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Graded contractions of Lie algebras and central extensions*

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Abstract. We use the concept of *grading* of Lie algebras to investigate the appearance of central charges during the contraction process. As for the usual graded contractions, one finds simultaneously the central extensions of classes of algebras, rather than specific Lie algebras. To illustrate the method, we consider in detail two physical applications: the kinematical algebras of spacetime and the $u(n)$-bosons limits of some Lie algebras.

1. Introduction

The objective of this paper is to generalize the method of graded contractions [1, 2] to include, using again the concept of Lie gradings [3], contractions with central charges. Although we will be primarily interested in contracting semisimple Lie algebras, our method can, in principle, be applied to non-semisimple Lie algebras, as well as Lie superalgebras and infinite-dimensional Lie algebras (just like the usual graded contractions [1]).

Contractions are important in physics as they explain formally why some theories arise as a limit regime of more 'complete' theories (see [4] and references therein). The paradigm is the passage from the Poincaré algebra to the Galilei algebra, in the limit where the speed of light approaches infinity [5]. Similarly, the de Sitter algebra can be contracted to the Poincaré algebra in the limit where the radius of the universe is large. Other examples include the $so(3)$ algebra of rotations, which contracts to translations for small angular displacements, and the dynamical algebra $sp(2n, \mathbb{R})$ of harmonic oscillators in n dimensions, which contracts to the $u(n) \oplus hw(\frac{1}{2}n(n+1))$ algebra (where $hw(m)$ is the mth Heisenberg–Weyl algebra) describing collective excitations at low energy: the $n = 3$ case is discussed in [6], whereas a realization applicable to specific problems in atomic physics was obtained in [7] for $n = 1$ and 2. From these examples, one can see that the existence of an 'approximate' or 'effective' theory can often be related to a contraction.

The importance of central charges in physics is likewise well appreciated [8–12]. Their appearance in a Lie algebra can be the counterpart of the presence of non-trivial phases in a projective representation of the corresponding group. We note, for instance, that the so-called 'Schwinger term' (associated with anomalies) of conformal field theories can be associated with the central charge of the current algebra of the underlying theory [13, 14]. This term is crucial, for example, in the construction of Wess–Zumino–Witten models [15].

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Mathematically, the general problem of finding the central charges 'sits' halfway between the contraction procedure, where commutators that were originally non-vanishing are set to zero, and the opposite procedure of *deformation*, where initially vanishing commutators become non-zero. No method of 'graded deformations' has been developed so far, and our scope is actually more modest.

Our first motivation is to understand in terms of gradings the central charges in kinematical algebras. It was found in [16] that all physically possible kinematics, described by Bacry and Lévy-Leblond using very general assumptions, are $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded contractions of either de Sitter algebras [17]. However, only some of the contractions can be extended in a non-trivial manner by a central element. We will show how one can anticipate this using grading arguments.

Perhaps the most pleasant result of our method occurs for the $(2+1)$-dimensional Galilei algebra. When considered on its own, this algebra admits three different central charges, one of which must eventually be eliminated by considering the transformations at the group level. Within our formalism, which considers the Galilei algebra as a contraction of the de Sitter algebra, we find that this same charge cannot possibly occur at the algebra level. This illustrates that our results will, in general, differ from those obtained had we considered the problem of finding central charges *in vacuo*, i.e. without reference to another, uncontracted algebra.

For a given fixed grading, not all contractions of a given algebra can be obtained as contractions preserving that chosen grading. In a similar way, we do not expect (and we must emphasize this) that our method will give all the central charges while simultaneously preserving the graded structure of the algebra. In fact, it is precisely the focus of this paper to determine how the graded structure constrains charges in an uncontracted algebra, such as de Sitter or Poincaré, to become (or remain) non-trivial in the contracted algebra, such as the Galilei algebra.

The second motivation is to understand in terms of gradings the contractions called in [6] $u(n)$-bosons limits, which include the families of contractions $su(n+1) \to u(n) \oplus hw(n)$, $sp(2n) \to u(n) \oplus hw(\frac{1}{2}n(n+1))$, $so(2n) \to u(n) \oplus hw(\frac{1}{2}n(n-1)$ and $so(2n+1) \to u(n) \oplus hw(n) \oplus hw(\frac{1}{2}n(n-1))$. An example, which has applications in nuclear collective motion [6], is the $sp(6,\mathbb{R}) \to u(3) \oplus hw(6)$ contraction. These contractions preserve a grading which is a $\mathbb{Z}_3$ grading for the A_n, B_n and C_n series, and a $\mathbb{Z}_2 \otimes \mathbb{Z}_3$ grading for the D_n series. As is tradition with graded contractions, our equations do not depend on the particular algebra but only on the graded structure, so that we need only consider in detail the $su(n) \to u(n-1) \oplus hw(n-1)$ and $so(2n+1) \to u(n) \oplus hw(n) \oplus hw(\frac{1}{2}n(n-1))$ contractions. The analysis for the $u(n)$-boson limits of $so(2n)$ and $sp(2n)$ does not differ from the $su(n)$ case, nor does the analysis of the contractions of the real forms of these algebras.

