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Clebsch–Gordan coefficients in the asymptotic limit

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We investigate the graded structure of the tensor product space $V^{\lambda_2}\otimes V^{\lambda_1}$, which arises in the coupling of two irreducible representations λ_1 and λ_2 of a semisimple Lie algebra $\mathscr{G}$, in the limit in which one of the highest weights becomes asymptotically large. The construction of asymptotic coupling coefficients is considered.
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I. INTRODUCTION

It is known that every unitary irreducible representation (unirrep) of a compact semisimple Lie group is characterized by a highest weight $\lambda=(\lambda^1,\lambda^2,\lambda^3,\ldots)$. Often, the unirreps of physical interest of noncompact semisimple Lie groups are likewise characterized by extremal (highest or lowest) weights.

In this paper, we consider the tensor product $V^{\lambda_2}\otimes V^{\lambda_1}$ of modules for two unirreps of highest weights λ_1 and λ_2 of a semisimple Lie algebra $\mathscr{G}$ in the limit in which λ_1 becomes asymptotically large. (A weight λ is said to be asymptotically large if one or more of its components is asymptotically large.) We show that $V^{\lambda_2}\otimes V^{\lambda_1}$ is expressible as a sum of graded subspaces which carry representations of a subalgebra $\mathscr{H}\subset\mathscr{G}$, where $\mathscr{H}$ is uniquely determined by specification of which components of λ_1 are asymptotically large. We also show that the graded subspaces of $V^{\lambda_2}\otimes V^{\lambda_1}$ are simple products of graded subspaces of V^{λ_1} and V^{λ_2}. This factorization implies simple expressions for the Clebsch–Gordan coupling coefficients of $\mathscr{G}$ in the asymptotic limit.

Our motivation for studying asymptotic limits arises because the limiting process is generally associated with group contractions. Thus if a noncompact group of interest is the asymptotic limit of a compact group, one can determine, for instance, Clebsch–Gordan coefficients for the noncompact group by taking the asymptotic limit of Clebsch–Gordan coefficients for the corresponding compact group.

An interesting example of this procedure can be found in the work of W. T. Sharp.[1] Sharp was able to obtain formulas for various products of Bessel functions by considering them as contractions of Legendre functions (see Ref. 2, Sec. 5.71). The coupling coefficients for Bessel functions then emerged naturally as asymptotic limits of ordinary SO(3) coupling coefficients.

Our analysis is appropriate for the less ambitious task of computing some coupling coefficients involving one representation of a group and one representation of its contraction. The need for such coupling coefficients arises in physics when quantum numbers become large. For example, it is well known, in nuclear physics, that the representations of su(3) approach those of the noncompact rigid rotor algebra for large values of the su(3) highest weight.[3] The asymptotic properties of tensor product spaces are also needed for other purposes. For example, they were recently considered by Rowe and Repka[4] in the construction of the coherent shift tensors of Flath and Towber.[5]

The nontrivial unitary representations of a noncompact group are usually infinite dimensional, even when their extremal weights are finite. To avoid confusion, we use the adjective ''asymptotic'' rather than ''infinite'' to describe properties which approach infinity as one or many components of an extremal weight become large. Thus, in the asymptotic limit, the number of basis states in a unirrep of a compact group becomes asymptotically large. Similarly, some chains of weights, which would be finite for finite highest weights, become asymptotic chains in the limit.

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The paper is organized as follows. Sections II and III A, respectively, deal with the basics of grade space decompositions and provide definitions and examples of asymptotic representations. We give, in Sec. III B, results for highest weight states of semisimple compact Lie algebras, starting with su(2) and followed by the general case of any semisimple compact rank N Lie algebras (with N finite) in III C. In Sec. III D, the theory of the previous section is applied to the Lie algebra su(3). The main result on asymptotic Clebsch–Gordan coefficients is presented in Sec. III E.

Although close in spirit to the corresponding compact cases, the results for the noncompact cases only apply to some of the representations appearing in the decomposition of the tensor product of two unirreps of the positive (or negative) discrete series. The results for noncompact cases are specialized to sp(m,R) and presented in Sec. IV, starting with sp(1,R) and followed by sp(m,R).

Section V deals with the coupling of a finite dimensional, nonunitary representation to an asymptotic unirrep.

II. GRADE-SPACE DECOMPOSITIONS

In calculating Clebsch–Gordan coefficients for a semisimple Lie algebra $\mathscr{G}$, in a basis which reduces a subalgebra $\mathscr{H}\subset\mathscr{G}$, one encounters a need to decompose tensor products of $\mathscr{G}$-invariant spaces into irreducible $\mathscr{G}$- and $\mathscr{H}$-invariant subspaces. Thus, if V^{λ} is a module for a unirrep of $\mathscr{G}$ with highest weight λ, we need to express $W=V^{\lambda_2}\otimes V^{\lambda_1}$ as a sum

$$W=\sum_{\rho\lambda} W^{\rho\lambda}=\sum_{\rho\lambda\,\gamma\omega} W^{\rho\lambda}_{\gamma\omega}, \tag{1}$$

where ρ indexes the multiplicity of $\mathscr{G}$-invariant subspaces $\{W^{\rho\lambda}\}$ of highest weight λ in W and γ indexes the multiplicity of $\mathscr{H}$-invariant subspaces of highest weight ω in $W^{\rho\lambda}$.

Such a decomposition is simplest when $\mathscr{H}$ is a reductive subalgebra of $\mathscr{G}$ containing the Cartan subalgebra $\mathscr{T}$ and a subset of simple root vectors for $\mathscr{G}$. The Lie algebra $\mathscr{G}$ can then be expressed as a sum of graded subspaces

$$\mathscr{G}=\sum_{k} \mathscr{G}_k, \tag{2}$$

in which $\mathscr{H}=\mathscr{G}_0$ is the zero-grade subspace and $\mathscr{G}_1$ is the $\mathscr{H}$-invariant subspace of $\mathscr{G}$ containing all simple root vectors not in $\mathscr{H}$. It then follows that

$$[\mathscr{G}_k,\mathscr{G}_l]\subseteq\mathscr{G}_{k+l}. \tag{3}$$

This grading induces a corresponding grading $V=\Sigma_l V_l$ of any module for a representation of $\mathscr{G}$ with the property

$$\mathscr{G}_k : V_l \rightarrow V_{k+l}. \tag{4}$$

It will be convenient to introduce the notation $[\omega]$ to denote the grade of a weight ω. Of particular importance is the highest grade subspace $V^{\lambda}_{[\lambda]}$, since it is the subspace of V^{λ} containing the highest weight state. This is because the highest grade subspace $V^{\lambda}_{[\lambda]}$ of an irreducible $\mathscr{G}$-module is an irreducible $\mathscr{H}$-module having the same highest weight.

We denote by $W^k\subset W$ the sum of $\mathscr{G}$-invariant subspaces of $W=V^{\lambda_2}\otimes V^{\lambda_1}$ whose highest weight states are of grade k. Let $W^k=\Sigma_l\, W^k_l$ be the grade-space decomposition of W^k so that

$$W=\sum_k W^k=\sum_{kl} W^k_l. \tag{5}$$

W^k can in turn be expressed as

$$W^k=\sum_{\rho\lambda} W^{\rho\lambda}\ , \qquad [\lambda]=k\ , \tag{6}$$

where $W^{\rho\lambda}$ is as in Eq. (1) but the sum is restricted to highest weights which satisfy $[\lambda]=k$, and

$$W_l^k=\sum_{\rho\lambda} W_l^{\rho\lambda}=\sum_{\rho\lambda}\sum_{\gamma\omega} W_{\gamma\omega}^{\rho\lambda}, \qquad [\lambda]=k\ ,[\omega]=l\ , \tag{7}$$

where $W_{\gamma\omega}^{\rho\lambda}$ is also as in Eq. (1) but the sum is now restricted by the condition $[\omega]=l$.

The grade-space decomposition of W is the first step towards obtaining the full decomposition of Eq. (1). Moreover, if one already knows the Clebsch–Gordan coefficients for the subalgebra $\mathscr{H}$, it is the most important step in the evaluation of asymptotic Clebsch–Gordan coefficients for, as we show in the following, the subspaces W_l^k acquire the very simple expressions

$$W_l^k\to V_{k-[\lambda_1]}^{\lambda_2}\otimes V_{[\lambda_1]-k+l}^{\lambda_1}\ , \tag{8}$$

in suitable asymptotic limits. A consequence of Eq. (8) is that asymptotic Clebsch–Gordan coefficients for $\mathscr{G}$ can be reduced to Clebsch–Gordan coefficients for $\mathscr{H}$.

III. COMPACT, SEMISIMPLE LIE ALGEBRAS

For a compact, semisimple Lie algebra, the highest weights are dominant integral (i.e., non-negative and integer valued). [For su(2) it will be convenient to follow the physics convention of labeling irreps by the angular momentum j which can take integer or half-odd integer values. Thus the su(2) angular momentum is half the standard, integer-valued, weight.]

A. Asymptotic representations

We follow the convention of saying that a unirrep is asymptotic if its highest weight is asymptotically large, and simply write $\lambda\to\infty$ if one or more components of λ are asymptotically large. We will denote by $\Delta_+=\{\alpha_1,\alpha_2,\ldots,\alpha_N,\alpha_{N+1},\ldots,\alpha_q\}$ an ordered set of positive roots of $\mathscr{G}$ such that the first N are simple, i.e., $\{\alpha_1,\ldots,\alpha_N\}$ is a basis for Δ_+, and denote by $\{e_1,e_2,\ldots,e_N,e_{N+1},\ldots,e_q\}$ the corresponding root vectors.

If α_k is a root and ω is a weight, then we shall refer to the set of weights of the type $\omega+n\alpha_k$ (n is an integer) that occur within the weight space of a given irrep as an α_k-string. An α_k-string can be labeled by its highest weight. Then, if an α_k-string has highest weight $2j$, its weights are those of an su(2) irrep of angular momentum j. An α_k-string will be said to be asymptotic if and only if $j\to\infty$ as $\lambda\to\infty$.

Let Δ_+ be expressed as the sum of two subsets $\Delta_+=\mathscr{L}_++\mathscr{F}_+$, defined as follows. A positive root $\alpha_k\in\Delta_+$ is an element of $\mathscr{L}_+$ if the α_k-string of weights through the highest weight state $|\lambda\rangle$ of V^λ is an asymptotic string, whereas it is an element of $\mathscr{F}_+$ if the α_k-string is not asymptotic. Corresponding sets, $\Delta_-,\mathscr{L}_-$, and $\mathscr{F}_-$, contain the negatives of the roots in $\Delta_+,\mathscr{L}_+$, and $\mathscr{F}_+$, respectively.

If all the components of λ are finite, then $\mathscr{L}_+=\varnothing$ and $\mathscr{F}_+=\Delta_+$. Figure 1(b) is an example of a representation of a rank 2 compact algebra where λ^2, the second component of the highest weight λ, is asymptotically large. In Fig. 1(a), $\alpha_1\in\mathscr{F}_+$, while α_2 and all the other positive roots belong to $\mathscr{L}_+$. In a situation where all the components of the weight are asymptotically large, $\mathscr{F}_+=\varnothing$ and $\mathscr{L}_+=\Delta_+$.

