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Some finite dimensional indecomposable representations of $E(2)$

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We describe the construction of some finite dimensional nonunitary representations of $E(2)$, the Lie group of Euclidean transformations in the plane. Some properties of these representations are also discussed, with emphasis on indecomposable representations. © *1999 American Institute of Physics.* [S0022-2488(99)02711-5]

I. INTRODUCTION

The group $E(2)$ of Euclidean transformations in two dimensions is the noncompact semidirect product group $[\mathbb{R}^2]$SO(2), which consists of Abelian translations in the plane together with rotations. Its unitary irreducible representations (unirreps) are either one-dimensional representations or infinite dimensional representations which can be constructed in the standard way by induction.[1] Much less is known about the finite dimensional, nonunitary representations of $E(2)$, the prototype of which is the ''natural'' representation

$$\pi:(R(\theta),x,y)\mapsto\begin{pmatrix}\cos\theta & -\sin\theta & x\\ \sin\theta & \cos\theta & y\\ 0 & 0 & 1\end{pmatrix} \tag{1}$$

in terms of 3×3 matrices, where $R(\theta)$ is the SO(2) rotation parametrized by the angle θ, and (x,y) is a vector describing the translation part of the transformation.

The representation of Eq. (1) was obtained in the familiar way from a 2×2 representation of SO(2), which is extended to a 3×3 matrix by addition of an extra line and an extra column with appropriate entries to account for the translation part of $E(2)$. This representation is not irreducible, but it is indecomposable.

It is the objective of this paper to present an explicit method of obtaining some finite-dimensional indecomposable representations of $E(2)$.

One can verify, using Eq. (1), the composition rule for $E(2)$ elements,

$$(R(\theta_1),x_1,y_1)\cdot(R(\theta_2),x_2,y_2)=(R(\theta_1+\theta_2),x_1+x_2\cos\theta_1-y_2\sin\theta_1,y_1+x_2\sin\theta_1+y_2\cos\theta_1). \tag{2}$$

From this composition rule, we can write a general element $(R(\theta),x,y)$ as the product $(1,x,y)\cdot(R(\theta),0,0)$, where $(1,0,0)=(R(\theta=0),0,0)$ is the unit element.

Throughout this paper, we will use complex coordinates, with $z=x+iy$. We can then obtain the 2×2 representations

$$\pi:(R(\theta),x,y)\equiv(R(\theta),z)\mapsto\begin{pmatrix}e^{i\theta} & z\\ 0 & 1\end{pmatrix},\quad \tilde{\pi}:(R(\theta),z)\mapsto\begin{pmatrix}1 & 0\\ \bar{z} & e^{-i\theta}\end{pmatrix}, \tag{3}$$

where $(R(\theta),z)$ now denotes an element of $E(2)$, and where the bar denotes complex conjugation. The composition rule now reads

$$(R(\theta_1),z_1)\cdot(R(\theta_2),z_2)=(R(\theta_1+\theta_2),z_1+z_2e^{i\theta_1}). \tag{4}$$

The full transformation in real space can be obtained from the real and imaginary parts of the complex transformation.

A motivation for our work is that $E(2)\sim[\mathbb{R}^2]\mathrm{SO}(2)$ represents the simplest nontrivial example of a semidirect product group, a family very useful in physics as it contains, amongst others, the rigid rotor group $[\mathbb{R}^5]\mathrm{SO}(3)$ of nuclear and molecular physics and the Poincaré group $[\mathbb{R}^4]\mathrm{SO}(3,1)$ of spacetime translations and boosts.

The starting point of our method is the Lie algebra $e(2)$ of the group $E(2)$. (We will jump freely between the algebra $e(2)$ and the group $E(2)$; all representations of $e(2)$ discussed here can be integrated to representations of $E(2)$.) Thus, suppose that $\pi(R(\theta),z)$ is a representation of $E(2)$ on a finite-dimensional space V. (It is a slight abuse of notation to write $\pi(R(\theta),z)$ because the representations will, in general, depend on both z and $\bar{z}$. However, this shorthand notation causes no problem. Technically speaking, we are thinking of z as an element of the complex plane, regarded as a *real* Lie group, not a complex Lie group.) Then, V decomposes into weight subspaces according to the action of SO(2),

$$V=\oplus W_k,$$

where

$$W_k=\{v\in V:\pi(R(\theta),0)v=e^{ik\theta}v\}, \tag{5}$$

where $k\in\mathbb{Z}$ so that $\pi(R(\theta+2\pi),z)=\pi(R(\theta),z)$ for representations of $E(2)$. We denote by

$$l_0=-i\frac{\partial}{\partial\theta}\pi(R(\theta),z)|_{\theta=z=0},\quad p_+=\frac{\partial}{\partial z}\pi(R(\theta),z)|_{\theta=z=0},\quad p_-=\frac{\partial}{\partial \bar{z}}\pi(R(\theta),z)|_{\theta=z=0}, \tag{6}$$

a basis for the $e(2)$ algebra, with nonzero commutation relations given by

$$[p_+,p_-]=0,\quad [l_0,p_\pm]=\pm p_\pm . \tag{7}$$

The elements p_+ and p_- are, respectively, ''raising'' and ''lowering'' operators, in the sense that

$$p_+W_k\subseteq W_{k+1},\quad p_-W_k\subseteq W_{k-1}. \tag{8}$$

In particular, for finite dimensional representations, they are nilpotent.

We have found that a useful and compact way of describing a representation of the $e(2)$ algebra is to display the result of Eqs. (7) and (8) in a graphical or diagrammatic form. We derive in Sec. II the rules for constructing representations of $e(2)$ that have no weight multiplicity. The tensor product of two such representations is simply obtained by combining their respective graphs in an appropriate way, as shown in Sec. II C. The resulting graph describes a representation of $e(2)$ which may or may not be decomposable; the problem of decomposing a tensor product turns out to be highly nontrivial, and we present in Sec. VIII some results on this issue.

A feature of tensor product representations and of certain other representations that we will present is that they typically contain indecomposable submodules with nontrivial weight multiplicities. One should recall that, thus far, the bulk of the results for $E(2)$ have dealt with unitary infinite dimensional representations, obtained either by induction or by the method of contraction,[2,3] where one considers representations of $E(2)$ as appropriate limits of representations of SU(2); in both cases, the weight multiplicity is never greater than 1. For the finite dimensional case, some of our representations can be thought of as smooth deformations of SU(2) representations. More generally, representations with trivial weight multiplicities are best accommodated inside the formalism of graded contractions[4] of SU(2), where the grading subgroup is the continuous subgroup SO(2)$\subset E(2)$. However, it is clear that the contraction of an SU(2) irrep cannot possibly yield a representation of $E(2)$ with nontrivial weight multiplicities. The possibility of constructing indecomposable modules containing arbitrarily high weight multiplicities is therefore, to our knowledge, completely new.

The representations of $E(2)$ that we construct belong to an identifiable family which, we think, is likely to contain many representations useful in physics. To illustrate this point, we give, in Sec. V, some explicit realizations of our representations. Moreover, the graphical method behind our results can certainly be adapted to more complicated semidirect product groups.[5]

II. STRING REPRESENTATIONS

In this section we discuss representations with weight multiplicities equal to 1, i.e., representations V for which, in the notation of (5), $\dim(W_k)\leqslant 1$, for all k. For such a representation, we let M and N be, respectively, the maximum and minimum nontrivial weights.

A. Some lemmas

Lemma 0: Every one-dimensional representation of $E(2)$ is of the form

$$\chi_k:(R(\theta),z)\mapsto e^{ik\theta}, \tag{9}$$

for some $k\in\mathbb{Z}$.

Proof: The translation subgroup T, i.e., the subgroup consisting of all elements of the form $(R(0),z)$, is the commutator subgroup of $E(2)$. So any one-dimensional representation of $E(2)$ must factor through the quotient $E(2)/T$, which is isomorphic to SO(2); the one-dimensional representations of SO(2) are of the specified form. □

Lemma 1: Let $0\neq|\varphi_k\rangle$ be an arbitrary vector in the one-dimensional subspace $W_k\subset V$. Then, at least one of $p_+|\varphi_k\rangle$ and $p_-(p_+|\varphi_k\rangle)$ must be zero for $[p_+,p_-]=0$ to be satisfied.

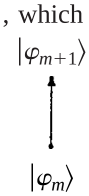
Proof: The raising and lowering operators p_+ and p_- are nilpotent and, since they commute, so is their product, the so(2)-invariant operator p_+p_-. The restriction of p_+p_- to any W_k subspace is therefore nilpotent. The only nilpotent operator on a one-dimensional space is the zero operator. If the W_k subspaces are all one-dimensional, this shows that $p_+p_-=0$ on V.

For an alternate, more explicit, proof, let $p_+p_-|\varphi_k\rangle=\alpha_k|\varphi_k\rangle$, where α_k is a proportionality constant. This holds since the subspace W_k is one-dimensional and p_+p_- is a weight-preserving operator. Since the representation is finite dimensional, there exists n such that $(p_+p_-)^n|\varphi_k\rangle=(p_+)^n(p_-)^n|\varphi_k\rangle=\alpha_k^n|\varphi_k\rangle=0$, from which it follows that $\alpha_k=0$. □

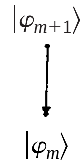
Proposition 1: If we specify on which subspaces W_m the raising and lowering operators are zero and on which they are nonzero, subject to the condition in lemma 1, this determines a unique representation of $e(2)$. The resulting representation is indecomposable if and only if p_+W_m and p_-W_{m+1} are not both zero for any m with $N\leqslant m<M$.

Proof: Because of the condition, we can choose a basis $\{|\varphi_m\rangle$ s.t. $l_0|\varphi_m\rangle=m|\varphi_m\rangle\}$ of eigenstates of l_0, with $|\varphi_m\rangle\in W_m$, for each m, and such that for each $m\in\{N,\ldots,M-1\}$, precisely one of the following holds:

(i) $p_+|\varphi_m\rangle=|\varphi_{m+1}\rangle$ and $p_-|\varphi_{m+1}\rangle=0$, which we represent by



(ii) $p_+|\varphi_m\rangle=0$ and $p_-|\varphi_{m+1}\rangle=|\varphi_m\rangle$, with graph,



or

(iii) $p_+|\varphi_m\rangle=0$ and $p_-|\varphi_{m+1}\rangle=0$, i.e., there is no arrow between $|\varphi_{m+1}\rangle$ and $|\varphi_m\rangle$,

$$\begin{array}{c}|\varphi_m\rangle\\ \vdots\\ |\varphi_{m+1}\rangle\end{array}.$$

Relative to the basis $\{|\varphi_M\rangle,|\varphi_{M-1}\rangle,\ldots,|\varphi_{N+1}\rangle,|\varphi_N\rangle\}$, so(2) acts diagonally, p_+ is represented by a matrix which is zero except for a 1 immediately above the diagonal corresponding to each m for which possibility (i) above holds, and p_- is represented by a matrix which is zero except for a 1 immediately below the diagonal corresponding to each $m+1$ for which possibility (ii) above holds.