Let us finally mention that the search for central charges within the framework of graded contractions has already been performed for some specific Lie algebras, and specific finest gradings by [18]. These authors have used a fixed finest grading in order to find all the graded contractions and then all their central extensions. They have then classified the extensions according to whether they can be obtained through a contraction or not. Our approach is different: we contract and look for central charges simultaneously by emphasizing the graded structure common to a family of algebras. We only obtain a subset of the solutions found in [18] because our gradings are, in general, coarser than theirs. The graded contraction equations that we find provide a preliminary sieve, as coarse or as fine as the grading itself, in the search for central extensions. This sieving process provides an interpretation, as coarse or as fine as the grading, as to why some charges do or do not appear during a contraction.

2. Graded contractions with central extensions

A *grading* of the Lie algebra $\mathcal{L}$ is a decomposition into subspaces labelled by μ:

$$\mathcal{L} = \bigoplus_{\mu\in\Gamma} \mathcal{L}_\mu \tag{1}$$

where μ takes on values in some index set Γ, such that [3]

$$[\mathcal{L}_\mu, \mathcal{L}_\nu] \subseteq \mathcal{L}_{\mu+\nu} \tag{2}$$

which means that if $x \in \mathcal{L}_\mu$ and $y \in \mathcal{L}_\nu$, then $[x, y]$ belongs to the subspace $\mathcal{L}_{\mu+\nu}$. The commutator of two elements $l_{(\mu,i)} \in \mathcal{L}_\mu$ (where μ is a grading index and i is a generator index) and $l_{(\nu,j)} \in \mathcal{L}_\nu$ is denoted by

$$[l_{(\mu,i)}, l_{(\nu,j)}] = \sum_k c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} l_{(\mu+\nu,k)} \tag{3}$$

where $c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)}$ are the structure constants of $\mathcal{L}$.

We now extend $\mathcal{L}$ to $\overline{\mathcal{L}}$ by adding the unit operator $\mathbb{1}$, so that the commutation relations for $\overline{\mathcal{L}}$ read

$$\begin{aligned} \left[l_{(\mu,i)}, \mathbb{1}\right] &= 0 \\ \left[l_{(\mu,i)}, l_{(\nu,j)}\right] &= \sum_k c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} l_{(\mu+\nu,k)} + \beta_{(\mu,i),(\nu,j)}\,\mathbb{1}. \end{aligned} \tag{4}$$

The *central charges* (or central parameters) $\beta_{(\mu,i),(\nu,j)}$ play the role of structure constants for the unit operator. The Jacobi identities force the structure constants to satisfy the quadratic conditions

$$\sum_l \left(c^{(\mu+\nu,l)}_{(\mu,i),(\nu,j)} c^{(\mu+\nu+\sigma,q)}_{(\mu+\nu,l),(\sigma,k)} + c^{(\nu+\sigma,l)}_{(\nu,j),(\sigma,k)} c^{(\mu+\nu+\sigma,q)}_{(\nu+\sigma,l),(\mu,i)} + c^{(\sigma+\mu,l)}_{(\sigma,k),(\mu,i)} c^{(\mu+\nu+\sigma,q)}_{(\sigma+\mu,l),(\nu,j)} \right) = 0 \tag{5}$$

and they constrain the β's to satisfy

$$\sum_l \left(c^{(\mu+\nu,l)}_{(\mu,i),(\nu,j)} \beta_{(\mu+\nu,l),(\sigma,k)} + c^{(\nu+\sigma,l)}_{(\nu,j),(\sigma,k)} \beta_{(\nu+\sigma,l),(\mu,i)} + c^{(\sigma+\mu,l)}_{(\sigma,k),(\mu,i)} \beta_{(\sigma+\mu,l),(\nu,j)} \right) = 0. \tag{6}$$