The separation of Δ_+ into subsets $\Delta_+=\mathscr{L}_++\mathscr{F}_+$ leads to a grading of $\mathscr{G}$ of the type given in Eq. (2), in which $\mathscr{H}=\mathscr{G}_0$ contains a Cartan subalgebra $\mathscr{T}$ for $\mathscr{G}$ as well as all root vectors whose roots are in $\mathscr{F}_+$ and $\mathscr{F}_-$. The complementary nilpotent subalgebras

$$\mathscr{N}_+=\sum_{k\geqslant 1}\mathscr{G}_k \quad \text{and} \quad \mathscr{N}_-=\sum_{k\geqslant 1}\mathscr{G}_{-k} \tag{9}$$

are spanned by root vectors whose roots are in $\mathscr{L}_+$ and $\mathscr{L}_-$, respectively.

For example, for an asymptotic representation of $\mathscr{G}=$ su(2), $\mathscr{N}_\pm=\mathscr{G}_\pm$ are the one-dimensional algebras spanned by $J_\pm$, respectively, and $\mathscr{H}$ is the Cartan subalgebra of su(2) spanned by J_0. In the example of Fig. 1(a), $\mathscr{H}$ is spanned by $\mathscr{T}$, e_1, and f_1 while $\mathscr{N}_\pm$ contain the remaining positive and negative root vectors, respectively.

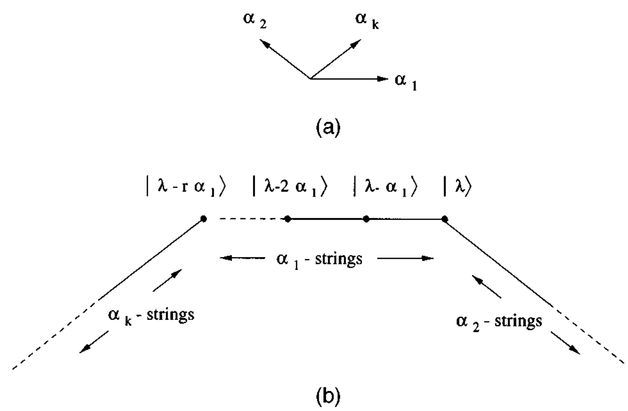

FIG. 1. Example of the weight diagram of a unirrep λ, with $\lambda^2\to\infty$. (a) Some roots of a rank 2 algebra; (b) An asymptotically large irrep of this algebra.

Throughout this paper, we will assume that the Lie algebra $\mathscr{G}=\Sigma_k\mathscr{G}_k$ is graded such that $\mathscr{G}_1$ is spanned by simple root vectors $\{e_i\}$ for which the corresponding α_i-chains are asymptotically long in the limit where $\lambda_1\to\infty$. Thus $\mathscr{L}_+$ will always contain positive roots of $\mathscr{G}$ that are asymptotic for the irrep λ_1.

B. Highest grade subspaces of su(2)

The generators of su(2) satisfy the well-known commutation relations

$$[J_0,J_\pm]=\pm J_\pm\,,\quad [J_+,J_-]=2J_0, \tag{10}$$

with matrix elements given by

$$J_0|jm\rangle=m|jm\rangle\ ,\quad m=j,j-1,\dots,-j\ ,$$
$$J_\pm|jm\rangle=\sqrt{(j\mp m)(j\pm m+1)}\ |j,m+1\rangle\ . \tag{11}$$

Lemma 1: The highest weight state $|jj\rangle$ of an su(2) unirrep j appearing in the decomposition of the tensor product $j_2\otimes j_1$ is given, when j_2 is finite and $j_1\to\infty$, by

$$\lim_{j,j_1\to\infty}|jj\rangle=|j_2\ \Delta\rangle\ |j_1j_1\rangle\ ,\qquad \Delta=j-j_1. \tag{12}$$

Proof: Consider the state

$$|jj\rangle=\sum_{m_1m_2}\ (j_1\ m_1;j_2\ m_2|j\ j)|j_2\ m_2\rangle|j_1\ m_1\rangle\ . \tag{13}$$

Since $m_1+m_2=j$, we must have $m_1=j_1,m_2=j_2$ when $j=j_1+j_2$. There is then only one term in the sum and the appropriate coupling coefficient is 1. Thus Eq. (12) holds trivially when $\Delta=j_2$.

When $\Delta=j-j_1\neq j_2$, we have, in general, more than one term in the sum (13). The equation

$$0=[\langle j_2m_2|\langle j_1m_1|]L_+|jj\rangle=(j_1\ m_1-1;j_2\ m_2|j\ j)\sqrt{j_1(j_1+1)-m_1(m_1-1)}$$
$$+\ (j_1\ m_1;j_2\ m_2-1|j\ j)\sqrt{j_2(j_2+1)-m_2(m_2-1)} \tag{14}$$

implies that the Clebsch–Gordan coefficients satisfy the equation

$$\frac{(j_1\ m_1-1;j_2\ m_2|j\ j)}{(j_1\ m_1;j_2\ m_2-1|j\ j)}=-\sqrt{\frac{j_2(j_2+1)-m_2(m_2-1)}{j_1(j_1+1)-m_1(m_1-1)}}\,, \tag{15}$$

and hence that

$$\frac{(j_1\ j_1-n;j_2\ m_2|j\ j)}{(j_1\ j_1-n+1;j_2\ m_2-1|j\ j)}=-\sqrt{\frac{j_2(j_2+1)-m_2(m_2-1)}{2nj_1-n(n-1)}}\ , \tag{16}$$

$$\lim_{j_1\to\infty}\frac{(j_1\ j_1-n;j_2\ m_2|j\ j)}{(j_1\ j_1-n+1;j_2\ m_2-1|j\ j)}=-\sqrt{\frac{j_2(j_2+1)-m_2(m_2-1)}{2nj_1}}\ , \tag{17}$$

where n is in the range $0\leqslant n\leqslant 2j_2$. It follows that, for $\Delta<j_2$,

$$\lim_{j_1\to\infty}\frac{(j_1\ j_1-n;j_2\ m_2|j\ j)}{(j_1\ j_1;j_2\ \Delta|j\ j)}<\frac{(-1)^n}{\sqrt{n!}}(2j_1)^{-n/2}[j_2(j_2+1)-\Delta(\Delta+1)]^{n/2}\ . \tag{18}$$

The coupling coefficients are normalized such that

$$\sum_{m_1m_2}|(j_1\ m_1;j_2\ m_2|j\ j)|^2=1\ . \tag{19}$$

This sum contains finitely many terms because m_2 ranges over finitely many values. Thus, using Eq. (18), the only nonvanishing contribution to the above sum, in the limit $j_1\to\infty$, is from the term $(j_1\ j_1;j_2\ \Delta|j\ j)$. Furthermore, we must have $|(j_1\ j_1;j_2\ \Delta|j\ j)|=1$. Hence the highest weight state approaches the simple form of Eq. (12), i.e.,

$$|j\ j\rangle \overset{j_1,j\to\infty}{\longrightarrow} |j_2\ \Delta\rangle|j_1\ j_1\rangle\ , \tag{20}$$

and the proof is complete.

The converse result, that every product state of the type $|j_2\Delta\rangle|j_1j_1\rangle$ becomes a highest weight state $|jj\rangle$ with $j=j_1+\Delta$ as $j_1\to\infty$, is easily shown starting from the general expression

$$|j_2\Delta\rangle|j_1j_1\rangle=\sum_j |j,j_1+\Delta\rangle\ (j_1\ j_1;j_2\ \Delta|j\ j_1+\Delta)\ . \tag{21}$$

All of these asymptotic properties can be verified directly from the exact expression[6]

$$(j_1\ m_1;j_2\ m_2|J\ J)=(-1)^{j_1-m_1}\times\sqrt{\frac{(2J+1)!(j_1+j_2-J)!(j_1+m_1)!(j_2+m_2)!}{(j_1+j_2+J+1)!(j_1-j_2+J)!(-j_1+j_2+J)!(j_1-m_1)!(j_2-m_2)!}}\ . \tag{22}$$

The above lemma can be seen as a special case of Eq. (8) by observing first that, for su(2), $\mathscr{H}$ is the u(1) ⊂ su(2) subalgebra spanned by J_0, so that the graded subspaces of W contain states of a given m value. Since the su(2) coupling is multiplicity free, the subspaces W^k of Eq. (5) contain a single su(2) unirrep and the subspace W_j^j contains only the highest weight state of the unirrep j. Thus, in the notation of Sec. II, Eq. (12) is equivalent to the statement

$$\lim_{j,j_1\to\infty} W_j^j=V_{j-j_1}^{j_2}\otimes V_{j_1}^{j_1}\ . \tag{23}$$

C. Highest grade subspaces of rank *N* Lie algebras

Lemma 2: Let V^λ denote the carrier space for a unirrep of $\mathscr{G}$ of highest weight λ and let W be the tensor product $W=V^{\lambda_2}\otimes V^{\lambda_1}$. Let W^k be the $\mathscr{G}$-invariant subspace of W containing the highest weight states of grade k. Then, the highest grade subspace $W_k^k\subset W^k\subset W$ is given asymptotically by

$$W_k^k \to V^{\lambda_2}_{k-[\lambda_1]} \otimes V^{\lambda_1}_{[\lambda_1]}\,, \quad \text{as} \quad \lambda_1 \to \infty\ . \tag{24}$$

Proof: Since W_k^k is the highest grade subspace of the $\mathscr{G}$-invariant subspace $W^k \subset W$, it follows that, if $|\Psi\rangle \in W_k^k$, then $e_i|\Psi\rangle = 0\,,\ \ \forall e_i \in \mathscr{N}_+$. Let $\mathscr{A}_i$ denote an su(2) subalgebra of $\mathscr{G}$ with a positive root $\alpha_i \in \mathscr{L}_+$, where we recall that $\mathscr{L}_+$ contains the positive roots of $\mathscr{G}$ which are asymptotic for the irrep λ_1.

Since $e_i|\Psi\rangle = 0$ for any $|\Psi\rangle \in W_k^k$, it follows that W_k^k is spanned by a set of states which are all highest weight states for unirreps of $\mathscr{A}_i$. Moreover, since $\alpha_i \in \mathscr{L}_+$, it follows that such a highest weight state $|\Psi\rangle$ is an asymptotic su(2) highest weight state for the su(2) subalgebra generated by e_i and f_i. Therefore, by *Lemma 1*, $|\Psi\rangle = |\varphi\rangle|\psi\rangle$ is a product of a state $|\varphi\rangle \in V^{\lambda_2}$ and a state $|\psi\rangle \in \sigma_i$, where

$$\sigma_i = \{|\psi\rangle \in V^{\lambda_1};\ \ e_i|\psi\rangle = 0\}\ . \tag{25}$$

Since this result holds for all $\mathscr{A}_i \subset \mathscr{G}$ with $\alpha_i \in \mathscr{L}_+$, and since

$$V^{\lambda_1}_{[\lambda_1]} = \{|\psi\rangle \in V^{\lambda_1}\ \ ; e_i|\psi\rangle = 0\,,\ \ \forall \alpha_i \in \mathscr{L}_+\} \tag{26}$$

it follows, in the asymptotic limit, that every state in W_k^k lies in $V^{\lambda_2}_{k-[\lambda_1]} \otimes V^{\lambda_1}_{[\lambda_1]}$. The converse likewise follows from the converse of *Lemma 1*. This completes the proof.