Clearly the matrices for p_+ and p_- commute. The remaining commutators $[l_0,p_\pm]=\pm p_\pm$ are satisfied since, for instance, $(l_0p_+-p_+l_0)|\varphi_m\rangle=l_0|\varphi_{m+1}\rangle-mp_+|\varphi_m\rangle=|\varphi_{m+1}\rangle=p_+|\varphi_m\rangle$ by construction.

If $p_+W_k=0=p_-W_{k+1}$, then

$$V=(\oplus_{m\leqslant k}W_m)\oplus(\oplus_{m>k}W_m) \tag{10}$$

is an $e(2)$-decomposition. Conversely, suppose $V=U\oplus U'$ is an $e(2)$-decomposition but that condition (iii) above does not hold for any $k\in\{N,N+1,...,M-1\}$. We can assume there exists $m\in\{N,N+1,...,M-1\}$ such that $W_m\subseteq U$, $W_{m+1}\subseteq U'$. Then either $p_+(W_m)$ or $p_-(W_{m+1})$ is nonzero. Since U and U' are both $e(2)$-spaces, this shows $U\cap U'\neq\{0\}$, a contradiction. □

Representations with weight multiplicities all equal to 1 will be called string representations.

B. String representations in graphical form

To a representation thus constructed, we can associate a graph as a mnemonic device to remember which of the conditions (i), (ii) or (iii) hold between two neighboring weight subspaces W_m and W_{m+1} by drawing an up arrow from W_m to W_{m+1} when (i) applies, a down arrow from W_{m+1} to W_m when (ii) applies, and no arrow when (iii) occurs; subgraphs of the type

for which $p_+|\varphi_m\rangle\neq0$ and $p_-|\varphi_{m+1}\rangle\neq0$, cannot occur.

To obtain a representation of $E(2)$ relative to the chosen basis, we start by exponentiating separately the diagonal matrix of l_0 to obtain the image of $(R(\theta),0)\in$ SO(2), and the off-diagonal matrix elements of the generators of translations p_+ and p_- to obtain $(1,z)$. The element $(R(\theta),z)$ is then constructed from the matrix multiplication of $(1,z)\cdot(R(\theta),0)$.

For instance, the ''raising string'' representation of $e(2)$, with a graph consisting only of up arrows, exponentiates to the $E(2)$ representation,

$$\Rightarrow \pi_1:(R(\theta),z)\mapsto\begin{pmatrix} e^{iM\theta} & e^{i(M-1)\theta}z & e^{i(M-2)\theta}\frac{1}{2}z^2 & \cdots & \frac{e^{i(N+1)\theta}}{(M-N+1)!}z^{M-N+1} & \frac{e^{iN\theta}}{(M-N)!}z^{M-N} \\ 0 & e^{i(M-1)\theta} & e^{i(M-2)\theta}z & \cdots & \frac{e^{i(N+1)\theta}}{(M-N+2)!}z^{M-N+2} & \frac{e^{iN\theta}}{(M-N+1)!}z^{M-N+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i(N+1)\theta} & e^{iN\theta}z \\ 0 & 0 & 0 & \cdots & 0 & e^{iN\theta} \end{pmatrix}, \tag{11}$$

containing the SO(2) unirreps $M,M-1,\ldots,N$ each with multiplicity 1. It is indecomposable.

The ''lowering string'' representation

$$\Rightarrow \pi_2:(R(\theta),z)\mapsto\begin{pmatrix} e^{iM\theta} & 0 & 0 & \cdots & 0 & 0 \\ e^{iM\theta}\overline{z} & e^{i(M-1)\theta} & 0 & \cdots & 0 & 0 \\ e^{iM\theta}\frac{1}{2}\overline{z}^2 & e^{i(M-1)\theta}\overline{z} & e^{i(M-2)\theta} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{e^{iM\theta}}{(M-N-1)!}\overline{z}^{M-N-1} & \frac{e^{i(M-1)\theta}}{(M-N-2)!}\overline{z}^{M-N-2} & \cdots & \cdots & e^{i(N+1)\theta} & 0 \\ \frac{e^{iM\theta}}{(M-N)!}\overline{z}^{M-N} & \frac{e^{i(M-1)\theta}}{(M-N-1)!}\overline{z}^{M-N-1} & \cdots & \cdots & e^{i(N+1)\theta}\overline{z} & e^{iN\theta} \end{pmatrix} \tag{12}$$

contains the SO(2) unirreps $M,M-1,\ldots,N$ each with multiplicity 1, and nontrivial lowering operators between each pair of adjacent SO(2) subspaces. It is also indecomposable.

The five-dimensional representation with graph

$$\Rightarrow \pi_3:(R(\theta),z)\mapsto\begin{pmatrix} e^{2i\theta} & 0 & 0 & 0 & 0 \\ e^{2i\theta}\overline{z} & e^{i\theta} & z & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\theta} & e^{-2i\theta}z \\ 0 & 0 & 0 & 0 & e^{-2i\theta} \end{pmatrix} \tag{13}$$

is decomposable into two subspaces $V_1\oplus V_2$, containing respectively the SO(2) irreps 2,1,0 and $-1,-2$.

The three-dimensional representation

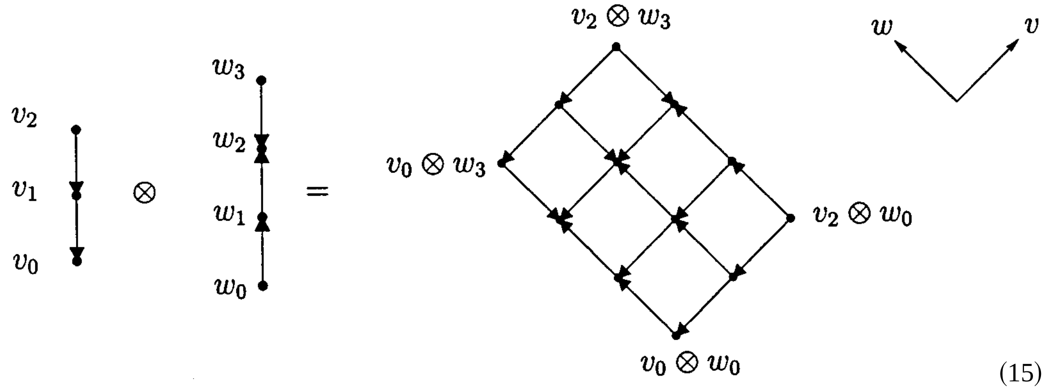
$$\Rightarrow \pi_4:(R(\theta),z)\mapsto\begin{pmatrix} e^{i\theta} & z & 0 \\ 0 & 1 & 0 \\ 0 & \overline{z} & e^{-i\theta} \end{pmatrix} \tag{14}$$

is indecomposable and equivalent to the ''natural'' representation of Eq. (1).

Note that, if π is an $E(2)$ representation containing the SO(2) irreps $M,M-1,\ldots,N$, then $\chi_k\cdot\pi$ is another (inequivalent) representation containing the SO(2) irreps $M+k,M-1+k,\ldots,N+k$.

C. Tensor product of two strings

Finally, it is also easy to represent the tensor product of two string representations in a graphical way. Thus, if V_1 and V_2 are two representations of $E(2)$ spanned, respectively, by $\{v_i, i=m_1, m_1-1, \ldots n_1\}$ and $\{w_j, j=m_2, m_2-1, \ldots, n_2\}$, then a basis for the tensor product representation $V_1 \otimes V_2$ is given by the points $v_i \otimes w_j$ having coordinates (i,j) on a two-dimensional grid. The arrows between point $v_i \otimes w_j$ and $v_k \otimes w_l$ are determined from the action of the $e(2)$ elements on v_i or w_j. Thus, for instance, consider the following tensor product:



(15)

where the final two-dimensional graph has been tilted so that states with the same weight occur at the same horizontal height. (The ''corner'' states on the graph have been explicitly indicated.)

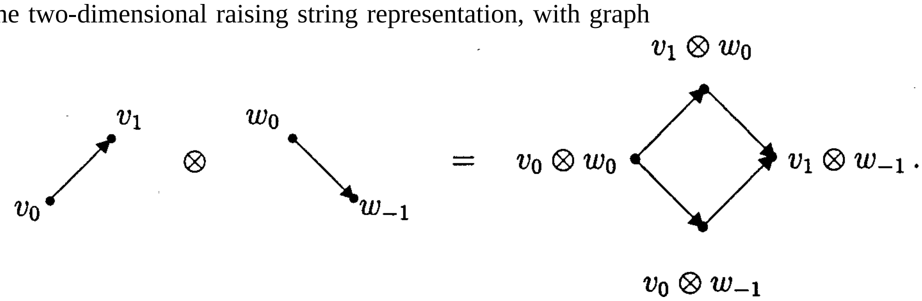
III. PARALLELOGRAM REPRESENTATIONS

A. The parallelogram representation as tensor product

Consider the representation

$$\pi_5:(R(\theta),z) \mapsto \begin{pmatrix} 1 & 0 \\ \bar{z} & e^{-i\theta} \end{pmatrix} \otimes \begin{pmatrix} e^{i\theta} & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{i\theta} & z & 0 & 0 \\ 0 & 1 & 0 & 0 \\ e^{i\theta}\bar{z} & z\bar{z} & 1 & e^{-i\theta}z \\ 0 & \bar{z} & 0 & e^{-i\theta} \end{pmatrix}, \tag{16}$$

which is obtained from the tensor product of the two-dimensional lowering string representation and the two-dimensional raising string representation, with graph



(17)

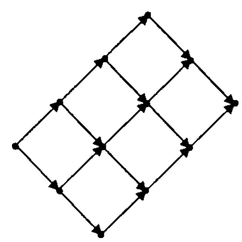
Claim: The representation π_5 is indecomposable.