Now recall that the solutions to equations (6) are not unique. If one shifts the infinitesimal generators to

$$\tilde{l}_{(\mu,i)} \equiv l_{(\mu,i)} + \alpha_{(\mu,i)}\,\mathbb{1} \tag{7}$$

and then compares the commutator (4) with its 'shifted' version

$$[\tilde{l}_{(\mu,i)}, \tilde{l}_{(\nu,j)}] = \sum_k \tilde{c}^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} \tilde{l}_{(\mu+\nu,k)} + \tilde{\beta}_{(\mu,i),(\nu,j)}\,\mathbb{1} \tag{8}$$

one can see that $\tilde{c}^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} = c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)}$. Furthermore, if the set $\{\beta_{(\mu,i),(\nu,j)}\}$ is a solution of (6), then so is the set $\{\tilde{\beta}_{(\mu,i),(\nu,j)}\}$ defined by

$$\tilde{\beta}_{(\mu,i),(\nu,j)} = \beta_{(\mu,i),(\nu,j)} - \sum_k \alpha_{(\mu+\nu,k)}\, c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} \tag{9}$$

for any values of the 'shift parameters' $\alpha_{(\mu,j)}$. Two sets of central parameters $\{\beta_{(\mu,i),(\nu,j)}\}$ and $\{\tilde{\beta}_{(\mu,i),(\nu,j)}\}$ which can be related through (9) are called *equivalent* and, in particular, a parameter $\beta_{(\mu,i),(\nu,j)}$ corresponds to a *trivial* charge if it is equivalent to zero, i.e. if there exist shift parameters which make the corresponding $\tilde{\beta}_{(\mu,i),(\nu,j)}$ in (9) zero.

The β's in equations (4) may or may not be trivial in $\overline{\mathcal{L}}$. Since we are seeking to find which charges in $\mathcal{L}$ become non-trivial during the graded contraction, we must, initially at least, explicitly keep even such trivial charges in $\overline{\mathcal{L}}$.

A graded contraction with central extension of the $\mathcal{L}$ to the Lie algebra $\overline{\mathcal{L}}_{\varepsilon,\eta}$ involves two types of parameters. The parameters $\varepsilon_{\mu,\nu}$ control the contraction by scaling commutators in $\mathcal{L}$. The parameters $\eta_{\mu,\nu}$, which again depend only on grading indices, scale all the β's in a family of commutators, thereby controlling the possible appearance of central charges. Put all altogether, the initial Lie algebra $\mathcal{L}$ is extended to $\overline{\mathcal{L}}$, whose commutators are deformed into those of the algebra $\overline{\mathcal{L}}_{\varepsilon,\eta}$.

The commutators in $\overline{\mathcal{L}}$ are redefined into those of $\overline{\mathcal{L}}_{\varepsilon,\eta}$ as follows:

$$
\begin{aligned}
[l_{(\mu,i)}, l_{(\nu,j)}] \to [l_{(\mu,i)}, l_{(\nu,j)}]_{\varepsilon,\eta} &= \varepsilon_{\mu,\nu}[l_{(\mu,i)}, l_{(\nu,j)}] + \eta_{\mu,\nu}\beta_{(\mu,i),(\nu,j)}\,\mathbb{1} \\
&= \varepsilon_{\mu,\nu}\left(\sum_k c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} l_{(\mu+\nu,k)}\right) + \eta_{\mu,\nu}\beta_{(\mu,i),(\nu,j)}\,\mathbb{1}.
\end{aligned} \tag{10}
$$

The commutators in $\overline{\mathcal{L}}_{\varepsilon,\eta}$ will henceforth be written with the subscripts $[\cdot\,,\cdot]_{\varepsilon,\eta}$ in order to clearly distinguish them from the commutators in $\overline{\mathcal{L}}$, which have no subscripts. Note that the parameters $\varepsilon_{\mu,\nu}$ and $\eta_{\mu,\nu}$ are *symmetric* under permutation of μ and ν. In order for $\overline{\mathcal{L}}_{\varepsilon,\eta}$ to be a Lie algebra, the parameters are subject to constraints derived from the Jacobi identity:

$$
\begin{aligned}
0 &= [[l_{(\mu,i)}, l_{(\nu,j)}]_{\varepsilon,\eta}, l_{(\sigma,k)}]_{\varepsilon,\eta} + \text{cyclic permutations} \\
&= [\varepsilon_{\mu,\nu}[l_{(\mu,i)}, l_{(\nu,j)}] + \eta_{\mu,\nu}\beta_{(\mu,i),(\nu,j)}\,\mathbb{1}, l_{(\sigma,k)}]_{\varepsilon,\eta} + \text{cyclic permutations} \\
&= \varepsilon_{\mu,\nu}[[l_{(\mu,i)}, l_{(\nu,j)}], l_{(\sigma,k)}]_{\varepsilon,\eta} + \text{cyclic permutations} \\
&= \varepsilon_{\mu,\nu}\left(\varepsilon_{\mu+\nu,\sigma}[[l_{(\mu,i)}, l_{(\nu,j)}], l_{(\sigma,k)}] + \eta_{\mu+\nu,\sigma}\left(\sum_l c^{(\mu+\nu,l)}_{(\mu,i),(\nu,j)}\beta_{(\mu+\nu,l),(\sigma,k)}\right)\mathbb{1}\right) \\
&\quad + \text{cyclic permutations.}
\end{aligned} \tag{11}
$$