The decomposition of a highest grade subspace W_k^k is a special case of Eq. (7) since, in W_k^k, a highest weight state for a unirrep of $\mathscr{G}$ is also a highest weight state for a unirrep of $\mathscr{H}$. Therefore, we have, in the notation of Eq. (7),

$$W_k^k = \sum_{\rho\lambda} W_\lambda^{\rho\lambda}\,, \qquad [\lambda] = k\ . \tag{27}$$

By the same argument, $V^{\lambda_1}_{[\lambda_1]}$ is irreducible under $\mathscr{H}$, since V^{λ_1} is irreducible under $\mathscr{G}$. This implies that $V^{\lambda_1}_{[\lambda_1]} = V^{\lambda_1}_{\lambda_1}$. On the other hand, $V^{\lambda_2}_{k-[\lambda_1]}$ is a sum of $\mathscr{H}$-irreducible subspaces

$$V^{\lambda_2}_{k-[\lambda_1]} = \sum_{\gamma\omega_2} V^{\lambda_2}_{\gamma\omega_2}\,, \qquad [\omega_2] = k - [\lambda_1]. \tag{28}$$

Now, let $[V^{\lambda_2}_{\gamma\omega_2} \otimes V^{\lambda_1}_{\lambda_1}]_{\alpha\omega}$ denote the $\mathscr{H}$-coupled tensor product of irreducible $\mathscr{H}$-modules, where ω is a highest weight for an $\mathscr{H}$-unirrep and α indexes its multiplicity in the tensor product space $V^{\lambda_2}_{\gamma\omega_2} \otimes V^{\lambda_1}_{\lambda_1}$. It follows from the lemma that

$$W_k^k = \sum_{\rho\lambda} W_\lambda^{\rho\lambda} \to \sum_{\lambda} \sum_{\gamma\omega_2\alpha} [V^{\lambda_2}_{\gamma\omega_2} \otimes V^{\lambda_1}_{\lambda_1}]_{\alpha\lambda}\,, \qquad [\lambda] = k \tag{29}$$

and

$$W_\lambda^{\rho\lambda} \to \sum_{\gamma\omega_2\alpha} C^\lambda_{\rho,\gamma\omega_2\alpha} [V^{\lambda_2}_{\gamma\omega_2} \otimes V^{\lambda_1}_{\lambda_1}]_{\alpha\lambda}\,, \tag{30}$$

where C^λ is a $\rho \times \rho$ matrix which combines equivalent unirreps of $\mathscr{H}$.

Now, if τ indexes a basis $\{|\Psi_\tau^{\rho\lambda}\rangle\}$ for $W_\lambda^{\rho\lambda}$, we can write

$$|\Psi_\tau^{\rho\lambda}\rangle = \sum_{\substack{\gamma\omega_2\alpha \\ ij}} C^\lambda_{\rho,\gamma\omega_2\alpha}\ (\lambda_1\ i;\omega_2\ j|\alpha\lambda\ \tau)|\varphi^{\lambda_2}_{\gamma\omega_2 j}\rangle|\psi_i^{\lambda_1}\rangle\ , \tag{31}$$

where i indexes a basis $\{|\psi_i^{\lambda_1}\rangle\}$ for $V^{\lambda_1}_{\lambda_1}$, j indexes a basis $\{|\varphi^{\lambda_2}_{\gamma\omega_2 j}\rangle\}$ for $V^{\lambda_2}_{\gamma\omega_2}$, and $(\lambda_1\ i;\omega_2\ j|\alpha\lambda\ \tau)$ is a Clebsch–Gordan coupling coefficient for $\mathscr{H}$.

The matrix C^{λ} depends only on the way in which the multiplicities ρ and α of the $\mathscr{H}$-unirrep λ are separated. Since this separation is arbitrary, there exists a choice of ρ and α that will make C^{λ} diagonal. This implies a remarkable property of asymptotic coupling coefficients for basis states of $\mathscr{G}$ in W_k^k: they can always be chosen to be equal to coupling coefficients for the subalgebra $\mathscr{H}\subset\mathscr{G}$.

D. An example: su(3)

The coupling of two su(3) unirreps provides the simplest application of Eq. (31). For su(3), we have a Cartan subalgebra $\mathscr{T}$ with basis $\{h_1,h_2\}$ and positive roots $\Delta_+=\{\alpha_1,\alpha_2,\alpha_3\}$, where α_1 and α_2 are simple roots, and $\{e_1,e_2,e_3\}$ are the corresponding raising operators. Let $|\psi^{\lambda}_{\beta\sigma}\rangle$ denote a state of weight $\sigma=(\sigma^1,\sigma^2)$ of an su(3) unirrep of highest weight λ, where β indexes multiple occurrences of the weight σ; the label β will be suppressed when not needed. The highest weight state of the unirrep $\lambda=(\lambda^1,\lambda^2)$ will be denoted by $|\lambda\rangle=|\psi^{\lambda}_{\lambda}\rangle$.

We will consider the tensor product of two su(3) unirreps $\lambda_2\otimes\lambda_1$ in the limit in which, say, the second component λ_1^2 of λ_1 is asymptotically large. Then, $\mathscr{F}_+=\{\alpha_1\}$, $\mathscr{L}_+=\{\alpha_2,\alpha_3\}$, and $\mathscr{H}$ is the subalgebra of $\mathscr{G}$ spanned by $\{e_1,f_1,h_1,h_2\}$, where e_1 and f_1 are root vectors corresponding to the roots α_1 and α_{-1}, respectively.

We choose h_1 and h_2 such that

$$[h_1,e_1]=2e_1,\quad [h_1,f_1]=-2f_1,\quad [e_1,f_1]=h_1, \tag{32}$$

and

$$h_1|\psi^{\lambda}_{\beta\sigma}\rangle=\sigma^1|\psi^{\lambda}_{\beta\sigma}\rangle\ ,\quad h_2|\psi^{\lambda}_{\beta\sigma}\rangle=\sigma^2|\psi^{\lambda}_{\beta\sigma}\rangle\ . \tag{33}$$

$\mathscr{H}$ is then the direct sum $\mathscr{H}=\text{su}(2)+\text{u}(1)$, where su(2) is the algebra spanned by e_1,f_1, and h_1, and u(1) is spanned by h_2. Since states of a given grade have identical values of σ^2, we can identify the grade with σ^2.

The subspace $V^{\lambda_1}_{[\lambda_1]}$ contains states with the property

$$V^{\lambda_1}_{[\lambda_1]}=\{|\psi\rangle\in V^{\lambda_1};e_i|\psi\rangle=0,\ \ i=2,3\}\ . \tag{34}$$

States in $V^{\lambda_1}_{[\lambda_1]}$ are generated from the highest weight state $|\lambda_1\rangle$ by repeatedly acting on $|\lambda_1\rangle$ with the lowering operator f_1. Thus they carry a unirrep of angular momentum $j_1=\lambda_1^1/2$ of su(2) $\subset\mathscr{H}$ and a unirrep λ_1^2 of u(1) $\subset\mathscr{H}$. The state $|\lambda_1-r\alpha_1\rangle$ [see Fig. 1(b)], which is obtained by lowering r times from $|\lambda_1\rangle$, can be written, in an $\mathscr{H}=$ su(2)+u(1) basis,

$$|\lambda_1-r\alpha_1\rangle=|\psi^{\lambda_1}_{j_1m_1\lambda_1^2}\rangle\ , \tag{35}$$

where $m_1=j_1-r$.

The subspace $V_l^{\lambda_2}\subset V^{\lambda_2}$ has decomposition into $\mathscr{H}$-irreducible subspaces given, in the notation of Eq. (28), by

$$V_l^{\lambda_2}=\sum_{j_2} V^{\lambda_2}_{j_2\sigma^2}\ ,\quad \sigma^2=l\ , \tag{36}$$

where the multiplicity label γ of Eq. (28) is suppressed because the unirrep $j_2\sigma^2$ occurs at most once in $V_l^{\lambda_2}$. An su(2)+u(1) basis for $V^{\lambda_2}_{j_2\sigma^2}$ is then given by $\{|\varphi^{\lambda_2}_{j_2m_2\sigma^2}\rangle\ ;m_2=-j_2,\ldots,j_2.\}$.

From *Lemma 2*, we know that a basis for W_k^k is given by products of the type $|\varphi^{\lambda_2}_{j_2m_2\sigma^2}\rangle|\psi^{\lambda_1}_{j_1m_1\lambda_1^2}\rangle$, with $k=\lambda_1^2+\sigma^2$. These states can be combined to form a good su(2)+u(1)-coupled basis $\{|\Psi_{JMk}\rangle\}$ for W_k^k with

$$|\Psi_{JMk}\rangle=[|\varphi^{\lambda_2}_{j_2\sigma^2}\rangle\otimes|\psi^{\lambda_1}_{j_1\lambda_1^2}\rangle]^{Jk}_M=\sum_{m_1m_2}(j_1\ m_1;j_2\ m_2|J\ M)|\varphi^{\lambda_2}_{j_2m_2\sigma^2}\rangle|\psi^{\lambda_1}_{j_1m_1\lambda_1^2}\rangle\ , \tag{37}$$

where $(j_1\ m_1;j_2\ m_2|J\ M)$ is an ordinary su(2) coupling coefficient. The u(1) coupling is fulfilled by the requirement that $\lambda_1^2+\sigma^2=k$.

Since W_k^k is a highest grade subspace, an su(2)+u(1) highest weight state $|\Psi_{JJk}\rangle$ is also a highest weight state for an su(3) unirrep of highest weight $\lambda=(\lambda^1=2J,\lambda^2=k)$. We then have the identification $|\Psi_{JJk}\rangle=|\lambda\rangle$.

Although the su(2)+u(1) coupling is multiplicity-free, there may be more than one $|\Psi_{JJk}\rangle$ (and hence more than one occurrence of $|\lambda\rangle$) because, in general, $V_l^{\lambda_2}$ contains more than one value of j_2 such that the coupling $j_2\otimes j_1\rightarrow J$ exists. Since there is one copy of $|\Psi_{JJk}\rangle$ for every j_2 satisfying the above condition, an obvious way to distinguish these multiple copies of $|\Psi_{JJk}\rangle$ is to label them with j_2. Highest weight states for the su(2)+u(1) unirrep Jk will henceforth be denoted $|\Psi_{j_2JJk}\rangle$.

If ρ labels multiple occurrences of the su(3) highest weight state $|\lambda\rangle$, the set $\{|\rho\lambda\rangle\}$ may still differ from $\{|\Psi_{j_2JJk}\rangle\}$ by an arbitrary unitary transformation; this is the matrix C^λ of Eq. (31). Thus, the identification $\rho\leftrightarrow j_2$, which provides a convenient resolution of the multiplicities in the $\lambda_2\otimes\lambda_1\rightarrow\lambda$ coupling, makes this matrix diagonal. Furthermore, for this choice, we have

$$|j_2\lambda\rangle=\sum_{m_1m_2}(j_1\ m_1;j_2\ m_2|J\ J)|\varphi^{\lambda_2}_{j_2m_2\sigma^2}\rangle|\psi^{\lambda_1}_{j_1m_1\lambda_1^2}\rangle\ , \tag{38}$$

where $2J=\lambda^1$ and $\lambda_1^2+\sigma^2=\lambda^2$. Other states with $M\neq J$ are obtained by lowering with f_1, and we have, in general,

$$|j_2\lambda\tau\rangle=\sum_{m_1m_2}(j_1\ m_1;j_2\ m_2|J\ M)|\varphi^{\lambda_2}_{j_2m_2\sigma^2}\rangle|\psi^{\lambda_1}_{j_1m_1\lambda_1^2}\rangle\ , \tag{39}$$

where $2M=\tau$.