Proof: Otherwise suppose $V=U \oplus U'$ is a nontrivial decomposition, and that one of the two subspaces, say U, contains a vector v in the two-dimensional subspace W_0 of weight 0 of the form $v=\alpha_1(v_0 \otimes w_0)+\alpha_2(v_1 \otimes w_{-1})$, with $\alpha_1 \neq 0$. By acting with p_+p_-, we find $p_+p_-v=\alpha_1(v_1 \otimes w_{-1})$ must also be in U. Thus, $v_1 \otimes w_{-1} \in W_0$ is also in U (since $\alpha_1 \neq 0$) and so $v-\alpha_2(v_1 \otimes w_{-1})=\alpha_1(v_0 \otimes w_0) \in U$. Since $p_+(v_0 \otimes w_0)=v_1 \otimes w_0 \in U$ and $p_-(v_0 \otimes w_0)=v_0 \otimes w_{-1} \in U$ as well, we find that $U=V, U'=\{0\}$, a contradiction. □

This generalizes to larger representations. For $M \geqslant 0$, let V_M be the ''raising-string'' representation with lowest weight 0 and highest weight M. It has a weight basis consisting of the lowest weight vector v_0 of weight 0, and nonzero weight vectors $v_k = (p_+)^k v_0$, for $k = 1,...,M$, with $p_+(v_M) = 0$; the dimension is $M+1$. Similarly, for $N \geqslant 0$, let V_{-N} be the ''lowering-string'' representation with lowest weight $-N$ and highest weight 0. It has a weight basis consisting of the highest weight vector w_0 of weight 0, and nonzero weight vectors $w_{-k} = (p_-)^k w_0$, for $k = 1,...,N$, with $p_-(w_{-N}) = 0$; the dimension is $N+1$. The action of p_- on V_M and the action of p_+ on V_{-N} are both trivial.

The ''parallelogram'' representation $V_{M,-N}$ is the tensor product $V_{M,-N} = V_M \otimes V_{-N}$; it has lowest weight $-N$, highest weight M, and dimension $(M+1)(N+1)$. Since $v_k \otimes w_{-l} = (p_+)^k (p_-)^l v_0 \otimes w_0 = (p_-)^l (p_+)^k v_0 \otimes w_0$, we see that $V_{M,-N}$ is generated by the weight vector $v_0 \otimes w_0$, which we call the ''initial vector.'' Twisting by the character $\chi_r : (R(\theta),z) \mapsto e^{ir\theta}$, $r \in \mathbb{Z}$, gives a parallelogram representation $V_{M,-N;r} = \chi_r \otimes V_{M,-N}$ with lowest weight $r-N$, highest weight $r+M$, and dimension $(M+1)(N+1)$; it is generated by the ''initial vector'' $\chi_r \otimes v_0 \otimes w_0$.

For instance, the graph



(18)

is associated with the $V_{3,-2;r}$ parallelogram representation.

Lemma 2: The parallelogram representation $V_{M,-N;r}$ is indecomposable.

Proof: Let $V_{M,-N;r} = U \oplus U'$ be a decomposition. Choose a basis T whose first vector is $\phi_0 = \chi_r \otimes v_0 \otimes w_0$ and whose first d_r vectors $\phi_0, p_+p_-\phi_0, (p_+p_-)^2\phi_0, \ldots, (p_+p_-)^{d_r-1}\phi_0$ span the weight subspace W_r (of dimension d_r). Writing vectors in terms of this basis, we have that the initial vector is $\phi_0 = \chi_r \otimes v_0 \otimes w_0 = (1,0,0,\ldots)^T$. Let $v = (\alpha_1, \alpha_2, \ldots, \alpha_{d_r}, 0, \ldots)^T, \alpha_1 \neq 0$, be an otherwise arbitrary vector in W_r. Assume that $v \in U$, and consider

$$p_+p_-v = (0,\alpha_1,\alpha_2,\ldots,\alpha_{d_r-1},0,\ldots)^T \in U,$$

$$(p_+p_-)^2 v = (0,0,\alpha_1,\alpha_2,\ldots,\alpha_{d_r-2},0,\ldots)^T \in U,$$

$$\vdots$$

$$(p_+p_-)^{d_r-3} v = (0,\ldots,\alpha_1,\alpha_2,\alpha_3,0,\ldots)^T \in U,$$

$$(p_+p_-)^{d_r-2} v = (0,\ldots,\alpha_1,\alpha_2,0,\ldots)^T \in U,$$

$$(p_+p_-)^{d_r-1} v = (0,\ldots,0,\alpha_1,0,\ldots)^T \in U. \tag{19}$$

Thus, the d_rth basis vector

$$v_{d_r} = \frac{1}{\alpha_1}(p_+p_-)^{d_r-1}(\chi_r \otimes v_0 \otimes w_0) = (0,\ldots,1,0,\ldots)^T \in U \tag{20}$$

(since $\alpha_1 \neq 0$ by assumption). If this vector is in U, then the vector

$$v_{d_r-1}=\frac{1}{\alpha_1}((p_+p_-)^{d_r-2}v-\alpha_2 v_{d_r})=(0,\ldots,1,0,0,\ldots)^T\in U \tag{21}$$

as well, and so is

$$v_{d_r-2}=\frac{1}{\alpha_1}((p_+p_-)^{d_r-3}v-\alpha_2 v_{d_r-1}-\alpha_3 v_{d_r})=(0,\ldots,1,0,0,0,\ldots)^T\in U, \tag{22}$$

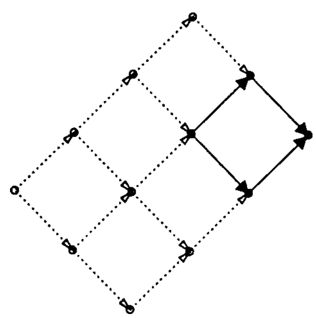
and so forth until one shows that all basis vectors in W_r are in U. In effect, this argument is based on the observation that the matrix of the restriction to each weight space of the operator p_+p_- is a triangular matrix.

In particular, the initial vector $\chi_r\otimes v_0\otimes w_0$ is in U and, since an arbitrary basis state $\chi_r\otimes v_s\otimes w_{-q}\in V$ can be obtained as $(p_+)^s(p_-)^q(\chi_r\otimes v_0\otimes w_0)$, it follows that $U=V$ and $U'=\{0\}$, which shows that V is indecomposable. □

In this way, we can construct indecomposable representations with arbitrarily high weight multiplicities.

B. Subrepresentations of the parallelogram

Observation: A vector $X_{k,-l,r}=\chi_r\otimes v_k\otimes w_{-l}=(p_+)^k(p_-)^l(\chi_r\otimes v_0\otimes w_0)$ generates a subrepresentation of $V_{M,-N;r}$ that is isomorphic to $V_{M-k,l-N;r+k-l}$. An example of this is given in Eq. (23), where, in $V_{3,-2;0}$, a 2×2 parallelogram subrepresentation $V_{1,-1;1}$ is generated by $X_{2,1,0}=(p_+)^2p_-(v_0\otimes w_0)$,



(23)

Proposition 2: Every subrepresentation of a parallelogram representation $V_{M,-N;r}$ can be expressed as the subrepresentation generated by finitely many ''initial vectors'' of the form $(p_+)^k(p_-)^l(\chi_r\otimes v_0\otimes w_0)$, for suitable k, l. The subrepresentations of $V_{M,-N;r}$ are all indecomposable.

Proof: First, note that the weight subspace $W_s\subset V_{M,-N;r}$ is spanned by vectors of the form $\{(p_+)^k(p_-)^l(\chi_r\otimes v_0\otimes w_0)|k-l=s-r\}$. The action of p_+p_- on this basis is to map

$$p_+p_-:(p_+)^k(p_-)^l(\chi_r\otimes v_0\otimes w_0)\mapsto(p_+)^{k+1}(p_-)^{l+1}(\chi_r\otimes v_0\otimes w_0). \tag{24}$$

Now, suppose $U\subset V_{M,-N;r}$ is an indecomposable subrepresentation with nontrivial intersection with the weight space W_s. Choose the minimal k such that U contains a nonzero vector $v\in W_s$ of the form

$$v=\sum_{i\geq0}\alpha_i(p_+)^{\tilde{k}+i}(p_-)^{\tilde{l}+i}(\chi_r\otimes v_0\otimes w_0),\ \alpha_0\neq0,\ \tilde{k}-\tilde{l}=s-r. \tag{25}$$

Then, by the p_+p_- argument of Lemma 2, all vectors of the form

$$(p_+)^{\tilde{k}+i}(p_-)^{\tilde{l}+i}(\chi_r \otimes v_0 \otimes w_0),\ i \geqslant 0, \tag{26}$$

must be in U, and $(p_+)^{\tilde{k}}(p_-)^{\tilde{l}}(\chi_r \otimes v_0 \otimes w_0)$ is an initial vector for the subspace of weight s in U.

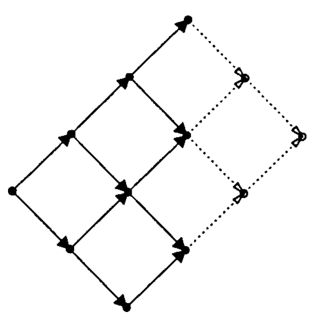
Do this for each weight s of U and discard redundant choices (vectors as above that are contained in the subspace generated by another such vector). The remaining finitely many ''initial vectors'' generate the whole space U. Note that the initial vectors have distinct weights.