Taking into account the fact that the commutators $[\cdot,\cdot]$ of the original algebra $\overline{\mathcal{L}}$ already satisfy the Jacobi identities (5), one finds the usual equations determining graded contractions [1]:

$$
\varepsilon_{\mu,\nu}\varepsilon_{\mu+\nu,\sigma} = \varepsilon_{\nu,\sigma}\varepsilon_{\nu+\sigma,\mu}. \tag{12}
$$

From equation (11) and (6), we also find

$$
\varepsilon_{\mu,\nu}\eta_{\mu+\nu,\sigma} = \varepsilon_{\nu,\sigma}\eta_{\nu+\sigma,\mu} \tag{13}
$$

as a set of solutions of (11). Equations (12) and (13), together with (10), are the central result of this paper.

Before solving (13), one should note that they must be slightly modified in three special cases:

(a) a subspace $\mathcal{L}_\mu$ is empty;
(b) $[\mathcal{L}_\mu, \mathcal{L}_\nu] = 0$ (so that $c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} = 0$, for all i, j); and
(c) the charges $\beta_{(\mu,i),(\nu,j)}$ are forced to be 0 for all i, j.

Under such circumstances, the relevant term in equation (11) is zero and does not contribute to the sum: any product containing $\varepsilon_{\mu,\nu}$ or $\eta_{\mu,\nu}$ must be taken out of the relations (12) and (13). The parameters $\varepsilon_{\mu,\nu}$ or $\eta_{\mu,\nu}$ are then referred to as *irrelevant*. In [1], a grading containing an irrelevant parameter is referred to as being *non-generic*. A *generic* grading is such that no commutator in (2) vanishes identically.

In [1], it was shown that the non-zero ε's can often be renormalized to 1 for complex Lie algebras, and to 1 or -1 for real Lie algebras. This fact provides us with the possibility of relating different real forms through a graded contraction, and this notion has proven useful in the context of kinematical groups [16]. Thus, the rescaling of the structure constants of $\overline{\mathcal{L}}$ through $\overline{\mathcal{L}}_\mu \to \overline{\mathcal{G}}_\mu = a_\mu \overline{\mathcal{L}}_\mu$ leads to a rescaling of the ε's and η's as

$$\varepsilon'_{\mu,\nu} \equiv \frac{a_\mu a_\nu}{a_{\mu+\nu}} \varepsilon_{\mu,\nu} \qquad \text{and} \qquad \eta'_{\mu,\nu} \equiv a_\mu a_\nu \eta_{\mu,\nu}. \tag{14}$$

Finally, once we have found those $\eta_{\mu,\nu}$'s that are not necessarily zero, one must remove the trivial parameters through a transformation (7). As for the contractions of Lie algebras, a *non-generic* grading will, in general, allow more non-trivial solutions for the η's, since (13) then contains fewer equations.

To summarize, the algorithm is as follows:

1. Choose a grading of some Lie algebra $\mathcal{L}$.
2. Extend $\mathcal{L}$ to $\overline{\mathcal{L}}$ but keep the same grading. (The unit, which commutes with everything, is added to the $\mathcal{L}_0$ subspace).
3. Remove the appropriate terms in (12) when the grading is non-generic, and then solving the equations for the ε's.
4. Given a set of solution ε's, solve for the η's using the linear equations (13), after removing therein the terms containing irrelevant η's. This contracts $\overline{\mathcal{L}}$ to $\overline{\mathcal{L}}_{\varepsilon,\eta}$.
5. Substitute in (10) the solutions η. The trivial charges are then eliminated using, in equation (9), the deformed structure constants $\varepsilon_{\mu,\nu}\, c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)}$ and central parameters $\eta_{\mu,\nu}\, \beta_{(\mu,i),(\nu,j)}$ of the contracted algebra $\overline{\mathcal{L}}_{\varepsilon,\eta}$. Non-trivial charges necessarily appear when $\eta_{\mu,\nu} \neq 0$.

Step 2 is easy when $\mathcal{L}$ is a semisimple Lie algebra because then all the charges must be trivial. In other words, the central charge $\beta_{(\mu,i),(\nu,j)}$ can be written, using (7) and (9) with $\tilde{\beta} = 0$, as

$$\beta_{(\mu,i),(\nu,j)} = \sum_l c^{(\mu+\nu,l)}_{(\mu,i),(\nu,j)}\, \alpha_{(\mu+\nu,l)} \tag{15}$$

where $\alpha_{(\mu+\nu,l)}$ are numbers chosen so that $\beta_{(\mu,i),(\nu,j)}$ is real.