Thus, in the limit where one component of λ_1 becomes asymptotically large, we find that su(3) coupling reduces to su(2)+u(1) coupling, with

$$(\lambda_1j_1m_1\sigma_1^2,\lambda_2j_2m_2\sigma_2^2|\rho\lambda\tau)\rightarrow\delta_{\rho,j_2}\delta_{\sigma_1^2,\lambda_1^2}(j_1m_1,j_2m_2|JM)\ . \tag{40}$$

This conclusion is confirmed when we take the limit of the expressions obtained, for instance, by Hecht[7] for some su(3) coupling coefficients (cf. also Rowe and Repka[8]).

Suppose now that both components $(\lambda_1^1,\lambda_1^2)$ of λ_1 are asymptotically large. Since $2j_1=\lambda_1^1$ and $2J=\lambda^1$, the coupling coefficient appropriate to this case is obtained from Eq. (40) by taking the limit of the su(2) coefficient:

$$\lim_{\lambda_1,\lambda\rightarrow\infty}(\lambda_1j_1m_1\sigma_1^2,\lambda_2j_2m_2\sigma_2^2|\rho\lambda\tau)\rightarrow\delta_{\rho,j_2}\delta_{\sigma_1^2,\lambda_1^2}\times\lim_{j_1,J\rightarrow\infty}(j_1m_1,j_2m_2|JM)\ . \tag{41}$$

In the next section, we show how Clebsch–Gordan coefficients of the type $(j_1m_1,j_2m_2|JM)$ can be evaluated in the $j_1\rightarrow\infty$ limit.

E. Asymptotic Clebsch–Gordan coefficients

Let $\mathscr{G}$ be a compact semisimple Lie algebra of rank N, and let V^{λ_1}, V^{λ_2} be defined as before.

Lemma 3: Let $|\varphi\rangle\in V^{\lambda_2},|\psi\rangle\in V^{\lambda_1}_{[\lambda_1]-p}$, and assume that p is finite. Then, in the limit in which $\lambda_1\rightarrow\infty$, the state $|\Psi\rangle=|\varphi\rangle|\psi\rangle\in V^{\lambda_2}\otimes V^{\lambda_1}$ is such that

$$f_k|\Psi\rangle\rightarrow|\varphi\rangle[f_k|\psi\rangle]\ ,\quad\forall f_k\in\dot{\mathscr{N}}_-\ . \tag{42}$$

Proof: Recall that $\dot{\mathscr{N}}_-$ contains root vectors whose corresponding roots are asymptotic for $\lambda_1\rightarrow\infty$. Then, we have

$$f_k|\Psi\rangle=[f_k|\varphi\rangle]|\psi\rangle+|\varphi\rangle[f_k|\psi\rangle]\ . \tag{43}$$

We may assume, without loss of generality, that $|\varphi\rangle$ and $|\psi\rangle$ are normalized states so that $\||\varphi\rangle\|^2=\||\psi\rangle\|^2=1$. Then

$$\|f_k|\Psi\rangle\|^2=\|f_k|\varphi\rangle\|^2+\|f_k|\psi\rangle\|^2 . \tag{44}$$

Now

$$\|f_k|\psi\rangle\|^2=\langle\psi|e_kf_k|\psi\rangle=\langle\psi|[e_k,f_k]|\psi\rangle+\|e_k|\psi\rangle\|^2=\langle\psi|h_k|\psi\rangle+\|e_k|\psi\rangle\|^2 , \tag{45}$$

which, since $\langle\psi|h_k|\psi\rangle\to\infty$ as $\lambda_1\to\infty$, implies that

$$\|f_k|\psi\rangle\|^2\to\infty \quad \text{as} \quad \lambda_1\to\infty . \tag{46}$$

On the other hand, $\|f_k|\varphi\rangle\|^2$ remains finite and the lemma is shown.

Two important properties for the norms of asymptotic states follow from this lemma. First, set $|\Psi_{ij}\rangle=|\varphi_i\rangle|\psi_j\rangle$, where i labels a basis $\{|\varphi_i\rangle\}$ for V^{λ_2}, and j labels a basis $\{|\psi_j\rangle\}$ for $V^{\lambda_1}_{[\lambda_1]-p}$, and consider the linear combination

$$|\Theta\rangle=\sum_{ij} c_{ij}|\Psi_{ij}\rangle , \tag{47}$$

where c_{ij} are complex coefficients satisfying $\Sigma_{ij}c_{ij}c^*_{ij}=1$, but otherwise arbitrary. Then, by *Lemma 3,* it follows that $\|f_k|\psi_j\rangle\|$ is independent of j. [*Proof:* act on $|\Theta\rangle$ to construct the normalized state $f_k|\Theta\rangle/\|f_k|\Theta\rangle\|$ in terms of $|\varphi_i\rangle$ and $|\psi_j\rangle$. Since $|\varphi_i\rangle$ is normalized, and since $\Sigma_{ij}c_{ij}c^*_{ij}=1$, we deduce that $\|f_k|\psi_j\rangle\|=\|f_k|\Theta\rangle\|\forall j$.]

Next, it follows immediately from Eq. (45) that $\|f_k|\Theta\rangle\|=\|f_k|\psi_j\rangle\|$ is also independent of any component of λ or λ_1 that is asymptotically large. [*Proof:* Eq. (45) is unchanged if we use the unirrep $\tilde{\lambda}_1$ whose highest weight is related to the highest weight of λ_1 by $\tilde{\lambda}_1=\lambda_1+\Delta$, where $\Delta=(\delta_1,\ldots,\delta_n)$ contains only finite integers.]

Theorem 1: Let p be a finite integer. Then,

$$W^k_{k-p}\to V^{\lambda_2}_{k-[\lambda_1]}\otimes V^{\lambda_1}_{[\lambda_1]-p}, \quad \text{as} \quad \lambda_1\to\infty . \tag{48}$$

Proof: Let $|\varphi\rangle\in V^{\lambda_2}_{k-[\lambda_1]}$ and $|\psi\rangle\in V^{\lambda_1}_{[\lambda_1]-p}$. Assume the theorem holds for some p, and act on $|\Psi\rangle=|\varphi\rangle|\psi\rangle\in W^k_{k-p}$ with any $f_r\in\mathscr{G}_{-q},q>0$. Since W^k is $\mathscr{G}$-invariant, we have, by Eq. (4), $f_r|\Psi\rangle\in W^k_{k-p-q}$, and, by *Lemma 3* and Eq. (4),

$$f_r|\Psi\rangle\to|\varphi\rangle[f_r|\psi\rangle]\in V^{\lambda_2}_{k-[\lambda_1]}\otimes V^{\lambda_1}_{[\lambda_1]-p-q} . \tag{49}$$

Hence, if the theorem holds for p, it holds for $p+q$. The seed of the recursion is Lemma 2, which is Theorem 1 for $p=q=0$. Going over all f_r in all $\mathscr{G}_{-q}$ completes the proof.

Consider, for example, the su(2) coupling $j_2\otimes j_1$ in the limit in which $j_1\to\infty$ but j_2 remains finite. Application of the theorem implies that, for a finite value of p, the state

$$|j,j-p\rangle=\sum_{m_1,m_2} (j_1\ m_1;j_2\ m_2|j\ j-p)|j_2,m_2\rangle|j_1m_1\rangle \tag{50}$$

becomes, in the $j_1\to\infty$ limit,

$$|j,j-p\rangle\to|j_2,j-j_1\rangle|j_1,j_1-p\rangle . \tag{51}$$

It follows that, for finite p, the asymptotic SU(2) coupling coefficients are given by

$$(j_1\ m_1;j_2\ m_2|j\ j-p)\to\delta_{m_1,j_1-p}\quad\delta_{m_2,j-j_1} . \tag{52}$$

For an arbitrary compact semisimple Lie algebra, the highest grade states in the tensor product space $V^{\lambda_2}\otimes V^{\lambda_1}$ become, by Theorem 1, of the form

$$|\Psi^{\rho\lambda}_{\lambda\tau}\rangle \rightarrow \sum_{\omega_2\sigma} C^{\lambda}_{\rho,\gamma_2\omega_2\sigma}\ [|\varphi^{\lambda_2}_{\gamma_2\omega_2}\rangle \otimes |\psi^{\lambda_1}_{\lambda_1}\rangle]_{\sigma\lambda\tau}, \tag{53}$$

in the asymptotic limit $\lambda_1 \rightarrow \infty$, where C^{λ} is a unitary transformation. If a particular resolution of the multiplicity is chosen, equivalent to setting $\rho \equiv (\gamma_2\omega_2\sigma)$, such that C^{λ} is an identity matrix, then we have

$$|\Psi^{\rho\lambda}_{\lambda\tau}\rangle \rightarrow [|\varphi^{\lambda_2}_{\gamma_2\omega_2}\rangle \otimes |\psi^{\lambda_1}_{\lambda_1}\rangle]_{\sigma\lambda\tau}, \quad \rho \equiv (\gamma_2\omega_2\sigma) \tag{54}$$

and the $\mathscr{H}$-reduced asymptotic CG coefficients

$$(\lambda_1\gamma_1\omega_1, \lambda_2\gamma_2\omega_2 \| \rho\lambda\lambda)_{\sigma} \rightarrow \delta_{\omega_1,\lambda_1}\delta_{\rho,\gamma_2\omega_2\sigma}, \tag{55}$$

in accordance with Eq. (31).