Now suppose $U = U_1 \oplus U_2$ is a decomposition. Fix one of the initial vectors described above. Then one of U_1, U_2, say U_1, contains a weight vector with nonzero overlap with this chosen initial vector. But then U_1 actually contains that initial vector, because of the triangularity of the operator p_+p_-, as above. If there is only one initial vector, then we are done, by Lemma 2 and the previous Observation. Otherwise, notice that once U_1 contains $(p_+)^k(p_-)^l(\chi_r \otimes v_0 \otimes v_0)$, it must also contain the ''final vector'' $(p_+)^m(p_-)^n(\chi_r \otimes v_0 \otimes v_0) = (p_+)^{m-k} \times (p_-)^{n-l}((p_+)^k(p_-)^l(\chi_r \otimes v_0 \otimes v_0))$. But then U_1 must contain all the initial vectors, and hence all of U, so U is indecomposable. □

Corollary: A basis for any subrepresentation of the parallelogram representation $V_{m,-n;r}$ *is given by vectors of the form* $(p_+)^k(p_-)^l(\chi_r \otimes v_0 \otimes v_0)$ *contained in this subrepresentation.*

C. Quotient representations

Consider the representation associated with the graph



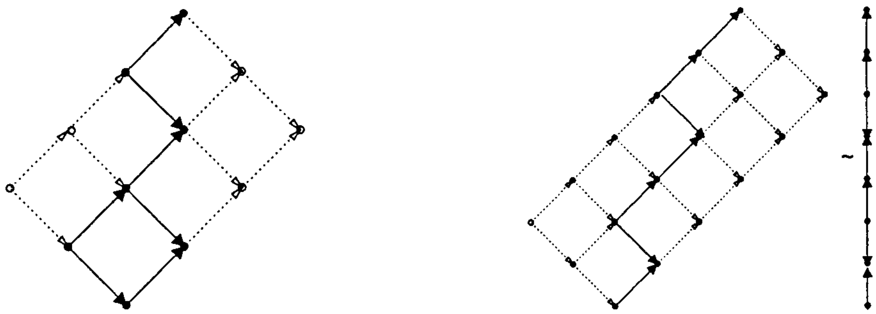
$$\Rightarrow \pi_6:(R(\theta),z) \mapsto \begin{pmatrix} e^{4i\theta} & ze^{3i\theta} & \frac{1}{2}z^2e^{2i\theta} & 0 & \frac{1}{6}z^3e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{3i\theta} & ze^{2i\theta} & 0 & \frac{1}{2}z^2e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{2i\theta} & 0 & ze^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & \overline{z}e^{3i\theta} & 0 & e^{2i\theta} & 0 & ze^{i\theta} & \frac{1}{2}z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{z}e^{2i\theta} & 0 & 0 & e^{i\theta} & z & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{z}e^{i\theta} & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}z^2e^{2i\theta} & 0 & 0 & ze^{i\theta} & 0 & 1 & ze^{-i\theta} \\ 0 & 0 & 0 & 0 & \frac{1}{2}\overline{z}^2e^{i\theta} & 0 & \overline{z} & 0 & e^{-i\theta} \end{pmatrix}. \tag{27}$$

This representation can be constructed by starting from the parallelogram $V_{3,-2;0}$, and by removing (or, equivalently, setting to 0) each line and column corresponding to a node in the subrepresentation with initial vectors $v_3 \otimes w_{-1}$ and $v_2 \otimes w_{-2}$. It is therefore a quotient of the parallelogram $V_{3,-2;r}$.

Lemma 3: Quotient representations of a parallelogram are indecomposable.

Proof: Such a quotient representation has a single initial vector, $v = v_0 \otimes w_0$. Let $V = V_1 \oplus V_2$ be a decomposition. By the p_+p_- triangularity argument, the whole weight space containing v must be in either V_1 or V_2; without loss of generality, we assume that it is in V_1, and thus v itself is in V_1. Since everything in V can be obtained from $v = v_0 \otimes w_0$ by acting with $e(2)$ generators, it follows that $V_1 = V, V_2 = \{0\}$, i.e., V is indecomposable. □

It is also possible to take quotients of subrepresentations of a parallelogram, as shown on the left-hand side of Eq. (28), or even to make a string representation out of a quotient of a subparallelogram, as shown on the right-hand side of that figure.



(28)

Thus, we can claim that

Proposition 3: A subquotient of a subrepresentation of the parallelogram $V_{m,-n;r}$ (i.e., the quotient of two subrepresentations of the parallelogram $V_{m,-n;r}$) is indecomposable, provided that its graph is connected.

Proof: Let i,j uniquely label a minimal set of initial vectors $X_{i,j,r}=(p_+)^i(p_-)^j(\chi_r)\otimes v_0 \otimes w_0)$ of weight $r+i-j$ in such a representation V. Because initial vectors cannot have the same weight, the elements in the set $\{X_{i,j,k}\}$ can be ordered by increasing weight. These initial vectors are also those of a subrepresentation of the parallelogram $V_{m,-n;r}$. Let $V=V_1\oplus V_2$. Using the p_+p_- triangularity argument, the vectors generated from an initial vector all belong to the same subspace, either V_1 or V_2, depending on whether the initial vector is in V_1 or V_2. However, the subspaces generated by two consecutive initial vectors must have a nontrivial intersection, for otherwise the graph would be disconnected (into the vectors in or above the higher subspace and the vectors in or below the lower subspace).

By the p_+p_- triangularity argument, each weight space in this intersection must belong to only one of V_1 or V_2. Thus, the subspaces generated by two consecutive initial vectors must belong to the same subspace, and, continuing this way, all subspaces must belong to the same subspace, say V_1. This means that V_2 is empty and we are done. □

The last noteworthy result on quotients of subparallelograms is as follows:

Proposition 4: Given a (connected) string representation, there exists a subquotient of a parallelogram which is isomorphic to this string.

Proof: In the string representation, let m be the number of up arrows, n the number of down arrows, and construct $V_{m,-n;r}$, with r adjusted so that the highest weight of $V_{m,-n;r}$ is equal to the highest weight of the string, i.e., $m+r$. If the topmost arrow of the string is a down arrow then the highest weight state of the representation, $\chi_r\otimes v_m\otimes w_0$ is the heaviest initial vector of the subrepresentation. Otherwise, the initial vectors are ''sources'' of the type

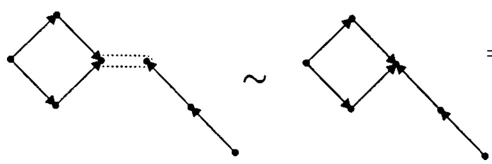
unless the bottom-most arrow of the string is an up arrow, in which case the lightest initial vector is the lowest weight vector of the representation, $\chi_r\otimes v_0\otimes w_{-N}$. The heaviest initial vector, for instance, is always of the form $\chi_r\otimes v_l\otimes w_0$. Next, we note that terminal vectors are either highest or lowest weight vectors or ''sinks'' of the type

If the topmost arrow in the string is a down arrow, then $\chi_r\otimes v_n\otimes w_0$ is an initial vector; if the bottommost arrow is an up arrow, then $\chi_r \otimes v_0 \otimes w_{-m}$ is an initial vector. For each ''source,'' as described above, V has an initial vector $\chi_r \otimes v_k \otimes w_{-l}$, where $m-k$ is the number of up arrows above the source and $n-l$ is the number of down arrows below it.

Having identified the initial vectors, one then constructs the corresponding subrepresentation V of $V_{m,-n;r}$. Consider now the subrepresentation $V' \subset V$ obtained by applying the operator p_+p_- to V, i.e., $V' = p_+p_-(V)$. The original string is the subquotient V/V'. $\square$

IV. GLUING

Not all representations need be quotients or subquotients of parallelograms. For instance, consider the representation



$$\Rightarrow \pi_7 : (R(\theta), z) \mapsto \begin{pmatrix} e^{i\theta} & z & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ e^{i\theta}\overline{z} & z\overline{z} & 1 & e^{-i\theta} z & e^{-i\theta} z & e^{-2i\theta}\frac{1}{2}z^2 \\ 0 & \overline{z} & 0 & e^{-i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i\theta} & e^{-2i\theta} z \\ 0 & 0 & 0 & 0 & 0 & e^{-2i\theta} \end{pmatrix} \tag{29}$$

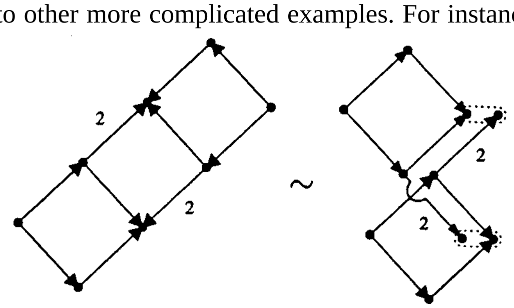
with $\dim(W_1) = 1$, $\dim(W_0) = \dim(W_{-1}) = 2$, $\dim(W_{-2}) = 1$. It is constructed by identifying the terminal vector of the 2×2 parallelogram representation of Eq. (17) with the terminal node of an extra two-element raising string ending at the terminal node, i.e., by ''gluing'' the parallelogram and the raising string at one node in the manner indicated by the graph.

Claim: The representation π_7 is indecomposable.

Proof: Otherwise suppose $V = U \oplus U'$ is a nontrivial decomposition. We can assume U contains the weight space W_{-2}. In this case, it also contains the standard basis vectors $\hat{e}_5 = (0,0,0,0,1,0)^T$ and $\hat{e}_4 = (0,0,0,1,0,0)^T$. But this last vector is the ''terminal node'' of the subspace which is isomorphic to π_5, and the argument given in connection to this representation shows that either U or U' must contain all of this subspace. Since we have already seen that U must contain e_4, it must contain all of the π_5 subspace, and therefore must be all of V. $\square$

This can be generalized to other more complicated examples. For instance, the representation

π_8 :



(30)

can be realized as shown as a two-point gluing. Note that the factors of 2 in the graph indicate that the corresponding matrix element is of strength 2 rather than 1, as has been thus far assumed throughout this paper. The representation π_8 can also be shown to be indecomposable.

Not all gluings are indecomposable. The representation of Fig. (53), which is obtained by gluing together two strings, is decomposable, as will be seen in Sec. VII.

V. EXPLICIT REALIZATIONS

In this section, we would like to exhibit some of the more interesting amongst the many explicit realizations of the $e(2)$ algebra, along with examples of the kind of representations that we have so far described.

Nonunitary representations occur in physics mostly as representations carried by tensor operators. Thus, if $\hat{T}_i^\lambda$ is the ith component of a tensor operator $\hat{T}^\lambda$ transforming by the representation λ, and if $\mathcal{O}$ is a representation of an element of $e(2)$ by linear operators, then the action of $\mathcal{O}$ on $\hat{T}^\lambda$ is given by

$$\mathcal{O}:\hat{T}_i^\lambda \mapsto [\mathcal{O},\hat{T}_i^\lambda] = \sum_j \alpha_{ij}\hat{T}_j^\lambda, \tag{31}$$

where α_{ij} are, in general, complex coefficients.

A. The adjoint representation

Let,

$$p_+ \mapsto \tfrac{1}{2}e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right),\ p_- \mapsto \tfrac{1}{2}e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right),\ l_0 \mapsto i\frac{\partial}{\partial\theta}. \tag{32}$$

If we let these operators act on one another, we obtain the adjoint representation of $e(2)$, with the graph given in Eq. (14).