In the literature (see, for instance, [20]), one often finds equation (15) written in the form

$$\beta_{(\mu,i),(\nu,j)} = \Lambda([l_{(\mu,i)}, l_{(\nu,j)}]) \tag{16}$$

where Λ is a linear functional defined so that $\Lambda(l_{(\mu,i)}) = \alpha_{(\mu,i)}$. Please observe that this last equation involves a commutator in the original algebra $\mathcal{L}$. Using this, equation (10) takes the more 'symmetric' form

$$[l_{(\mu,i)}, l_{(\nu,j)}]_{\varepsilon,\eta} = \varepsilon_{\mu,\nu}[l_{(\mu,i)}, l_{(\nu,j)}] + \eta_{\mu,\nu}\, \Lambda([l_{(\mu,i)}, l_{(\nu,j)}])\, \mathbb{1}.$$

Although $\eta_{\mu,\nu} = \varepsilon_{\mu,\nu}$ is a solution to equation (6), it is not the *only* solution: we have a non-trivial central extension when the scalings $\varepsilon_{\mu,\nu}$ and $\eta_{\mu,\nu}$ are not the same so that the central parameter does not 'follow' the commutator.

We deduce from equation (15) that $\beta_{(\mu,i),(\nu,j)} = 0$ if $[l_{(\mu,i)}, l_{(\nu,j)}] = 0$ and that, if two subspaces commute, i.e. if $[\mathcal{L}_\mu, \mathcal{L}_\nu] = 0$, then the corresponding $\eta_{\mu,\nu}$ and $\varepsilon_{\mu,\nu}$ are irrelevant. Moreover, equation (15) also provides relations between charges. For instance, consider the following $\mathbb{Z}_3$ grading of $su(3)$:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_+ + \mathcal{L}_- \tag{17}$$

where

$$\begin{aligned} \mathcal{L}_0 &= \{E_{12}, E_{21}, E_{11} - E_{22}, 2E_{33} - E_{22} - E_{11}\} \\ \mathcal{L}_+ &= \{E_{23}, E_{13}\} \\ \mathcal{L}_- &= \{E_{32}, E_{31}\}. \end{aligned} \tag{18}$$

The commutation relations are given in equation (44). The commutator $[E_{13}, E_{21}]$, for instance, gives, in the notation of equation (4),

$$[E_{13}, E_{21}] = -E_{23} \rightarrow [l_{(1,13)}, l_{(0,21)}] = -l_{(1,23)} + \beta_{(1,13),(0,21)}\, \mathbb{1}. \tag{19}$$

From this, $\beta_{(1,13),(0,21)} = -\alpha_{(1,23)}$. Let $E_{11} - E_{22} = h_1$. We also have

$$[h_1, E_{23}] = -E_{23} \rightarrow [l_{(0,h_1)}, l_{(1,23)}] = -l_{(1,23)} + \beta_{(0,h_1),(1,23)}\, \mathbb{1} \tag{20}$$

and hence the relation

$$\beta_{(0,h_1),(1,23)} = -\alpha_{(1,23)} = \beta_{(1,13),(0,21)} \tag{21}$$

between these central parameters.

This can be generalized. We know, from the root diagram of a semisimple Lie algebra, that all the weight subspaces, with the exception of the zero-weight subspace, are of dimension $\leqslant 1$. Thus, if $[l_{(\mu,i)}, l_{(\nu,j)}] \neq 0$ and does not lie in the zero-weight subspace, the sum in (15) contains exactly one term. Quite generally then, if two commutators are proportional to the same element not in the zero-weight subspace, e.g. if $[l_{(\mu,i)}, l_{(\nu,j)}] = c^{(\mu+\nu,k)}_{(\mu,i),(\nu,j)} l_{(\mu+\nu,k)}$ and $[l_{(\mu',i')}, l_{(\nu',j')}] = c^{(\mu'+\nu',k)}_{(\mu',i),(\nu',j)} l_{(\mu'+\nu',k)}$ with $\mu + \nu = \mu' + \nu'$, then their respective charges $\beta_{(\mu,i),(\nu,j)}$ and $\beta_{(\mu',i'),(\nu',j')}$ are proportional to one another, as in equation (21). As all the applications discussed in this paper have as a starting point a semisimple Lie algebra, such relations will be extremely useful.