To obtain the asymptotic limits of coefficients $(\lambda_1\gamma_1\omega_1, \lambda_2\gamma_2\omega_2 \| \rho\lambda\gamma\omega)_{\sigma}$ for finite values of $[\lambda]-[\omega]$, let the state $|\Psi^{\rho\lambda}_{\gamma\omega\tau}\rangle$ be expressed in the form

$$|\Psi^{\rho\lambda}_{\gamma\omega\tau}\rangle = [P_n(f) \otimes |\Psi^{\rho\lambda}_{\lambda}\rangle]_{\alpha\omega\tau}, \tag{56}$$

where $P_n(f)$ is an $\mathscr{H}$-tensor of highest weight n whose components are polynomials in the elements of $\mathscr{N}_-$ and we identify the multiplicity index γ with the pair of indices $(n\alpha)$. Then, in the asymptotic limit,

$$|\Psi^{\rho\lambda}_{\gamma\omega\tau}\rangle \rightarrow [P_n(f) \otimes [|\varphi^{\lambda_2}_{\gamma_2\omega_2}\rangle \otimes |\psi^{\lambda_1}_{\lambda_1}\rangle]_{\sigma\lambda}]_{\alpha\omega\tau}. \tag{57}$$

Let $\phi(\lambda_1, \omega_2; \sigma\lambda)$ be the phase factor for which

$$[|\varphi^{\lambda_2}_{\gamma_2\omega_2}\rangle \otimes |\psi^{\lambda_1}_{\lambda_1}\rangle]_{\sigma\lambda} = \phi(\lambda_1, \lambda_2; \sigma\lambda)[|\psi^{\lambda_1}_{\lambda_1}\rangle \otimes |\varphi^{\lambda_2}_{\gamma_2\omega_2}\rangle]_{\sigma\lambda}. \tag{58}$$

Using Theorem 1 and setting $\rho \equiv (\gamma_2\omega_2\sigma)$, we have

$$\begin{aligned} |\Psi^{\rho\lambda}_{\gamma\omega\tau}\rangle &\rightarrow \phi(\lambda_1, \omega_2; \sigma\lambda) \sum_{\beta\kappa\omega_1} U(\omega_2\lambda_1\omega n; \lambda\sigma\alpha, \omega_1\beta\kappa)[[P_n(f) \otimes |\psi^{\lambda_1}_{\lambda_1}\rangle]_{\beta\omega_1} \otimes |\varphi^{\lambda_2}_{\gamma_2\omega_2}\rangle]_{\kappa\omega\tau}, \\ &\rightarrow \phi(\lambda_1, \omega_2; \sigma\lambda) \sum_{\beta\kappa\omega_1} \phi(\omega_2, \omega_1; \kappa\omega) U(\omega_2\lambda_1\omega n; \lambda\sigma\alpha, \omega_1\beta\kappa)[|\varphi^{\lambda_2}_{\gamma_2\omega_2}\rangle \otimes |\psi^{\lambda_1}_{n\beta\omega_1}\rangle]_{\kappa\omega\tau}, \end{aligned} \tag{59}$$

where $U(\omega_2\lambda_1\omega n; \lambda\sigma\alpha, \omega_1\beta\kappa)$ is a Racah recoupling coefficient for $\mathscr{H}$, and κ labels multiple copies of ω in the coupling $\omega_1 \otimes \omega_2$. In deriving Eq. (59), we have used properties of the norms of asymptotic states discussed as corollaries of Lemma 3. Furthermore, we use $(n\beta)$ to label multiple occurrences of γ_1. Thus we find

$$\begin{aligned} (\lambda_1\gamma_1\omega_1, \lambda_2\gamma_2\omega_2 \| \rho\lambda\gamma\omega)_{\sigma} \rightarrow \sum_{\kappa}\ & \phi(\lambda_1, \omega_2; \sigma\lambda)\phi(\omega_2, \omega_1; \alpha\omega)\ \ U(\omega_2\lambda_1\omega n; \lambda\sigma\alpha, \omega_1\beta\kappa) \\ & \times\ M^{\omega}_{\gamma\kappa}\ \times\ \delta_{\rho,\gamma_2\omega_2\sigma}\ \ \delta_{\gamma,n\alpha}\ \ \delta_{\gamma_1,n\beta}, \quad \text{as}\ \ \lambda_1 \rightarrow \infty\ , \end{aligned} \tag{60}$$

where the matrix M^{ω} combines the equivalent $\mathscr{H}$-representations ω. Since this matrix depends only on the arbitrary way in which the multiplicities γ and κ of the $\mathscr{H}$-unirrep ω are separated, we set $\kappa = \gamma$ and choose the phases so that M^{ω} is the unit matrix. It then follows that the $\mathscr{H}$-reduced asymptotic coupling coefficients for basis states of $\mathscr{G}$ in $W^{k}_{[\lambda]-[\omega]}$, for finite values of $[\lambda]-[\omega]$, are given simply by a recoupling coefficient for the subalgebra $\mathscr{H}$ times some phases, i.e.,

$$(\lambda_1\gamma_1\omega_1,\lambda_2\gamma_2\omega_2\|\rho\lambda\gamma\omega)_\sigma\to\phi(\lambda_1,\omega_2;\sigma\lambda)\phi(\omega_2,\omega_1;\alpha\omega)\times U(\omega_2\lambda_1\omega n;\lambda\sigma\alpha,\omega_1\beta\gamma)\delta_{\rho,\gamma_2\omega_2\sigma}\quad\delta_{\gamma,n\alpha}\quad\delta_{\gamma_1,n\beta},\quad\text{as }\lambda_1\to\infty. \tag{61}$$

Combining Eq. (59) with Eq. (61), we finally obtain the simple expression

$$|\Psi^{\rho\lambda}_{\gamma\omega\tau}\rangle\to\phi(\lambda_1,\omega_2;\sigma\lambda)\sum_{\beta\omega_1}\phi(\omega_2,\omega_1;\kappa\omega)U(\omega_2\lambda_1\omega n;\lambda\sigma\alpha,\omega_1\beta\omega)\times[|\varphi^{\lambda_2}_{\gamma_2\omega_2}\rangle\otimes|\psi^{\lambda_1}_{n\beta\omega_1}\rangle]_{\gamma\omega\tau}\delta_{\rho,\gamma_2\omega_2\sigma}\quad\delta_{\gamma,n\alpha},\quad\text{as }\lambda_1\to\infty\ . \tag{62}$$

IV. THE REAL SYMPLECTIC sp(m,R) ALGEBRAS

In this section, we discuss the generalization of the above results to noncompact semisimple Lie algebras. For simplicity we restrict considerations to sp(m,R), although the results are much more widely applicable. We also restrict considerations to products of harmonic series of representations.[9,10] These representations, which include the positive discrete series, have lowest weights but no highest weights. Thus we consider subspaces of $W=V^{\lambda_2}\otimes V^{\lambda_1}$ that are lowest (rather than highest) in grade. $W^k\subset W$ is now the sp(m,R)-invariant subspace containing the lowest weight states of grade k, and W^k_{k+p} is the subspace of W^k of grade $k+p$.

A. Factorization for sp(1,R)

The elements of sp(1,R) satisfy

$$[J_-,J_+]=2J_0,\quad [J_0,J_\pm]=\pm J_\pm, \tag{63}$$

with matrix elements given by

$$\begin{aligned}J_0|jm\rangle&=m|jm\rangle\ ,\quad m=j,j+1,j+2,\dots\ ,\\ J_+|jm\rangle&=\sqrt{(m+j)(m-j+1)}|j,m+1\rangle\ ,\\ J_-|jm\rangle&=\sqrt{(m-j)(m+j-1)}|j,m-1\rangle\ ,\end{aligned} \tag{64}$$

where $j=\frac{1}{4},\frac{3}{4},\frac{5}{4},\dots$. Now, consider the coupling of two harmonic series unirreps $j_2\otimes j_1$ with $j_1\to\infty$ and j_2 finite. It can be verified, e.g., by comparing the number of states of weight $m=m_1+m_2$ with the number of states of weight $m-1$, that

$$j_2\otimes j_1=(j_1+j_2)\oplus(j_1+j_2+1)\oplus(j_1+j_2+2)\oplus\cdots\ . \tag{65}$$

Next, consider the lowest weight state $|jj\rangle$ of a subrepresentation in Eq. (65), with $j=j_1+j_2+s$, where j_2 and $s\geq 0$ are finite, in the limit $j_1\to\infty$. Write

$$|jj\rangle=\sum_{m_1m_2}\ (j_1\ m_1;j_2\ m_2|j\ j)\ \ |j_2m_2\rangle|j_1m_1\rangle\ , \tag{66}$$

where $(j_1\ m_1;j_2\ m_2|j\ j)$ is now an sp(1,R) coupling coefficient. If we use $J_-|jj\rangle=0$, we get

$$\begin{aligned}0=&\ (j_1\ m_1+1;j_2\ m_2-1|j\ j)\ \ \sqrt{(m_1+1-j_1)(m_1+j_1)}\\ &+(j_1\ m_1;j_2\ m_2|j\ j)\ \ \sqrt{(m_2-j_2)(m_2+j_2-1)}\ .\end{aligned}$$

This equation can then be rewritten as

$$\frac{(j_1\ m_1+1;j_2\ m_2-1|j\ j)}{(j_1\ m_1;j_2\ m_2|j\ j)}=-\sqrt{\frac{(m_2-j_2)(m_2+j_2-1)}{(m_1+1-j_1)(m_1+j_1)}}\ . \tag{67}$$

Since $m_1+j_1\geq 2j_1$ and $j_1\to\infty$ while j_2 and m_2 remain finite, it follows that

$$\frac{(j_1\ m_1+1;j_2\ m_2-1|j\ j)}{(j_1\ m_1;j_2\ m_2|j\ j)}\rightarrow 0 \quad \text{as } j_1\rightarrow\infty\ . \tag{68}$$

If we now suppose that $|j,j+\delta\rangle\rightarrow|j_2,j-j_1\rangle|j_1,j_1+\delta\rangle$ as $j_1\rightarrow\infty$, when δ is some finite integer, then, by acting on $|j,j+\delta\rangle$ with J_+, we find, on the one hand,

$$J_+|j,j+\delta\rangle=\sqrt{(2j+\delta)(\delta+1)}|j,j+\delta+1\rangle\ , \tag{69}$$

while, on the other hand,

$$\begin{aligned}J_+[|j_2,j-j_1\rangle|j_1,j_1+\delta\rangle]=&\sqrt{(2j_1+\delta)(\delta+1)}|j_2,j-j_1\rangle|j_1,j_1+\delta+1\rangle\\&+\sqrt{(2j_2+s)(s+1)}|j_2,j-j_1+1\rangle|j_1,j_1+\delta\rangle\ .\end{aligned} \tag{70}$$

Comparison of these equations gives

$$\begin{aligned}|j,j+\delta+1\rangle=&\sqrt{\frac{(2j_1+\delta)}{(2j+\delta)}}|j_2,j-j_1\rangle|j_1,j_1+\delta+1\rangle\\&+\sqrt{\frac{(2j_2+s)(s+1)}{(2j+\delta)(\delta+1)}}|j_2,j-j_1+1\rangle|j_1,j_1+\delta\rangle\ ,\end{aligned} \tag{71}$$

and implies that

$$|j,j+\delta+1\rangle\rightarrow|j_2,j-j_1\rangle|j_1,j_1+\delta+1\rangle\ ,\quad \text{as}\ \ j_1\rightarrow\infty\ . \tag{72}$$

Hence we have shown by recursion that the following holds:

Lemma 4: Let $j=j_1+j_2+s$ with $s\geqslant 0$. Then, if j_2 and s are finite,

$$|j,j+\delta\rangle\rightarrow|j_2,j-j_1\rangle|j_1,j_1+\delta\rangle\ ,\quad \text{as}\ \ j_1\rightarrow\infty\ . \tag{73}$$

B. Asymptotic unirreps of sp(m,R)

Every harmonic series representation of sp(m,R) is contained within the space of some A-particle harmonic oscillator in m dimensions.[9,10] For these representations, the sp(m,R) algebra is realized as the set of bilinear products of creation and destruction operators:

$$\begin{aligned}A_{ij}&=\sum_{a=1}^{A} b_{ai}^\dagger b_{aj}^\dagger,\\ C_{ij}&=\sum_{a=1}^{A}\frac{1}{2}\,(b_{ai}^\dagger b_{aj}+b_{aj}b_{ai}^\dagger)\ ,\\ B_{ij}&=\sum_{a=1}^{A} b_{ai}b_{aj},i,\ \ j=1,\ldots,m\ .\end{aligned} \tag{74}$$

Sp(m,R) and u(m) have a common Cartan algebra, spanned by $\{C_{ii},i=1,\ldots,m\}$. Thus a u(m) unirrep is labeled by a highest weight $\lambda=(\lambda^1,\ldots,\lambda^m)$ where λ^i is an eigenvalue of C_{ii} and $\lambda^i\geqslant\lambda^{i+1}$. However, the sp($m$,R) unirreps we consider have no highest weight. The carrier spaces of these sp(m,R) unirreps comprise infinitely many u(m) invariant subspaces each labeled by a u(m) highest weight. The lowest of these u(m) highest weights uniquely characterizes the sp(m,R) unirrep and will be referred to as the lowest weight of the sp(m,R) irrep.