B. Raising string or lowering string representations

Let

$$p_+ \mapsto e^{i\theta}\frac{\partial}{\partial r},\ p_- \mapsto 0,\ l_0 \mapsto -i\frac{\partial}{\partial\theta}, \tag{33}$$

and consider the set of polynomials $\{e^{ik\theta}r^q, e^{i(k+1)\theta}r^{q-1}, e^{i(k+2)\theta}r^{q-2},\ldots,e^{i(k+q)\theta}\}$, with $k \in \mathbb{Z}, q \in \mathbb{Z}^+$.

Using Eq. (31) and the realization of Eq. (33), one sees that these polynomials carry a representation equivalent to a raising string representation of dimension $q+1$ with lowest weight k.

Similarly, the realization

$$p_+ \mapsto 0,\ p_- \mapsto e^{-i\theta}\frac{\partial}{\partial r},\ l_0 \mapsto -i\frac{\partial}{\partial\theta}, \tag{34}$$

action on the polynomials $\{e^{in\theta}r^q, e^{i(n-1)\theta}r^{q-1},\ldots,e^{i(n-q)\theta}\}$, with $n \in \mathbb{Z}, q \in \mathbb{Z}^+$, is a lowering string representation of dimension $q+1$ with lowest weight $n-q$.

We obtain a less trivial example if we let the operators of Eq. (32) act on the polynomials $\{r^t e^{it\theta}, r^{t-1}e^{i(t-1)\theta},\ldots,1\}$, which yields a raising string representation with highest weight vector $r^t e^{it\theta}$ of weight t.

In a similar way, the polynomials $\{r^t e^{-it\theta}, r^{t-1}e^{-i(t-1)\theta},\ldots,1\}$ span a lowering string representation of $E(2)$ under the action of the operators of Eq. (32).

C. Parallelograms

Consider the three operators

$$p_+ = \frac{\partial^2}{\partial x^2} + 2i\frac{\partial^2}{\partial x\,\partial y} - \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \overline{w}^2},$$

$$p_- = \frac{\partial^2}{\partial x^2} - 2i\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial w^2}, \tag{35}$$

$$l_0 = -\frac{i}{2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = w\frac{\partial}{\partial w} - \overline{w}\frac{\partial}{\partial \overline{w}}. \tag{36}$$

With the identification $p_\pm \leftrightarrow \hat{Q}_{\pm 2}, l_0 \leftrightarrow \frac{1}{2}\hat{l}_z$, these are easily recognized as forming an $e(2)$ subalgebra of $[\mathbb{R}]^5$so(3), the algebra of the rigid rotor.

We now consider the operators

$$\hat{X}_{LM} = \sqrt{\frac{2^{L+M}(L+M)!(L-M)!L!}{(2L)!}} \sum_t \frac{\omega^t \tilde{\omega}^{t-M} z^{L+M-2t}}{2^t t!(t-M)!(L+M-2t)!}, \tag{37}$$

which can be recognized as proportional to the spherical harmonics

$$Y_{LM}(\theta,\varphi) = \sqrt{\frac{(2L+1)!}{4\pi 2^L L!}}\hat{X}_{LM} \tag{38}$$

if we make the identifications

$$\omega = \frac{-e^{i\varphi}\sin\theta}{\sqrt{2}}, \quad z = \cos\theta, \quad \tilde{\omega} = \frac{e^{-i\varphi}\sin\theta}{\sqrt{2}}. \tag{39}$$

In general, we have

$$[p_\pm, \hat{X}_{LM}] = \alpha_{LM}\hat{X}_{L-2,M\pm 2}, \tag{40}$$

with

$$\alpha_{LM} = \langle \hat{X}_{L-2,M\pm 2} | p_\pm | \hat{X}_{LM} \rangle, \tag{41}$$

with the upper and lower sign valid for $p_\pm$, respectively, and where the number $\langle \hat{X}_{L-2,M\pm 2} | p_\pm | \hat{X}_{LM} \rangle$ is computed using the standard boson inner product. With these we can build a variety of parallelograms. (Note that the $e(2)$ weight is $\frac{1}{2}M$.)

Thus, for instance, the set of nine operators $\{\hat{X}_{80}, \hat{X}_{6,\pm 2}, \hat{X}_{4,\pm 4}, \hat{X}_{40}, \hat{X}_{2,\pm 2}, \hat{X}_{00}\}$ are the components of a tensor which carries the parallelogram representation $V_{2,-2,0}$ with

$$\langle \hat{X}_{L-2,M\pm 2} | p_\pm | \hat{X}_{LM} \rangle = \sqrt{\frac{(2L+1)L(L-1)}{2L-3}}(L,M;2,\pm 2|L-2,M\pm 2), \tag{42}$$

where $(L,M;2,\pm 2|L-2,M\pm 2)$ is an SO(3) Clebsch–Gordan coefficient.

Similarly, the set $\{\hat{X}_{71}, \hat{X}_{51}, \hat{X}_{5,-3}, \hat{X}_{33}, \hat{X}_{3,-1}, X_{11}\}$ are the six components of a tensor which carries the parallelogram representation $V_{2,-1;1}$.

Subparallelograms of these or of any parallelogram are obtained by simply removing from the original parallelogram all the basis polynomials not contained in the subrepresentations.

It is also possible to obtain parallelograms from the realization of Eq. (32). Repeated action of these operators on the initial vector $r^p e^{in\theta}$ produce the parallelogram representation $V_{(1/2)(p+n),-(1/2)(p-n);n}$. Note that, since $\frac{1}{2}(p+n)$ and $\frac{1}{2}(p-n)$ must be integers, $p+n$ must be even for the representation to remain finite dimensional.

The realization of Eq. (32) is a special case of the more general realization

$$p_+ \mapsto r^x e^{i(x-1)\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right), p_- \mapsto r^{-y} e^{i(y+1)\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right),$$

$$l_0 \mapsto i\frac{(x-y-2)}{2(x-1)(y+1)}\frac{\partial}{\partial\theta} + \frac{(x+y)}{2(x-1)(y+1)} r\frac{\partial}{\partial r}, \tag{43}$$

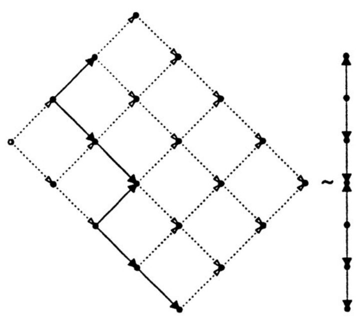
from which we can extract various parallelograms. (The exponents m,n of the initial vector $r^m e^{in\theta}$ must satisfy some conditions if the representation is to remain finite dimensional.)

D. Quotients and strings

Consider now the realization

$$p_+ \mapsto e^{i\theta}\frac{\partial^2}{\partial r^2},\ p_- \mapsto e^{-i\theta}\frac{\partial}{\partial r},\ l_0 = -i\frac{\partial}{\partial\theta}. \tag{44}$$

Acting on the initial vector r^4 to generate a parallelogram, and removing from this parallelogram the basis states r^4 and $r^3 e^{-i\theta}$, we obtain the string representation



$$\Rightarrow \pi_9 : (R(\theta), w) \mapsto \begin{pmatrix} e^{-4i\theta} & \bar{w}e^{-3i\theta} & \frac{1}{2}\bar{w}^2 e^{-2i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{-3i\theta} & \bar{w}e^{-2i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-2i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & we^{-2i\theta} & e^{-i\theta} & \bar{w} & \frac{1}{2}\bar{w}^2 e^{i\theta} & 0 \\ 0 & 0 & 0 & 0 & 1 & \bar{w}e^{i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & we^{i\theta} & e^{2i\theta} \end{pmatrix}, \tag{45}$$

spanned by the basis states $\{e^{2i\theta}, e^{i\theta}r^2, r, e^{-i\theta}, e^{-2i\theta}r^2, e^{-3i\theta}r, e^{-4i\theta}\}$.

A nice feature of this string is that it represents an example of a $\mathbb{Z}_3$ graded-contraction of su(2) into $e(2)$, as discussed in Ref. 4. The grading is generated by the SO(2) subgroup $e^{(2\pi/3)l_0}$. The carrier space V decomposes into $V = V_0 \oplus V_1 \oplus V_2$, which contain the basis states

$$V_0 = \{r, e^{-3i\theta}r\},\ V_1 = \{e^{i\theta}r^2, e^{-2i\theta}r^2\},\ V_2 = \{e^{2i\theta}, e^{-i\theta}, e^{-4i\theta}\}, \tag{46}$$

while the basis elements of Eq. (44) have grades 1, 2, and 0, respectively.

VI. TENSORING TWO RAISING OR TWO LOWERING STRINGS

In Sec. III A, we considered the tensoring of a raising and a lowering string representation. The resulting parallelogram, as well as all its subrepresentations and ''connected'' subquotients, were shown to be indecomposable.

The next simplest example of tensor product is the product of two raising or two lowering string representations. This yields a tensor product representation that is always decomposable into a sum of raising or lowering string representations. The decomposition is closely related to the decomposition of SU(2) tensor products.

In this section, we discuss only the tensor product of two raising string representations since the case of two lowering strings is handled in the exact same way.

A. Raising strings as contractions of su(2) irreps

Consider the su(2) algebra spanned by the operators $\{L_+, L_-, L_0\}$, with the usual nonzero commutation relations

$$[L_0, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = 2L_0. \tag{47}$$

If we now rescale the su(2) generators to

$$L_- \to \mathcal{L}_- = \epsilon L_- , \quad L_+ \to \mathcal{L}_+ = L_+ , \quad L_0 \to \mathcal{L}_0 = L_0 , \tag{48}$$

express the su(2) commutation relations in terms of the $\mathcal{L}$ generators, and take the limit as $\epsilon \to 0$, we see that

$$\begin{aligned} [\mathcal{L}_+ ,\mathcal{L}_-] &= \lim_{\epsilon\to 0} \epsilon [L_+ ,L_-] = \lim_{\epsilon\to 0} \epsilon 2 L_0 = \lim_{\epsilon\to 0} 2\epsilon \mathcal{L}_0 = 0, \\ [\mathcal{L}_0 ,\mathcal{L}_-] &= \lim_{\epsilon\to 0} \epsilon [L_0 ,L_-] = \lim_{\epsilon\to 0} -\epsilon L_- = -\mathcal{L}_- , \end{aligned} \tag{49}$$

with the commutator $[\mathcal{L}_+ ,\mathcal{L}_0]$ remaining unchanged from the su(2) commutator. We recover the algebra $e(2)$ by the identification $\mathcal{L}_\pm = p_\pm ,\mathcal{L}_0 = l_0$, but, because the generators L_- and L_+ have been treated asymmetrically, the resulting representation cannot be unitary.