Before turning our attention to physical applications, we conclude this section with the simplest example of a solution, a $\mathbb{Z}_2$ grading, for which a general Lie algebra $\mathcal{L}$ decomposes into

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \tag{22}$$

with commutation relations expressed symbolically as

$$[\mathcal{L}_0, \mathcal{L}_0] = \mathcal{L}_0 \qquad [\mathcal{L}_0, \mathcal{L}_1] = \mathcal{L}_1 \qquad [\mathcal{L}_1, \mathcal{L}_1] = \mathcal{L}_0. \tag{23}$$

Then, equations (12) give the following possible solutions [1]:

$$\begin{aligned} &\varepsilon^I = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \qquad \varepsilon^{II} = \begin{pmatrix} 0 & 0 \\ & 0 \end{pmatrix} \qquad \varepsilon^{III} = \begin{pmatrix} 1 & 1 \\ & 0 \end{pmatrix} \\ &\varepsilon^{IV} = \begin{pmatrix} 0 & 0 \\ & 1 \end{pmatrix} \qquad \varepsilon^{V} = \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} \end{aligned} \tag{24}$$

where we have taken advantage of the symmetry $\varepsilon_{\mu,\nu} = \varepsilon_{\nu,\mu}$ to write ε in the form

$$\varepsilon = \begin{pmatrix} \varepsilon_{0,0} & \varepsilon_{0,1} \\ & \varepsilon_{1,1} \end{pmatrix}.$$

Equation (13) then reduces to

$$\begin{aligned} \varepsilon_{0,0}\eta_{0,1} &= \varepsilon_{0,1}\eta_{0,1} \\ \varepsilon_{0,1}\eta_{1,1} &= \varepsilon_{1,1}\eta_{0,0} \end{aligned} \tag{25}$$

and, solving for the η's, we obtain

$$\begin{aligned} &\eta^{I} = \begin{pmatrix} a & b \\ & a \end{pmatrix} \qquad \eta^{II} = \begin{pmatrix} a & b \\ & c \end{pmatrix} \qquad \eta^{III} = \begin{pmatrix} a & b \\ & 0 \end{pmatrix} \\ &\eta^{IV} = \begin{pmatrix} 0 & a \\ & b \end{pmatrix} \qquad \eta^{V} = \begin{pmatrix} a & 0 \\ & b \end{pmatrix} \end{aligned} \tag{26}$$

where a, b and c are free parameters. As it may be possible to eliminate some β's through a transformation of the type found in equation (9), two different sets of η's, *a priori* inequivalent, could yield identically extended algebras once such shift transformations have been performed.

3. $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ gradings and kinematical algebras

In this section, we determine the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ centrally extended contractions of the de Sitter algebras (in (2 + 1) and (3 + 1) dimensions) that lead to the kinematical algebras.

3.1. Generic $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded contractions

A $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ grading is a decomposition

$$\mathcal{L} = \mathcal{L}_{00} + \mathcal{L}_{01} + \mathcal{L}_{10} + \mathcal{L}_{11}. \tag{27}$$

In the applications that we have in mind, the first $\mathbb{Z}_2$ index gives the transformation properties of the generators under space inversion, whereas the second $\mathbb{Z}_2$ refers to the time reversal.

Equation (12) is given in [1] for the generic $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ grading, whereas (13) takes the form

$$\begin{aligned} \varepsilon_{00,00}\eta_{00,k} &= \varepsilon_{00,k}\eta_{00,k} \\ \varepsilon_{00,k}\eta_{k,k} &= \varepsilon_{k,k}\eta_{00,00} \\ \varepsilon_{00,01}\eta_{01,10} &= \varepsilon_{01,10}\eta_{00,11} = \varepsilon_{00,10}\eta_{01,10} \\ \varepsilon_{00,10}\eta_{10,11} &= \varepsilon_{10,11}\eta_{00,01} = \varepsilon_{00,11}\eta_{10,11} \\ \varepsilon_{00,11}\eta_{01,11} &= \varepsilon_{01,11}\eta_{00,10} = \varepsilon_{00,01}\eta_{01,11} \\ \varepsilon_{01,10}\eta_{11,11} &= \varepsilon_{10,11}\eta_{01,01} = \varepsilon_{01,11}\eta_{10,10} \\ \varepsilon_{01,01}\eta_{00,10} &= \varepsilon_{01,10}\eta_{01,11} \\ \varepsilon_{01,01}\eta_{00,11} &= \varepsilon_{01,11}\eta_{01,10} \\ \varepsilon_{10,10}\eta_{00,01} &= \varepsilon_{01,10}\eta_{10,11} \\ \varepsilon_{10,10}\eta_{00,11} &= \varepsilon_{10,11}\eta_{01,10} \\ \varepsilon_{11,11}\eta_{00,01} &= \varepsilon_{01,11}\eta_{10,11} \\ \varepsilon_{11,11}\eta_{00,10} &= \varepsilon_{10,11}\eta_{01,11} \end{aligned} \tag{28}$$