We will consider states of sp(m,R) unirreps in the limit where $\lambda^m\rightarrow\infty$. This implies that all the other components of λ are also asymptotically large.

There are two types of sp(m,R) raising operators: the compact su(m) raising operators $\{C_{ij};1\leqslant j<i\leqslant m\}$ and the noncompact raising operators $\{A_{ij};1\leqslant i,j\leqslant m\}$. The action of an A_{ij} operator connects weights belonging to an sp(1,R) chain. States of an sp(1,R) chain carry a unirrep

of an sp(1,R) subalgebra of sp(m,R). If such a unirrep is labeled by j, we define the corresponding sp(1,R) chain of weights to be an asymptotic chain if and only if $j\rightarrow\infty$ as $\lambda\rightarrow\infty$. Moreover, we note that, when $\lambda^m\rightarrow\infty$, all the $\{A_{ij}\}$ operators generate asymptotic chains. Thus, when $\lambda^m\rightarrow\infty$, all the noncompact roots are in $\mathscr{L}_+$.

Chains of weights associated with the compact su(m) roots are conveniently analyzed by reexpressing a u(m) weight λ as an su(m)+u(1) weight with components $(\lambda^1-\lambda^2,\lambda^2-\lambda^3,\ldots,\lambda^{m-1}-\lambda^m)\lambda^m$. The chains of weights associated with the compact su(m) roots then become asymptotic when the differences $\lambda^i-\lambda^{i+1}$ of consecutive components of a u(m) weight become large.

We assume, for the moment, that $\lambda^i-\lambda^{i+1}$ remains finite for all i, so that $\mathscr{F}_+$ contains all the positive su(m) roots; i.e., $\mathscr{F}_+$ contains the compact positive roots and $\mathscr{L}_+$ contains the noncompact positive roots. We then have $\mathscr{H}=$ u(m), and sp(m,R) (or, more precisely, its complex extension) has the graded decomposition

$$\mathrm{sp}(m,\mathrm{R})=\mathscr{G}_0+\mathscr{G}_++\mathscr{G}_-, \tag{75}$$

where $\mathscr{G}_0$ is the u(m) subalgebra spanned by the $\{C_{ij}\}$, $\mathscr{G}_+$ is spanned by the raising operators $\{A_{ij}\}$, and $\mathscr{G}_-$ is spanned by the lowering operators $\{B_{ij}\}$. The grade is conveniently identified with the eigenvalue of the operator

$$\hat{N}=\sum_{i=1}^{m} C_{ii}. \tag{76}$$

The lowest grade subspace $V^\lambda_{[\lambda]}$ of an sp(m,R)-irreducible vector space V^λ is defined as the subset of states that are annihilated by the elements of $\mathscr{G}_-$:

$$V^\lambda_{[\lambda]}=\{|\psi\rangle\in V^\lambda;\ \ B_{ij}|\psi\rangle=0,\ \forall B_{ij}\in\mathscr{G}_-\}. \tag{77}$$

If one or more of the su(m) labels $\lambda^i-\lambda^{i+1}$ becomes asymptotically large, then $\mathscr{F}_+$ and $\mathscr{L}_+$ must be shrunk and expanded accordingly. However, as the process of taking $\lambda^m\rightarrow\infty$ ''commutes'' with the process of taking any su(m) label to ∞, we will assume henceforth that all the $\lambda^i-\lambda^{i+1}$ are finite, knowing that the extra simplifications that arise should one or more su(m) labels become asymptotically large can be made once the analysis of the large-λ^m limit has been completed.

C. Lowest grade subspaces of sp(m,R)

Lemma 5: Let V^λ denote the carrier space for a unirrep of sp(m,R) of lowest weight λ. Let $W^k\subset W$ be the sp(m,R) invariant subspace of $W=V^{\lambda_2}\otimes V^{\lambda_1}$ containing the lowest weight states of grade k, where $k=[\lambda_1]+[\lambda_2]+s$, with s a finite integer. Then, the lowest grade subspace $W^k_k\subset W^k$ is given asymptotically by

$$W^k_k\rightarrow V^{\lambda_2}_{k-[\lambda_1]}\otimes V^{\lambda_1}_{[\lambda_1]},\qquad \text{as } \lambda_1^m\rightarrow\infty. \tag{78}$$

Proof: The proof parallels that given for Lemma 2 and is omitted.

It is possible to use this lemma to derive asymptotic sp(m,R) coupling coefficients. Since $\mathscr{H}=$ u(m) in the $\lambda_1^m\rightarrow\infty$ limit, it follows, in the notation of Eq. (31), that a basis $\{|\Psi^{\rho\lambda}_\tau\rangle\}$ for $W^{\rho\lambda}_\lambda\subset W^k_k$ is given by the states

$$|\Psi^{\rho\lambda}_\tau\rangle=\sum_{\substack{\gamma\omega_2\alpha\\ ij}} C^\lambda_{\rho,\gamma\omega_2\alpha}\ (\lambda_1\ i;\omega_2\ j|\alpha\omega\ \tau)|\varphi^{\lambda_2}_{\gamma\omega_2 j}\rangle|\psi^{\lambda_1}_i\rangle, \tag{79}$$

where α labels the multiplicity of the u(m) coupling $\omega_2\otimes\lambda_1\rightarrow\lambda$, i indexes a basis $\{|\psi^{\lambda_1}_i\rangle\}$ for the u(m) unirrep λ_1 in $V^{\lambda_1}_{[\lambda_1]}$, j indexes a basis $\{|\varphi^{\lambda_2}_{\gamma\omega_2 j}\rangle\}$ for the γth copy of the u(m) unirrep ω_2 in $V^{\lambda_2}_{k-[\lambda_1]}$, and $(\lambda_1\ i;\omega_2\ j|\alpha\omega\ \tau)$ is a u(m) Clebsch–Gordan coupling coefficient.

The derivation of Eq. (79) is identical to the derivation of Eq. (31) and therefore omitted.

D. Asymptotic states of sp(m,R)

Lemma 6: Let $|\psi\rangle \in V^{\lambda_1}$ and $|\varphi\rangle \in V^{\lambda_2}_{[\lambda_2]+q}$ be normalized states, where q is a finite positive integer. Then, in the $\lambda_1^m \to \infty$ limit, the state $|\Psi\rangle = |\varphi\rangle |\psi\rangle \in V^{\lambda_2} \otimes V^{\lambda_1}$ satisfies

$$A_{ij}|\Psi\rangle \to |\varphi\rangle [A_{ij}|\psi\rangle] \ , \quad \forall ij \ . \tag{80}$$

Proof: We have

$$A_{ij}|\Psi\rangle = [A_{ij}|\varphi\rangle] \ |\psi\rangle + |\varphi\rangle [A_{ij}|\psi\rangle] \tag{81}$$

and, since $|\varphi\rangle$ and $|\psi\rangle$ are normalized states,

$$\|A_{ij}|\Psi\rangle\| = \|A_{ij}|\varphi\rangle\| + \|A_{ij}|\psi\rangle\| \ . \tag{82}$$

To evaluate $\|A_{ij}|\psi\rangle\|$, observe that

$$\begin{aligned} \|A_{ij}|\psi\rangle\|^2 &= \langle \psi|[B_{ij}, A_{ij}]|\psi\rangle + \|B_{ij}|\psi\rangle\|^2 \\ &= \langle \psi|C_{ii} + C_{jj} + 2\delta_{ij}C_{ii}|\psi\rangle + \|B_{ij}|\psi\rangle\|^2 \ . \end{aligned} \tag{83}$$

Since

$$\langle \psi|C_{ii}|\psi\rangle \geqslant \langle \psi|C_{mm}|\psi\rangle \geqslant \lambda_1^m , \tag{84}$$

it follows that $\|A_{ij}|\psi\rangle\|^2 \to \infty$ as $\lambda_1^m \to \infty$. On the other hand, for $|\varphi\rangle \in V^{\lambda_2}_{[\lambda_2]+q}$, with q finite, the norm $\|A_{ij}|\varphi\rangle\|$ remains finite. Hence we have $\|A_{ij}|\Psi\rangle\| \to \|A_{ij}|\psi\rangle\|$ and the proof is complete.

Theorem 2: In the $\lambda_1^m \to \infty$ limit,

$$W^k_{k+p} \to V^{\lambda_2}_{k-[\lambda_1]} \otimes V^{\lambda_1}_{[\lambda_1]+p} \tag{85}$$

for finite values of p.

Proof: The proof is by induction, starting with Lemma 5. It parallels that for Theorem 1, but uses Lemma 6 rather than Lemma 3 to iterate between the graded subspaces.

Asymptotic coupling coefficients for the large-λ_1^m limit of sp(m,R) can be derived in the manner of Sec. III E, provided that the polynomial $P_n(f)$ appearing in Eqs. (56), (57), and (59) is replaced by the corresponding polynomial $P_n(A)$ in terms of raising operators. Thus in the $\lambda_1^m \to \infty$ limit and for finite values of $[\omega]-[\lambda]$, where ω and λ label unirreps of u(m), the state $|\Psi^{\rho\lambda}_{\gamma\omega\tau}\rangle$ is given by Eq. (62), where all coupling and recoupling coefficients are u(m) coefficients.

V. sp(m,R) TENSOR OPERATORS

In this section, we extend the results on sp(m,R) to the coupling of a nonunitary finite-dimensional representation of sp(m,R) to a unitary infinite-dimensional representation. Interest in such couplings arises because tensor operators often belong to finite-dimensional irreps.