This is an example of a contraction.[2] The more familiar example of rescaling, where L_- and L_+ are treated on the same footing and both multiplied by the scale factor ε, leads to unitary infinite dimensional representations. Coming back to Eq. (49), suppose that we are given a standard unitary representation Γ_j of the su(2) algebra, of dimension, say, $2j+1$. The effect of taking the $\epsilon \to 0$ limit of the asymmetric scaling is to leave matrix elements of $\mathcal{L}_+ = L_+$ and $\mathcal{L}_0 = L_0$ unchanged, while setting to 0 the matrix elements of $\mathcal{L}_- = \epsilon L_-$. The resulting representation, where $p_- = \mathcal{L}_- = 0$ everywhere but where $p_+ = \mathcal{L}_+$ acts by raising the weight of the su(2) states, is clearly equivalent to a $(2j+1)$-dimensional raising string representation γ_j. The equivalence relation just rescales the nonzero su(2) matrix elements of L_+ to the standard $e(2)$ matrix elements of p_+, which are 1. Thus, we can write, for a raising string representation

$$\gamma_j = \lim_{\epsilon\to 0} \Gamma_j . \tag{50}$$

The lowest weight of γ_j is $-j$. A general raising string representation $\gamma_{j;k}$ of dimension $2j+1$ with lowest weight $-j+k$ can be obtained by twisting γ_j by a character χ_k.

(Note that one can obtain a lowering string representation from an su(2) representation by scaling $L_+ \to \mathcal{L}_+ = \epsilon L_+$ and leaving the other two generators unchanged.)

The advantage of introducing limits in such a fashion is that one can then think of raising string representations as smooth deformations of su(2) representations.

B. Decomposing tensor products of two raising strings

First, recall that the tensor product $j_1 \otimes j_2$ of two su(2) representations j_1 and j_2 of dimensions $2j_1+1$ and $2j_2+1$, respectively, decomposes into a sum of su(2) representations of dimension $2j+1$, with possible values of j given by $|j_1-j_2|,|j_1-j_2|+1,|j_1-j_2|+2, \ldots ,j_1+j_2$, and where each value of j allowed by the inequality occurs exactly once.

Proposition 5: The tensor product of two raising string representations $\gamma_{j_1;r_1} \otimes \gamma_{j_2;r_2}$ *of dimensions* $2j_1+1$ *and* $2j_2+1$, *respectively, decomposes into a sum of raising string representations. The representations occuring in this tensor product have the dimensions* $2j+1$, *where* j *takes the possible values* $|j_1-j_2|,|j_1-j_2|+1,|j_1-j_2|+2, \ldots ,j_1+j_2$, *and where each value of* j *allowed by the inequality occurs exactly once.*

Proof: Write

$$\begin{aligned}\gamma_{j_1;r_1}\otimes\gamma_{j_2;r_2} &= \chi_{r_1}\otimes(\lim_{\epsilon\to 0}\Gamma_{j_1})\otimes\chi_{r_2}\otimes(\lim_{\epsilon\to 0}\Gamma_{j_2}) \\ &= \chi_{r_1+r_2}\otimes(\lim_{\epsilon\to 0}\Gamma_{j_1}\otimes\Gamma_{j_2}) \\ &= \chi_{r_1+r_2}\otimes\left(\lim_{\epsilon\to 0}\sum_j \Gamma_j\right), \quad |j_1-j_2|\leqslant j\leqslant j_1+j_2, \\ &= \chi_{r_1+r_2}\otimes\left(\sum_j \gamma_j\right), \\ &= \sum_j \gamma_{j;r_1+r_2}. \end{aligned} \tag{51}$$

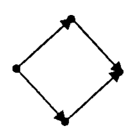
□

Note that, because the limiting process by which we transform the su(2) irrep into an $e(2)$ representation is smooth, i.e., because it is possible to define a sequence of su(2) representations parametrized by ϵ such that the limit when $\epsilon\to 0$ of this sequence corresponds to a raising string representation, it is possible to interchange the process of taking the limit with the process of taking the tensor product.

One may further remark that this limiting process cannot be used to analyze parallelogram representations, since those correspond to tensor products of a raising and a lowering string representations, i.e., a tensor product of ''different'' contractions.

VII. ACYCLIC REPRESENTATIONS AS SUMS OF STRINGS

The decomposition of the tensor product of two raising or two lowering representations is a special case of a more general theorem regarding ''acyclic'' representations, i.e., representations containing no ''cycles'' of the form



An algebraic characterization of such acyclic representations is that the operator p_+p_- is 0 everywhere. The main result of this section is that acyclic representations are always decomposable into sums of string representations.

Definition: A finite-dimensional representation V of $E(2)$ is said to be *acyclic* if $p_+p_-=0$ on V.

Clearly a string is acyclic, by Proposition 1. For that matter, a direct sum of strings must be acyclic. The goal of this section is to prove the converse: that any acyclic representation must be a direct sum of strings. We begin by establishing some machinery.

We need first a concept which is not restricted to acyclic representations, that of a ''chain.'' We define a ''chain'' to be a finite sequence of strictly increasing weight vectors $v_r, v_{r+1},\ldots,v_s$ in V, where each v_j has weight j, and such that for each $r\leqslant j<s$, either $p_+(v_j)=v_{j+1}$ or $p_-(v_{j+1})=v_j$.

Thus, for instance, in the 2×2 parallelogram representation of Eq. (17), which is not acyclic, there are infinitely many chains, each containing $v_1\otimes w_0$ and $v_0\otimes w_{-1}$ but each containing as middle element an otherwise arbitrary nonzero linear combination of $v_0\otimes w_0$ and $v_1\otimes w_{-1}$.

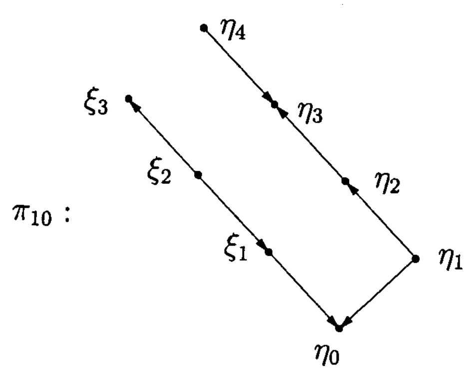
Specializing now to acyclic representations, we see that the condition $p_+p_-=0$ implies that it is not possible to have both $p_+(v_j)\neq 0$ *and* $p_-(v_{j+1})\neq 0$. A chain is ''maximal'' if it cannot be extended by including additional vectors (necessarily at the top or the bottom). The space spanned by the vectors in a maximal chain is a subrepresentation of V, necessarily a string.

Begin by observing that any weight vector can always be embedded in a maximal chain, i.e., it is always possible to find a maximal chain which contains a specified weight vector. Suppose that we have already defined N maximal chains,

$$\begin{aligned} & v^1_{r_1}, v^1_{r_1+1}, \ldots, v^1_{s_1}; \\ & \quad v^2_{r_2}, v^2_{r_2+1}, \ldots, v^2_{s_2}; \\ & \qquad \ldots \\ & \qquad \ldots \\ & \quad v^N_{r_N}, v^N_{r_N+1}, \ldots, v^N_{s_N}. \end{aligned} \tag{52}$$

Note that we have displayed the chains horizontally rather than vertically for convenience. The chains need not all start nor end at the same weight nor all have the same length. Suppose that they are ''fully independent,'' in the sense that the vectors $\{v_i^m\}$ form a linearly independent set. Note that if v is any vector in the ''span'' of the N chains, i.e., the space spanned by the vectors $\{v_i^m\}$, then $p_+(v)$ and $p_-(v)$ are also in the span of these chains, again because of the assumption that $p_+p_-=0$.

As an inductive step, we have to show how to find $N+1$ fully independent maximal chains in V. It will be fruitful to illustrate the various steps of the induction with the following example.



(53)

This representation is found by gluing the string representations containing $\{\eta_4, \eta_3, \eta_2, \eta_1, \eta_0\}$ and $\{\xi_3, \xi_2, \xi_1, \eta_0\}$ at the common node η_0. In the example, we assume that $N=1$ and that the first maximal chain in π_{10} contains $\{v_4^1=\eta_4, v_3^1=\eta_3, v_2^1=\eta_2, v_1^1=\eta_1, v_0^1=\eta_0\}$.

Let k be the highest weight for which the kth weight space is not spanned by the vectors $\{v_k^m : m=1,...,N\}$ that lie in the kth weight space of the first N chains. (In π_{10}, we have $k=3$.)

If there is a vector v in the kth weight space which satisfies $p_+(v)=0$ but which is not in the span of the $\{v_k^m\}$, we will choose it as the top vector in a new chain. Because the $(k+1)$th weight space is by construction contained in the span of the N chains, the action of p_- on vectors in the $(k+1)$th weight space must also take them into the span of these N chains, so that v cannot be in the image of p_-.

The other possibility is that every v in the kth weight space which is not in the span of the $\{v_k^m\}$ satisfies $p_+(v)\neq 0$. Choose such a v. But the intersection of the $(k+1)$th weight space with the image of p_+ is spanned by the vectors $\{p_+(v_k^m)\}$. It is therefore possible to find a (nonzero) vector of the form $v'=v-\Sigma c_m v_k^m$ satisfying $p_+(v')=0$; since v' is not in the span of the $\{v_k^m\}$, this is a contradiction.

In the example, this first step could yield $v_3^2=\xi_3$ as the top state of the second chain in π_{10}.

Having found the top vector for a new chain, $v_{s_{N+1}}^{N+1}$, with $s_{N+1}=k$, we can attempt to construct the rest of the chain inductively by extending it from below.

Suppose we have constructed a chain $v_{s_{N+1}}^{N+1},\ldots,v_{l+1}^{N+1},v_l^{N+1}$ so that it and the original N maximal chains form a fully independent set, in the sense defined above.