where $k = (01), (10), (11)$. This set of equations is maximal, in the sense that whenever one considers a non-generic grading, then one just has to remove the corresponding terms from the equations above. Taking into account the symmetry of ε and η, we shall write the solutions in the form

$$\varepsilon = \begin{pmatrix} \varepsilon_{00,00} & \varepsilon_{00,01} & \varepsilon_{00,10} & \varepsilon_{00,11} \\ & \varepsilon_{01,01} & \varepsilon_{01,10} & \varepsilon_{01,11} \\ & & \varepsilon_{10,10} & \varepsilon_{10,11} \\ & & & \varepsilon_{11,11} \end{pmatrix} \qquad \eta = \begin{pmatrix} \eta_{00,00} & \eta_{00,01} & \eta_{00,10} & \eta_{00,11} \\ & \eta_{01,01} & \eta_{01,10} & \eta_{01,11} \\ & & \eta_{10,10} & \eta_{10,11} \\ & & & \eta_{11,11} \end{pmatrix}. \tag{29}$$

3.2. Galilei algebra in (2 + 1) dimensions

In (2 + 1) dimensions, the de Sitter algebra $so(3, 1)$ is six dimensional, with commutation relations

$$\begin{aligned} &[J, P_1] = P_2 && [J, P_2] = -P_1 \\ &[J, K_1] = K_2 && [J, K_2] = -K_1 \\ &[H, P_i] = -K_i && [H, K_i] = -P_i \\ &[P_1, P_2] = J && [K_1, K_2] = -J \\ &[P_i, K_j] = -\delta_{ij} H \end{aligned} \tag{30}$$

where J is the angular momentum, H the energy, and P_i and K_i are the generators of translations and inertial transformations, respectively. Our $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ grading (27) decomposes $so(3, 1)$ into the subspaces

$$\mathcal{L}_{00} = \{J\} \qquad \mathcal{L}_{01} = \{H\} \qquad \mathcal{L}_{10} = \{P_1, P_2\} \qquad \mathcal{L}_{11} = \{K_1, K_2\}. \tag{31}$$

Then, using equation (10), one obtains $\overline{so(3, 1)}_{\varepsilon,\eta}$, with the deformed commutation relations

$$\begin{aligned} &[J, P_1]_{\varepsilon,\eta} = \varepsilon_{00,10} P_2 + \eta_{00,10}\alpha_{P_2} \mathbb{1} && [J, P_2]_{\varepsilon,\eta} = -\varepsilon_{00,10} P_1 - \eta_{00,10}\alpha_{P_1} \mathbb{1} \\ &[J, K_1]_{\varepsilon,\eta} = \varepsilon_{00,11} K_2 + \eta_{00,11}\alpha_{K_2} \mathbb{1} && [J, K_2]_{\varepsilon,\eta} = -\varepsilon_{00,11} K_1 - \eta_{00,11}\alpha_{K_1} \mathbb{1} \\ &[H, P_i]_{\varepsilon,\eta} = -\varepsilon_{01,10} K_i - \eta_{01,10}\alpha_{K_i} \mathbb{1} && [H, K_i]_{\varepsilon,\eta} = -\varepsilon_{01,11} P_i - \eta_{01,11}\alpha_{P_i} \mathbb{1} \\ &[P_1, P_2]_{\varepsilon,\eta} = \varepsilon_{10,10} J + \eta_{10,10}\alpha_J \mathbb{1} && [K_1, K_2]_{\varepsilon,\eta} = -\varepsilon_{11,11} J - \eta_{11,11}\alpha_J \mathbb{1} \\ &[P_i, K_j]_{\varepsilon,\eta} = -\varepsilon_{10,11}\delta_{i,j} H - \eta_{10,11}\delta_{i,j}\alpha_H \mathbb{1} \end{aligned} \tag{32}$$

where we have used the notational shortcuts $\alpha_{P_2} = \alpha_{(10,2)}$ and so forth, to denote the shift parameters.

The parameters $\varepsilon_{00,00}, \varepsilon_{00,01}, \varepsilon_{01,01}$ and their corresponding η's are irrelevant, and the appropriate terms must be removed from (28). The possible kinematical algebras found in [17] all have $\varepsilon_{00,10}$ and $\varepsilon_{00,11}$ equal to 1. Hence, equations (28) simplify to

$$\begin{aligned} &\eta_{01,10} = \varepsilon_{01,10}\eta_{00,11} \\ &\eta_{01,11} = \varepsilon_{01,11}\eta_{00,10} \\ &\varepsilon_{01,10}\eta_{11,11} = \varepsilon_{01,11}\eta_{10,10} \\ &\varepsilon_{10,10}\eta_{00,11} = \varepsilon_{10,11}\eta_{01,10} \\ &\varepsilon_{11,11}\eta_{00,10} = \varepsilon_{10,11}\eta_{01,11}. \end{aligned} \tag{33}$$

There is no restriction on the parameter $\eta_{10,11}$, which occurs in commutators of the type $[P_i, K_j]$.

