mathrm{i}\alpha\hat{K}_y\\ &= \hat{K}_-\cosh\tau - \hat{K}_z\sinh\tau\end{aligned} \tag{C.3}$$

which leads to

$$\begin{aligned}&\langle k, k+r|(\hat{K}_-\cosh\tau - \hat{K}_0(\sinh\tau))|h_{n_1}(\tau); h_{n_2}(\tau))\\ &\quad= \cosh\tau\sqrt{(2k+r)(r+1)}\kappa_{r+1}^{k,n_1,n_2}(\tau) - (k+r)(\sinh\tau)\kappa_r^{k,n_1,n_2}(\tau).\end{aligned} \tag{C.4}$$

Combining and simplifying equations (C.2) and (C.4) yields equation (45).

For $0 \leqslant \alpha \leqslant 1$, start from

$$\langle k, k+r|\,\mathrm{e}^{\mathrm{i}\tau\hat{K}_x}(\hat{K}_x - \mathrm{i}\alpha\hat{K}_y)\,\mathrm{e}^{-\mathrm{i}\tau\hat{K}_x}|h_{n_1}(\tau); h_{n_2}(\tau)) \tag{C.5}$$

and recall that $\mathrm{e}^{-\mathrm{i}\tau\hat{K}_x}|h_{n_1}(\tau); h_{n_2}(\tau))$ is just $|\psi_{n_1}(\tau)\rangle|\psi_{n_2}(\tau)\rangle$. Thus,

$$\begin{aligned}&\langle k, k+r|\,\mathrm{e}^{\mathrm{i}\tau\hat{K}_x}(\hat{K}_x - \mathrm{i}\alpha\hat{K}_y)\,\mathrm{e}^{-\mathrm{i}\tau\hat{K}_x}|h_{n_1}(\tau); h_{n_2}(\tau))\\ &\quad= \lambda\langle k, k+r|h_{n_1}(\tau); h_{n_2}(\tau)) = -\frac{\mathrm{i}}{2}(n_1+n_2+1)\tanh\tau\,\kappa_r^{k,n_1,n_2}(\tau)\end{aligned} \tag{C.6}$$

where equation (B.2) has been used.

Using $\alpha = 1/\cosh\tau$, we rewrite

$$\mathrm{e}^{\mathrm{i}\tau\hat{K}_x}(\hat{K}_x - \mathrm{i}\alpha\hat{K}_y)\,\mathrm{e}^{-\mathrm{i}\tau\hat{K}_x} = \hat{K}_-\mathrm{i}\tanh\tau\hat{K}_z. \tag{C.7}$$

Finally,

$$\begin{aligned}\langle k, k+r|\,\mathrm{e}^{\mathrm{i}\tau\hat{K}_x}(\hat{K}_x - \mathrm{i}\alpha\hat{K}_y)\,\mathrm{e}^{-\mathrm{i}\tau\hat{K}_x}|h_{n_1}(\tau); h_{n_2}(\tau)\rangle \\ = \langle k, k+r|(\hat{K}_- - \mathrm{i}(\tanh\tau)\hat{K}_z)|h_{n_1}(\tau); h_{n_2}(\tau)\rangle, \\ = \sqrt{(r+1)(2k+r)}\,\kappa_{r+1}^{k,n_1,n_2}(\tau) - \mathrm{i}(k+r)(\tanh\tau)\kappa_r^{k,n_1,n_2}(\tau).\end{aligned} \tag{C.8}$$

Combining equations (C.8) and (C.6) yields

$$\kappa_{r+1}^{k,n_1,n_2}(\tau) = \mathrm{i}\frac{\left(k+r-\frac{1}{2}(n_1+n_2+1)\right)}{\sqrt{(r+1)(2k+r)}}\tanh(\tau)\kappa_r^{k,n_1,n_2}(\tau). \tag{C.9}$$

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