If $p_-(v_l^{N+1})=0$, and if v_l^{N+1} is not in the image of p_+, then we have constructed a maximal chain, with $r_{N+1}=l$, completing the inductive step. In all other cases, it necessary to perform an induction to extend the chain.

In general, the situation is that the induction leaves us with the possibility of constructing $N+1$ fully independent chains, none of which can be extended at the ''top,'' meaning that for each $m=1,\ldots,N+1$, $p_+(v_{s_m}^m)=0$ and $v_{s_m}^m$ is not in the image of p_-. At least one of the chains is nonmaximal; we will assume that the $(N+1)$th chain is nonmaximal, with lowest weight l, and that l is the maximum of the lowest weights of the nonmaximal chains. (Note that, although the $(N+1)$th chain has so far only been extended as low as weight l, it is, by assumption, not maximal and can therefore certainly be extended below l.) Let L be the number of nonmaximal chains with lowest weight l.

We will produce $N+1$ chains, none of which can be extended at the top, so that the number of nonmaximal chains with lowest weight l is strictly less than L, and so that none of the chains are nonmaximal with lowest weight greater than l. Induction on L will then allow us to construct $N+1$ chains so that any nonmaximal chains among them have lowest weight below l, and, continuing in this way, we can produce $N+1$ maximal chains.

If $p_-(v_l^{N+1})=0$ and $v_l^{N+1}=p_+(v'')$, for some weight vector v'' of weight $l-1$, then choose such a v'' and let $v_{l-1}^{N+1}=v''$. Note that the resulting chain and the chains labeled 1 to N still form a fully independent set, because v_{l-1}^{N+1} cannot possibly be in the span of the chains labeled 1 to N, since $p_+(v_{l-1}^{N+1})=v_l^{N+1}$ is not in those N chains. Extending the $(N+1)$th chain by adding v_{l-1}^{N+1} gives a chain which has lowest weight $l-1$, which reduces the number of nonmaximal chains with lowest weight l and completes the inductive step in this case.

If $p_-(v_l^{N+1})$ is nonzero but linearly independent of $\{v_{l-1}^1,v_{l-1}^2,\ldots,v_{l-1}^N\}$, then let $v_{l-1}^{N+1}=p_-(v_l^{N+1})$. The resulting $(N+1)$th chain and the original N maximal chains still form a fully independent set, and the $(N+1)$th chain has lowest weight $l-1$, which completes the inductive step in this case.

In the example of π_{10}, these two situations occur. Starting with the top vector ξ_3, we see that $p_-\xi_3=0$ but that $p_+\xi_2=\xi_3$, so that the chain containing ξ_3 can be extended to include ξ_2. Since $p_-\xi_2=\xi_1$, which is linearly independent of η_1, the vector of weight 1 which is in the first maximal chain, we can again extend the chain containing $\{\xi_3,\xi_2\}$ to include ξ_1. However, $p_-\xi_1=\eta_0$, which is already in the first maximal chain. We therefore have $l=1$. The second chain contains $\{\xi_3,\xi_2,\xi_1\}$, and it is not maximal.

It could be, (as in the case with ξ_1 in π_1), that $p_-(v_l^{N+1})$ is in the span of $\{v_{l-1}^1,v_{l-1}^2,\ldots,v_{l-1}^N\}$. Suppose

$$p_-(v_l^{N+1})=\sum_{m=1}^{N} a_m v_{l-1}^m=\sum_{m=1}^{N} a_m p_-(v_l^m). \tag{54}$$

(The last equality holds because a linear combination of vectors from the N chains that is in the image of p_- must be the image under p_- of a linear combination of vectors from the N chains, because of our assumption that $p_+p_-=0$.) Then let $a_{N+1}=-1$ and consider the vector

$$u_l=-v_l^{N+1}+\sum_{m=1}^{N} a_m v_l^m=\sum_{m=1}^{N+1} a_m v_l^m. \tag{55}$$

It satisfies $p_-(u_l)=0$. Note that, although v_l^{N+1} is not in the image of p_+, this is not necessarily true of u_l. Reordering if necessary, we can assume that $u_l = \Sigma_{m=1}^{N_0} a_m v_l^m$, with $1 \leqslant N_0 \leqslant N+1$ and $a_m \neq 0$, $p_-(v_l^m) \neq 0$, for $1 \leqslant m \leqslant N_0$ (i.e., we reorder so that the first m coefficients in u_l are nonzero.)

In the example of π_{10}, Eq. (55) produces the vector $u_1 = -\xi_1 + \eta_1$, which is not in the first (maximal) nor in the second (nonmaximal) chain. There is no reordering necessary as u_1 is a linear combination of the two vectors in the first and second chains, i.e., $a_1, a_2 \neq 0$, so that $N_0 = 2 = N$.

The situation is as follows. We have $N+1$ ''bottom'' vectors; u_l and the bottom vectors of the original N maximal chains. We have $N+1$ top vectors and the subspace spanned by the vectors in the $N+1$ chains, but no chains containing u_l. It is then a matter of reorganizing the states in the subspace so as to replace one of the existing $N+1$ chains with one that contains u_l. First we construct a chain containing u_l. There is a unique integer $\mu_1 \geqslant l$ so that $(p_+)^{\mu_1 - l}(u_l) \neq 0$ and $(p_+)^{\mu_1 - l+1}(u_l) = 0$; let $u_i = (p_+)^{i-l}(u_l)$, for $l \leqslant i \leqslant \mu_1$.

In the example of π_{10}, this integer is $\mu_1 = 3$ since it is possible to act $\mu_1 - l = 3 - 1 = 2$ times on $u_1 = -\xi_1 + \eta_1$ before getting $(p_+)^3(-\xi_1 + \eta_1) = 0$. Thus we have $u_1 = -\xi_1 + \eta_1, u_2 = \eta_2, u_3 = \eta_3$.

Reordering the chains if necessary, we can assume that $u_{\mu_1} = \Sigma_{m=1}^{N_1} a_m v_{\mu_1}^m$, with $a_m \neq 0$ whenever $1 \leqslant m \leqslant N_1$, for some $N_1 \leqslant N_0$. (In π_1, we have $N_1 = 1$ since u_3 can be expressed as a linear combination of a single vector.)

If u_{μ_1} is not in the image of p_-, then the chain $u_l, \ldots, u_{\mu_1}$ cannot be extended further at the top; it is a ''raising chain.'' If u_{μ_1} is in the image of p_-, then it is possible to add a ''lowering chain'' above it. Indeed, in this case there is some $\nu_1 \geqslant 1$ so that

$$(p_-)^{i-\mu_1}\left(\sum_{m=1}^{N_1} a_m v_i^m\right) = u_{\mu_1}, \tag{56}$$

for all i with $\mu_1 \leqslant i \leqslant \nu_1$, and so that $\Sigma_{m=1}^{N_1} a_m v_{\nu_1}^m$ is not in the image of p_-. For all i with $\mu_1 \leqslant i \leqslant \nu_1$, let $u_i = \Sigma_{m=1}^{N_1} a_m v_i^m$. These vectors form a lowering chain.

In the example of π_{10}, we have $\nu_1 = 4$ since $p_- \eta_4 = \eta_3 = u_3$. The chain now contains $u_1 = -\xi_1 + \eta_1$, $u_2 = \eta_2$, $u_3 = \eta_3$, $u_4 = \eta_4$.

We continue in the same way, finding positive integers $N_0 \geqslant N_1 \geqslant N_2 \geqslant \cdots \geqslant N_t$ and integers $l = \mu_0 = \nu_0 \leqslant \mu_1 < \nu_1 < \mu_2 < \nu_2 < \cdots < \mu_t \leqslant \nu_t$; making a suitable rearrangement of the $N+1$ chains; and for each $1 \leqslant j \leqslant t$, and for each i with $\nu_{j-1} \leqslant i \leqslant \mu_j$, letting

$$u_i = (p_+)^{i-\nu_{j-1}}(u_{\nu_{j-1}}), \tag{57}$$

and for each i with $\mu_j \leqslant i \leqslant \nu_j$, letting

$$u_i = \sum_{m=1}^{N_j} a_m v_i^m, \tag{58}$$

so that for each i with $\mu_j < i \leqslant \nu_j$,

$$p_-(u_i) = u_{i-1}. \tag{59}$$

In this way we have constructed a lowering chain between each μ_j and ν_j. Because

$$p_+(u_i) = u_{i+1}, \tag{60}$$

for each i with $\nu_{j-1} \leqslant i < \mu_j$, $j = 1, \ldots, t$, there is a raising chain between each ν_{j-1} and μ_j.

In the example of π_{10}, we have $t = 1$, as there is only one raising and one lowering chain to be glued to the vector $u_1 = -\xi_1 + \eta_1$. Our process therefore stops at $\nu_1 = 4$.

We can continue the construction until we reach a point where $p_+(u_{\nu_t})=0$, so the chain cannot continue up to higher weights. Note that for

$$u_{\nu_t}=\sum_{m=1}^{N_t} a_m v_{\nu_t}^m \tag{61}$$

to be in the kernel of p_+, it must be true that $p_+(v_{\nu_t}^m)=0$, for each $1\leqslant m\leqslant N_t$. And if u_{ν_t} is not in the image of p_-, then at least one of the vectors $v_{\nu_t}^m$, for $1\leqslant m\leqslant N_t$, must not be in the image of p_-. Renumbering yet again, if necessary, we can assume that this is true for $m=1$, which means that ν_t is the highest weight of the chain $\{v_i^1\}$.

In the example of π_{10}, we have $u_4=\eta_4$ as the top vector; η_4 is in the kernel of p_+ and not in the image of p_-. Thus, with the renumbering, the chain $\{u_4,u_3,u_2,u_1\}$ becomes the first chain.

The vectors $u_l,u_{l+1},\ldots,u_{\nu_t}$ form a chain. They all lie in the span of the linearly independent vectors making up the $N+1$ chains $\{v_i^m\}$, for $m=1,...,N+1$. Since $p_-(u_l)=0$, either u_l is not in the image of p_+, in which case the chain $\{u_l,u_{l+1},\ldots,u_{\nu_t}\}$ is maximal, or $u_l=p_+(u_{l-1})$, for some weight vector u_{l-1} of weight $l-1$. It can be added to produce a longer chain $\{u_{l-1},u_l,u_{l+1},\ldots,u_{\nu_t}\}$. Observe that u_{l-1} cannot be in the space spanned by the chains $\{v_i^1\},\ldots,\{v_i^{N+1}\}$. We have constructed a chain $\{u_i\}$; in one case it is maximal, and in the other it has lowest weight $l-1$.