A. sp(1,R) tensor operators

The components $\{T^{\tilde{j}}_{\tilde{m}}; \tilde{m} = -\tilde{j}, \ldots, +\tilde{j}\}$ of a finite-dimensional (nonunitary) sp(1,R) tensor operator $T^{\tilde{j}}$ are related[10] by the equations

$$\begin{aligned} [J_\pm, T^{\tilde{j}}_{\tilde{m}}] &= \mp \sqrt{(\tilde{j} \mp \tilde{m})(\tilde{j} \pm \tilde{m} + 1)}\, T^{\tilde{j}}_{\tilde{m}+1} , \\ [J_0, T^{\tilde{j}}_{\tilde{m}}] &= \tilde{m} T^{\tilde{j}}_{\tilde{m}} . \end{aligned} \tag{86}$$

The decomposition of the coupling $\tilde{j} \otimes j$, where j is the lowest weight for a unirrep, can be inferred by simply counting the number of states with a given weight $M = \tilde{m} + m$. This shows that $\tilde{j} \otimes j$ is the finite sum of unitary irreducible representations

$$\tilde{j}\otimes j=|j-\tilde{j}|\oplus|j-\tilde{j}+1|\oplus\cdots\oplus|j+\tilde{j}| \ . \tag{87}$$

We are interested in the lowest weight states of these couplings, which we express in the form

$$[T^{\tilde{j}}\otimes|j\rangle]^J_J=\sum_{m,\tilde{m}}(j\ m;\tilde{j}\ \tilde{m}|J\ J)T^{\tilde{j}}_{\tilde{m}}\ |jm\rangle \ . \tag{88}$$

Applying the lowering operator J_- to Eq. (88), we get

$$0=\sum_{m,\tilde{m}}(j\ m-1;\tilde{j}\ \tilde{m}+1|J\ J)[J_-,T^{\tilde{j}}_{\tilde{m}+1}]|jm-1\rangle$$
$$+\sum_{m,\tilde{m}}(j\ m;\tilde{j}\ \tilde{m}|J\ J)T^{\tilde{j}}_{\tilde{m}}\ [J_-|jm\rangle]\ , \tag{89}$$

from which it follows, using Eqs. (64) and (86), that

$$0=\sqrt{(\tilde{j}+\tilde{m}+1)(\tilde{j}-\tilde{m})}\,(j\ m-1;\tilde{j}\ \tilde{m}+1|J\ J)$$
$$+\sqrt{(m-j)(m+j-1)}\,(j\ m;\tilde{j}\ \tilde{m}|J\ J)\ . \tag{90}$$

Combined with the fact that there are finitely many coupling coefficients when $\tilde{j}$ is finite, Eq. (90) can be used to establish that

$$(j\ m;\tilde{j}\ \tilde{m}|J\ J)\rightarrow k\left(\frac{1}{\sqrt{2j}}\right)^{m-j}(j\ j;\tilde{j}\ J-j|J\ J)\ ,\quad \text{as } j\rightarrow\infty\ , \tag{91}$$

where k is some finite constant. It then follows that

$$[T^{\tilde{j}}\otimes|j\rangle]^J_J\rightarrow T^{\tilde{j}}_{J-j}|jj\rangle\ ,\quad \text{as } j\rightarrow\infty\ . \tag{92}$$

Furthermore, since

$$J_+[T^{\tilde{j}}\otimes|j\rangle]^J_J=\sqrt{2J}[T^{\tilde{j}}\otimes|j\rangle]^J_{J+1}\rightarrow[J_+,T^{\tilde{j}}_{J-j}]|jj\rangle+T^{\tilde{j}}_{J-j}[J_+|jj\rangle]$$
$$=-\sqrt{(\tilde{j}-J+j)(\tilde{j}+J-j+1)}T^{\tilde{j}}_{J-j+1}|jj\rangle+\sqrt{2j}T^{\tilde{j}}_{J-j}|j,j+1\rangle\ , \tag{93}$$

we have

$$[T^{\tilde{j}}\otimes|j\rangle]^J_{J+1}\rightarrow T^{\tilde{j}}_{J-j}|j,j+1\rangle\ ,\quad \text{as}\quad j\rightarrow\infty\ . \tag{94}$$

By acting now on $[T^{\tilde{j}}\otimes|j\rangle]^J_{J+1}$ with J_+, we find, in the same manner, that

$$[T^{\tilde{j}}\otimes|j\rangle]^J_{J+2}\rightarrow T^{\tilde{j}}_{J-j}|j_1,j_1+2\rangle\ ,\quad \text{as } j\rightarrow\infty\ . \tag{95}$$

These are the first two steps of an inductive proof of the following lemma.

Lemma 7: For finite values of $\tilde{j}$,

$$[T^{\tilde{j}}\otimes|j\rangle]^J_{J+\delta}\rightarrow T^{\tilde{j}}_{J-j}|j,j+\delta\rangle\ ,\quad \text{as}\quad j\rightarrow\infty\ . \tag{96}$$

B. sp(3,R) tensor operators

The coupling of an sp(3,R) tensor operator $T^{\tilde{\lambda}}$, that transforms according to a nonunitary, finite dimensional representation $\tilde{\lambda}$, to a unitary asymptotic irrep λ of sp(3,R) can be reduced as follows.

Let us denote by $\langle\lambda\rangle$ the character of the Sp(3,R) irrep with lowest weight λ and by $\{\lambda\}=\{\lambda^1,\lambda^2,\lambda^3\}$ the character of the U(3) irrep with highest weight $\lambda=(\lambda^1,\lambda^2,\lambda^3)$. Then, for $\lambda^3\geqslant 3$, the Sp(3,R) $\rightarrow$ U(3) branching rule is given in terms of characters,[10] by

$$\text{Sp}(3,\text{R})\rightarrow \text{U}(3) \quad ; \quad \langle\lambda\rangle\mapsto\{\lambda\}\cdot\{D\} , \tag{97}$$

where D denotes the sum of U(3) irreps with character

$$\{D\}=\{0\}+\{2\}+\{4\}+\{2,2\}+\{6\}+\{4,2\}+\{2,2,2\}+\cdots . \tag{98}$$

Let

$$\langle\lambda\rangle\mapsto\sum_p m_p\{\omega_p\}=\sum_p m_p\{\omega_p^1,\omega_p^2,\omega_p^3\} , \tag{99}$$

denote the decomposition of the character of $\tilde{\lambda}$ into its U(3) characters $\{\omega_p\}=\{\omega_p^1,\omega_p^2,\omega_p^3\}$, where m_p is the multiplicity of ω_p in $\tilde{\lambda}$. Since $\tilde{\lambda}$ is finite dimensional, the sum in Eq. (99) will contain finitely many U(3) unirreps. Then, if $\langle\tilde{\lambda}\rangle\cdot\langle\lambda_1\rangle$ denotes the character for the tensor product $\tilde{\lambda}\otimes\lambda_1$ of a finite Sp(3,R) irrep $(\tilde{\lambda})$ and an infinite unitary irrep (λ_1), we have

$$\text{Sp}(3,\text{R})\rightarrow \text{U}(3) \quad ; \quad \langle\tilde{\lambda}\rangle\cdot\langle\lambda_1\rangle\mapsto\sum_p m_p\{\omega_p\}\cdot\{\lambda_1\}\cdot\{D\} . \tag{100}$$

By comparing this expression with Eq. (97), we deduce that an Sp(3,R) unirrep with lowest weight λ will occur $\Sigma_p m_p\times\rho_\lambda$ times, where ρ_λ is the multiplicity of the U(3) highest weight λ in the reduction of the U(3) product $\omega_p\otimes\lambda_1$; i.e., if

$$\langle\lambda\rangle\mapsto\sum_p m_p\{\omega_p\} \quad \text{and} \quad \{\omega_p\}\cdot\{\lambda_1\}=\sum_\lambda \rho_\lambda\{\lambda\} , \tag{101}$$

then

$$\langle\tilde{\lambda}\rangle\cdot\langle\lambda_1\rangle=\sum_{p\lambda} m_p\rho_\lambda\langle\lambda\rangle . \tag{102}$$

Lemma 8: Let $T^{\tilde{\lambda}}$ be an sp(3,R) tensor operator, and let V^{λ_1} be defined as usual. Let $W^k\subset W$ be the sp(3,R) invariant subspace of $W=T^{\tilde{\lambda}}\otimes V^{\lambda_1}$ containing the lowest weight of grade k. Then, the lowest grade subspace $W_k^k\subset W^k$ is given, in the limit where $\lambda_1^3\rightarrow\infty$, by

$$W_k^k\rightarrow T_{k-[\lambda_1]}^{\tilde{\lambda}}\otimes V_{[\lambda_1]}^{\lambda_1} , \tag{103}$$

where $T_{k-[\lambda_1]}^{\tilde{\lambda}}$ is the subspace of $T^{\tilde{\lambda}}$ spanned by the components of $T^{\tilde{\lambda}}$ with grade $k-[\lambda_1]$.

Proof: Let $\{A_{ij},B_{ij},[A_{ij},B_{ij}]\}$ denote a basis for $\mathscr{A}_{ij}$, the sp(1,R) subalgebra of sp(3,R) generated by A_{ij} and B_{ij}. Since W_k^k is a lowest grade subspace of the sp(3,R) invariant subspace $W^k\subset W$, it follows that $B_{ij}|\Psi\rangle=0$ for any $|\Psi\rangle\in W_k^k$. Thus W_k^k is spanned by a set of states which are all lowest weight states for unirreps of $\mathscr{A}_{ij}$. If $|\Psi\rangle$ is such a lowest weight state, it is, by Lemma 7, a product $|\Psi\rangle=T_q^{\tilde{\lambda}}\ |\psi\rangle$ of a state $|\psi\rangle\in\sigma_{ij}$ and a component of $T^{\tilde{\lambda}}$, where

$$\sigma_{ij}=\{|\psi\rangle\in V^{\lambda_1};\ \ B_{ij}|\psi\rangle=0\} . \tag{104}$$

Since this result holds for all $\mathscr{A}_{ij}$, and since

$$V_{[\lambda_1]}^{\lambda_1}=\{|\psi\rangle\ ;B_{ij}|\psi\rangle=0\ ;\forall\ ij\} , \tag{105}$$

it follows that, in the asymptotic limit, every state in W_k^k lies in $T_{k-[\lambda_1]}^{\tilde{\lambda}}\otimes V_{[\lambda_1]}^{\lambda_1}$, and the proof is complete.

Now, if α labels the components $\{T^{\tilde{\lambda}}_{\gamma_p\omega_p\alpha}\}$ of the U(3) tensor $T^{\tilde{\lambda}}_{\gamma_p\omega_p}$ whose components are a subset of components of the sp(3,R) tensor $T^{\tilde{\lambda}}$, we have

$$T^{\tilde{\lambda}}_{\gamma_p\omega_p\alpha}|\psi^{\lambda_1}_\beta\rangle \in W^k_k, \quad k=[\omega_p]+[\lambda_1] , \tag{106}$$

where β labels basis states $\{|\psi^{\lambda_1}_\beta\rangle\}$ in $V^{\lambda_1}_{[\lambda_1]}$. Furthermore, since

$$[A_{ij},T^{\tilde{\lambda}}_{\gamma_p\omega_p\alpha}]=\sum_{\gamma'\omega'\beta'} c_{\gamma_p\omega_p\alpha,\gamma'\omega'\beta'}T^{\tilde{\lambda}}_{\gamma'\omega'\beta'} , \tag{107}$$

and, for finite values of $\tilde{\lambda}$, the coefficients $c_{\gamma_p\omega_p\alpha,\gamma'\omega'\beta'}$ are finite for all values of the indices, we immediately find that

$$A_{ij}T^{\tilde{\lambda}}_{\gamma_p\omega_p\alpha}|\psi^{\lambda_1}_\beta\rangle \rightarrow T^{\tilde{\lambda}}_{\gamma_p\omega_p\alpha}\, A_{ij}|\psi^{\lambda_1}_\beta\rangle , \quad \forall|\psi^{\lambda_1}_\beta\rangle \in V^{\lambda_1} , \tag{108}$$

since, as seen before, $\|A_{ij}|\psi\rangle\| \rightarrow \infty$ in the $\lambda_1^3 \rightarrow \infty$ limit. From this, we now have

Theorem 3: In the $\lambda_1^3 \rightarrow \infty$ limit of λ_1,

$$W^k_{k+q} \rightarrow T^{\tilde{\lambda}}_{k-[\lambda_1]} \otimes V^{\lambda_1}_{[\lambda_1]+q} . \tag{109}$$

Proof: The proof is once again inductive starting this time with Lemma 8 and using Eq. (108) to step between the graded subspaces. The details are omitted.