In the example of π_{10}, the chain is maximal since $u_1=-\xi_1+\eta_1$ is not in the image of p_+.

Since every one of the vectors u_i, for $i\geqslant l$, contains a nonzero component in what is now labeled as the first chain $\{v_i^1\}$, we can replace the chain $\{v_i^1\}$ with the chain $\{u_i\}$ and the resulting $N+1$ chains will still be fully independent.

At this point there are different possibilities. One is that the chain labeled $\{v_i^1\}$ that was removed was the original nonmaximal chain $\{v_i^{N+1}\}$, and it has just been replaced by the chain $\{u_i\}$. This does not apply to π_{10}; the original second chain is not identical to the newly constructed maximal chain.

Otherwise, the original nonmaximal chain $\{v_i^{N+1}\}$ with lowest weight l is still present. In this case we can change its label back to $v_l^{N+1},\ldots,v_{s_{N+1}}^{N+1}$. But now $p_-(v_l^{N+1})$ is not in the span of the vectors in the other N chains, since $p_-(v_l^1)$ is no longer present in the other N chains. This means that the $(N+1)$th chain can be extended to include $p_-(v_l^{N+1})$ as its ''bottom'' vector.

In the example of π_{10}, this is what happens. The original second chain contained $\{\xi_3,\xi_2,\xi_1\}$. Since $p_-\xi_1=\eta_0$, which is no longer in the span of the newly constructed first maximal chain, we can extend this second chain to include η_0.

In either case, the number of nonmaximal chains with lowest weight l has been reduced.

In the example of π_{10}, we now restart the induction with $\{\eta_4,\eta_3,\eta_2,-\xi_1+\eta_1\}$ as the first maximal chain and $\{\xi_3,\xi_2,\xi_1,\eta_0\}$ as the second chain, we find that now $k=3$, $N=1$ but $l=0$. However, $p_-\eta_0=0$ and η_0 is not in the image of p_+. Thus, the second chain is maximal as it is. This concludes the inductions: we have found the decomposition of π_{10} as the sum of two strings.

Theorem 1: A finite-dimensional representation V of $E(2)$ is acyclic if and only if V is a direct sum of indecomposable representations, in each of which the weight spaces are all of dimension 1.

Proof: $(\Leftarrow)$ Trivial.

$(\Rightarrow)$ The above argument shows that, given N fully independent maximal chains that do not span all of V, it is possible to construct $N+1$ fully independent maximal chains.

In attempting this construction, we may reach a situation where there are $N+1$ fully independent chains, but not all of them are maximal. We let l be the maximum of the lowest weights of the nonmaximal strings. The inductive step described above reduces the number of nonmaximal strings with lowest weight l. Repeated application of this procedure will eventually reduce l, the maximum of the lowest weights of the nonmaximal strings.

Continuing with an induction on l, we can eventually eliminate all the nonmaximal chains, producing $N+1$ fully independent maximal chains, as required.

Then, since the number of fully independent chains is certainly bounded by the dimension of the whole space V, we will eventually be able to construct enough chains that they span the whole space. □

VIII. DISCUSSION AND CONCLUSION

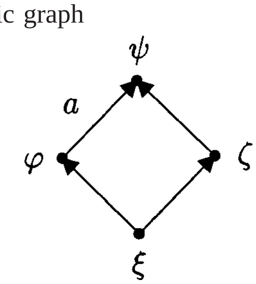
In this paper, we have described numerous finite dimensional indecomposable representations of $E(2)$ by means of a method which encapsulates in graphical form all the necessary information to explicitly construct, up to a character, a representation.

The basic type representation is the string, in which all the weight subspaces are of dimension one. Using lemma 1 and proposition 1, we can associate to a string representation a graph, from which it is easy to determine if the representation is decomposable or not. In an indecomposable string representation, the ''strength'' of the $e(2)$ matrix element connecting two states is irrelevant; all indecomposable strings representations are equivalent to representations for which this matrix element is 1.

We have been successful in showing the indecomposability of another very important class of representations, the parallelograms and all their subrepresentations and quotients. Parallelograms and their subrepresentations may contain nontrivial weight multiplicities, an unusual feature for representations of $E(2)$. We have also shown how acyclic representations can be decomposed into sums of string representations.

The problem of decomposing a general graph containing nontrivial weight multiplicities arising either per se or as from the tensor product of two general string representations is difficult.

Consider for instance the acyclic graph



(62)

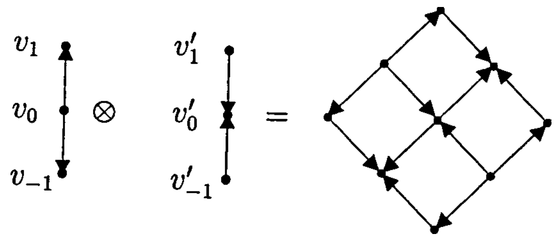
with basis states $\{\xi,\varphi,\zeta,\psi\}$, in which $p_+\varphi=a\psi$, with all other nonzero matrix elements being 1. When $a=-1$, the representation decomposes into a sum containing two (inequivalent) two-dimensional subrepresentations.

When $a\neq-1$, however, this can be decomposed into a sum of a three-dimensional and a one-dimensional string. The special case where $a=1$ corresponds to a tensor product of the two-dimensional raising string with itself.

The decomposability of some graphs can be understood in terms of representations of S_n, the permutation group of n objects. Unfortunately, arguments based on the permutation group are of limited use because (i) the S_n-invariant subspaces may themselves decompose further (for instance, in the tensor product of a three-dimensional raising string with itself, the six-dimensional subspace that carries the fully symmetric representation of S_2 can be divided into a five-dimensional and a one-dimensional indecomposable raising string), (ii) experience has shown that the problem of deciding if a given graph (string or otherwise) can be obtained by tensoring n copies of a given string is nontrivial.

There is, however, one case which we would like to mention. Consider the tensor product of an indecomposable string V, with a basis of weight vectors $v_l,\ldots,v_m$ with $l\leq m$, with another indecomposable string V' With weight vectors $v'_{-m},\ldots,v'_{-l}$ such that $p_+v'_{-k-1}=v'_{-k}$ if and only if $p_+v_k=v_{k+1}$ and $p_-v'_{-k}=v_{-k-1}$ if and only if $p_-v_{k+1}=v_k$. The tensor product $V\otimes V'$ is decomposable into two parts, one of which is the one-dimensional indecomposable representation with basis vector $v=\Sigma_{k=l}{}^m(-1)^k v_k\otimes v'_{-k}$, because V' occurs when we tensor together $(l-m-1)$ copies of V.

The simplest example of this is found in (63).



(63)

In this example, $V \otimes V'$ decomposes into an eight-dimensional representation isomorphic to (30) and a one-dimensional subspace $v = v_0 \otimes v'_0 - v_1 \otimes v'_{-1} + v_2 \otimes v'_{-2}$. Note that, obviously, the weights of V' could be shifted up or down and the tensor product would still be decomposable. What is important is the relative positions of the arrows, not the actual weights.

This family of decomposable tensor products can be related to the symmetric group as follows. It can be shown that, if the dimension of V is d, then the $(d-1)$-fold tensor product of V with itself contains, up to a character, V' in the S_{d-1}-invariant subspace labeled by a Young tableau containing a single column of $d-1$ boxes.

In the example of (63), V is of dimension $d=3$, and a basis for the three-dimensional representation of $V \otimes V$ which carries the S_2 representation labeled by 1 column of 2 boxes is given by

$$v'_{-2} = \chi_{-3} \otimes \det \begin{vmatrix} w_1 & x_1 \\ w_0 & x_0 \end{vmatrix}, \quad v'_{-1} = \chi_{-3} \otimes \det \begin{vmatrix} w_2 & x_2 \\ w_0 & x_0 \end{vmatrix}, \quad v'_0 = \chi_{-3} \otimes \det \begin{vmatrix} w_2 & x_2 \\ w_1 & x_1 \end{vmatrix}, \tag{64}$$

where $w_i, x_j, i,j = 0,1,2$ are basis states for the first and second copy of V in the tensor product $V \otimes V$, respectively. The representation V' can be reconstructed if we observe that the nonzero matrix elements of $p_\pm$ are given by

$$\begin{aligned} p_+ v'_{-2} &= \chi_{-3} \otimes \det \begin{vmatrix} p_+ w_1 & p_+ x_1 \\ w_0 & x_0 \end{vmatrix} + \det \begin{vmatrix} w_1 & x_1 \\ p_+ w_0 & p_+ x_0 \end{vmatrix} = \chi_{-3} \otimes \det \begin{vmatrix} w_2 & x_2 \\ w_0 & x_0 \end{vmatrix} = v'_{-1}, \\ p_- v'_0 &= \chi_{-3} \otimes \det \begin{vmatrix} p_- w_2 & p_- x_2 \\ w_1 & x_1 \end{vmatrix} + \chi_{-3} \otimes \det \begin{vmatrix} w_2 & x_2 \\ p_- w_1 & p_- x_1 \end{vmatrix} = \chi_{-3} \otimes \det \begin{vmatrix} w_2 & x_2 \\ w_0 & x_0 \end{vmatrix} = v'_0. \end{aligned} \tag{65}$$

From this, it can be seen how the decomposition of $V \otimes V'$ is related to the action of symmetric group on $(V)^d$, and why the scalar v is alternating in nature.

Finally, even if all the examples of decomposable tensor products discussed in this paper can ultimately be related to the symmetric group, we believe that there very likely exist decomposable graphs with nontrivial weight multiplicities which are unrelated to S_n. We have, unfortunately, been unable to isolate a provable conjecture on this matter. The low-dimensional examples of this section are sufficiently complex to illustrate the difficulty of the general problem.

In a subsequent publication, we will investigate the role of gluings of the type found in (30) in the construction of finite dimensional representations.

There is no doubt that results similar to Lemma 1 and Proposition 1 can be extended to other groups,[5] in particular within the context of graded contractions.[4] It is also reasonable to expect that the method can be generalized to the construction of finite dimensional representations of other semidirect product groups. In that regard, one should observe that the operator $p_+ p_-$ is in fact, the $e(2)$ Casimir operator, so that one way of generalizing the concept of string representations to other groups is to require that the appropriate Casimir be 0. It remains to see how other concepts, such as parallelograms, can be generalized to other examples.

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