Using the theorem, we find, in the notation of Eq. (62), that

$$|\Psi^{\rho\lambda}_{\gamma\omega\tau}\rangle \rightarrow \phi(\lambda_1,\omega_p;\sigma\lambda)\sum_{\beta\omega_1} \phi(\omega_p,\omega_1;\gamma\omega) \quad U(\omega_2\lambda_1\omega n;\lambda\sigma\alpha,\omega_1\beta\gamma)$$
$$\times[T^{\tilde{\lambda}}_{\gamma_p\omega_p}\otimes|\psi^{\lambda_1}_{n\beta\omega_1}\rangle]_{\gamma\omega\tau} \quad \delta_{\rho,\gamma_p\omega_p\sigma} \quad \delta_{\gamma,n\alpha}, \quad \text{as} \quad \lambda_1^3 \rightarrow \infty , \tag{110}$$

where the coupling is a u(3) coupling, and U is a Racah recoupling coefficient for u(3).

VI. DISCUSSION AND CONCLUSION

In this paper we have investigated the properties of asymptotic representations by looking at matrix elements of some ladder operators of semisimple Lie algebras. Ladder operators are naturally associated with gradings of representations. Thus we have made use of the fact that a contraction, corresponding to an asymptotic limit $\lambda \rightarrow \infty$, preserves a suitably defined graded structure of a Lie algebra and its ladder representations.

The formalism singles out a subalgebra $\mathscr{H} \subset \mathscr{G}$ which, by construction, contains all the ladder operators of $\mathscr{G}$ with finite matrix elements. We have shown that, in the asymptotic limit as $\lambda_1 \rightarrow \infty$, the basis states $\{|\Psi^{\rho\lambda}_{\gamma\omega\tau}\rangle\}$, for which $[\lambda]-[\omega]$ is finite, for the tensor product space $W=V^{\lambda_2} \otimes V^{\lambda_1}$, depend only on the coupling and recoupling coefficients of $\mathscr{H}$. Thus a major result of this paper is given by Eq. (62)

$$|\Psi^{\rho\lambda}_{\gamma\omega\tau}\rangle \rightarrow \phi(\lambda_1,\omega_2;\sigma\lambda)\sum_{\beta\omega_1} \phi(\omega_2,\omega_1;\kappa\omega)U(\omega_2\lambda_1\omega n;\lambda\sigma\alpha,\omega_1\beta\omega)$$
$$\times[|\varphi^{\lambda_2}_{\gamma_2\omega_2}\rangle\otimes|\psi^{\lambda_1}_{n\beta\omega_1}\rangle]_{\gamma\omega\tau} \quad \delta_{\rho,\gamma_2\omega_2\sigma} \quad \delta_{\gamma,n\alpha}, \text{ as } \lambda_1 \rightarrow \infty ,$$

where $\phi(\lambda_1,\omega_2;\sigma\lambda)$ and $\phi(\omega_2,\omega_1;\gamma\omega)$ are phase factors and $U(\omega_2\lambda_1\omega n;\lambda\sigma\alpha,\omega_1\beta\gamma)$ is a Racah coefficient for $\mathscr{H}$. In deriving this explicit expression, we have made use of a result, given in Sec. III C, that the arbitrariness of separating multiple copies of equivalent irreducible subrepresentations in the tensor product $\lambda_2\otimes\lambda_1$ has a natural resolution in the $\lambda_1 \rightarrow \infty$ limit.

In the $\lambda_1 \to \infty$ limit, a unirrep of a semisimple Lie algebra $\mathcal{G}$ approaches a (possibly reducible) representation of a contraction $\mathcal{G}^c$ of $\mathcal{G}$. Consider, for example, a representation of the su(2) Lie algebra

$$[J_+, J_-] = 2J_0, \quad [J_0, J_\pm] = \pm J_\pm \tag{111}$$

with angular momentum j and basis states $\{|jm\rangle\}$. Let $\mathcal{J}_\pm$ and $\mathcal{I}$ denote the renormalized operators

$$\mathcal{J}_\pm = \frac{J_\pm}{\sqrt{2j}}, \quad \mathcal{I} = \frac{J_0}{j}. \tag{112}$$

In terms of these operators, the su(2) commutation relations become

$$[\mathcal{J}_+, \mathcal{J}_-] = \mathcal{I}, \quad [\mathcal{I}, \mathcal{J}_\pm] = \pm \frac{\mathcal{J}_\pm}{j}. \tag{113}$$

Thus, in the asymptotic limit as $j \to \infty$, we obtain the İnönü–Wigner[12] contraction of su(2) to a Heisenberg–Weyl algebra with

$$[\mathcal{J}_+, \mathcal{J}_-] = \mathcal{I}, \quad [\mathcal{I}, \mathcal{J}_\pm] = 0. \tag{114}$$

The latter algebra is more usually expressed in terms of harmonic oscillator raising and lowering operators

$$[c, c^\dagger] = I, \quad [\mathcal{I}, c] = [I, c^\dagger] = 0. \tag{115}$$

Note also that we still have the commutation relation

$$[J_0, \mathcal{J}_\pm] = \pm \mathcal{J}_\pm \tag{116}$$

which can be compared with the harmonic oscillator equations

$$[H, c^\dagger] = c^\dagger, \quad [H, c] = -c, \tag{117}$$

with $H = c^\dagger c$.

The way in which the states $\{|jm\rangle\}$ of the su(2) representation approach those of a harmonic oscillator is given by the identification

$$|jm\rangle \equiv |n\rangle, \quad \text{with} \quad n = j - m. \tag{118}$$

As $j \to \infty$, we have, for small values of n,

$$\mathcal{J}_+ |n\rangle = \sqrt{\frac{1}{2j} n(2j+1)}\, |n-1\rangle \to \sqrt{n}\, |n-1\rangle,$$

$$\mathcal{J}_- |n\rangle = \sqrt{\frac{1}{2j}(2j-n)(n+1)}\, |n+1\rangle \to \sqrt{n+1}\, |n+1\rangle, \qquad \mathcal{J}_0 |n\rangle = n|n\rangle. \tag{119}$$

Thus we obtain the correspondence

$$\mathcal{J}_+ \to c, \quad \mathcal{J}_- \to c^\dagger, \quad \mathcal{I} \to I, \quad \mathcal{J}_0 \to c^\dagger c \tag{120}$$

valid whenever $n \ll j$. In terms of special functions, the contraction of spherical harmonics to Hermite polynomials can be found in Ref. 13.

Note that such a contraction of a semisimple Lie algebra is not semisimple. Since representations of the non-semisimple algebra can be obtained as asymptotic limits of those of the semisimple algebra $\mathcal{G}$, we can *define* the coupling of an irrep of $\mathcal{G}^c$ to an irrep of $\mathcal{G}$ as the asymptotic limit of the coupling of irreps of $\mathcal{G}$. Asymptotic coupling coefficients of $\mathcal{G}$ are precisely the coupling coefficients for the coupling $\mathscr{G}^c \times \mathscr{G} \rightarrow \mathscr{G}^c$. Furthermore, the renormalization of the Lie algebra implies a renormalization of the representation labels of $\mathscr{G}$, so that the labels specifying representations of $\mathscr{G}^c$ can take finite values. Thus we have the remarkable fact that the asymptotic coupling coefficients become exact for $\mathscr{G}^c \times \mathscr{G} \rightarrow \mathscr{G}^c$, even for finite values of the representation labels of $\mathscr{G}^c$.

The Clebsch–Gordan coefficients for the coupling of two irreps, $\lambda_1 \otimes \lambda_2$, of $\mathscr{G}^c$ are given precisely by the double limit $\lambda_1, \lambda_2 \rightarrow \infty$ of Clebsch–Gordan coefficients for $\mathscr{G}$. Unfortunately, it appears difficult to compute such coefficients within the current formalism.

In a previous work,[13] we obtained asymptotic Clebsch–Gordan coefficients for the su(3) $\supset$ so(3) subalgebra chain by a projection method. This projection method is different from the technique presented in Sec. III E. Nevertheless, it shares a ''factorization'' property with the current work. To be precise, it is implicit in Ref. 14 that

$$\frac{\| Q_0 | \varphi \rangle \|}{\| Q_0 | \psi \rangle \|} \rightarrow 0 \qquad \text{as } \lambda_1^1 \rightarrow \infty , \tag{121}$$

where $|\psi\rangle \in V^{\lambda_1}$ and $|\varphi\rangle \in V^{\lambda_2}$ as usual, and where Q_0 is the $\nu = 0$ component of an so(3) $\subset$ su(3) tensor operator. This leads immediately to the factorization

$$Q_0 | \Psi \rangle \rightarrow | \varphi \rangle [Q_0 | \psi \rangle] \qquad \text{as } \lambda_1^1 \rightarrow \infty , \tag{122}$$

which can be compared with Eq. (42) for components of $\mathscr{H} \subset \mathscr{G}$ tensors. However, unlike the $\mathscr{H} \subset \mathscr{G}$ tensors, not all components of the su(3) quadrupole moment Q_ν satisfy Eq. (121) in general. Thus it would be interesting to know if asymptotic coupling always implies such a factorization of some matrix elements and if, conversely, such a factorization implies asymptotic coupling.

Asymptotic Clebsch–Gordan coefficients have so far been used to analyze coupled systems having two different scales. For instance, in some core-plus-particle models of the nucleus, the so-called ''collective'' part of multipole operators can dwarf the single particle contribution, so that Eqs. (42) or (122) are true to a first approximation. The eigenfunctions of such two-scale systems are often found, to leading order, by using the Born–Oppenheimer (BO) approximation.[15] This suggests a useful parallel between the physically insightful BO approach and the mathematical technique of asymptotic coupling.[16] It remains to see if this parallel extends to higher orders, i.e., if corrections to the BO wave functions have a corresponding group-theoretical interpretation in terms of corrections to the asymptotic limit.

Finally, an obvious question which remains unanswered in the present work is the evaluation and properties of asymptotic coupling coefficients for finite values of the grade $[\omega]$. In Ref. 14, we computed su(2)$\supset$u(1) Clebsch–Gordan coefficients for finite value of the projection m in the limit where $j \rightarrow \infty$. This was done by embedding the u(1) subalgebra so that the set of states obtained from the highest weight state by the action of all elements of the corresponding U(1) subgroup spanned the whole su(2) representation. In contrast, the theory and examples presented in the present paper are closely related to the work found in Ref. 8, where the subalgebra $\mathscr{H}$ does not connect states in a unitary representation of $\mathscr{G}$ having different grades. This suggests that different embeddings of $\mathscr{H}$, when equivalent, will yield results applicable to different ranges of values of $[\omega]$.

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