The Galilei algebra is obtained from the de Sitter algebra by using the matrix

$$\varepsilon_{\mathrm{Gal}} = \begin{pmatrix} \emptyset & \emptyset & 1 & 1 \\ & \emptyset & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}.$$

This, in turns, leads to the matrix

$$\eta_{\mathrm{Gal}} = \begin{pmatrix} \emptyset & \emptyset & a & b \\ & \emptyset & 0 & a \\ & & 0 & m \\ & & & k \end{pmatrix}.$$

(As in [1], $\emptyset$ denotes an entry associated with an irrelevant parameter.) The labelling of lines and column in each matrix is (00), (01), (10), (11). In terms of explicit commutation relations, the use of $\varepsilon_{\mathrm{Gal}}$ and η_{Gal} transforms (32) into

$$\begin{aligned} &[J, P_1]_{\varepsilon,\eta} = P_2 + a\alpha_{P_2}\mathbb{1} && [J, P_2]_{\varepsilon,\eta} = -P_1 - a\alpha_{P_1}\mathbb{1} \\ &[J, K_1]_{\varepsilon,\eta} = K_2 + b\alpha_{K_2}\mathbb{1} && [J, K_2]_{\varepsilon,\eta} = -K_1 - b\alpha_{K_1}\mathbb{1} \\ &[H, P_i]_{\varepsilon,\eta} = 0 && [H, K_i]_{\varepsilon,\eta} = -P_i - a\alpha_{P_i}\mathbb{1} \\ &[P_1, P_2]_{\varepsilon,\eta} = 0 && [K_1, K_2]_{\varepsilon,\eta} = -k\,\alpha_J\mathbb{1} \\ &[P_i, K_j]_{\varepsilon,\eta} = -m\,\delta_{ij}\alpha_H\mathbb{1}. \end{aligned} \tag{34}$$

The charges $a\alpha_{P_i}$ and $b\alpha_{K_i}$ are clearly equivalent to trivial charges as they can be eliminated by shifting the translation P_i and K_i generators using (7). This leaves the non-trivial charges of the extended Galilei algebra as $k\,\alpha_J$ and $m\alpha_H\delta_{ij}$. Thus, we finally have

$$\begin{aligned} &[J, P_1]_{\varepsilon,\eta} = P_2 && [J, P_2]_{\varepsilon,\eta} = -P_1 \\ &[J, K_1]_{\varepsilon,\eta} = K_2 && [J, K_2]_{\varepsilon,\eta} = -K_1 \\ &[H, P_i]_{\varepsilon,\eta} = 0 && [H, K_i]_{\varepsilon,\eta} = -P_i \\ &[P_1, P_2]_{\varepsilon,\eta} = 0 && [K_1, K_2]_{\varepsilon,\eta} = -k\,\alpha_J\mathbb{1} \\ &[P_i, K_j]_{\varepsilon,\eta} = -m\,\delta_{ij}\alpha_H\mathbb{1}. \end{aligned} \tag{35}$$

This result is in accordance with the extended commutation relations found on p 240 of [20] (note that equation (3.29f) of this reference should read $[K_i, H] = P_i$), *except* in the following respect. In [20], the commutator $[J, H]$ in the Galilei algebra can be extended to $[J, H] = h\mathbb{1}$. However, it is shown that h must be zero if a finite rotation by $2\pi + \theta$ is to coincide with a rotation by θ. When the Galilei algebra is considered as a contraction of the de Sitter algebra, however, the central parameter $\eta_{00,11}\,\beta_{(J,H)}$ vanishes immediately because $\beta_{(J,H)}$, the central parameter for the commutator $[J, H]$ in the de Sitter algebra, is found to be necessarily zero from equation (15).

3.3. Kinematical algebras in (3 + 1) dimensions

The $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ grading of the de Sitter algebras $so(4, 1)$ and $so(3, 2)$ is

$$\begin{aligned} &\mathcal{L}_{00} = \{\boldsymbol{J}\}: && \text{three angular momentum operators} \\ &\mathcal{L}_{01} = \{H\}: && \text{one energy operator} \\ &\mathcal{L}_{10} = \{\boldsymbol{P}\}: && \text{three translation operators} \\ &\mathcal{L}_{11} = \{\boldsymbol{K}\}: && \text{three inertial transformations.} \end{aligned} \tag{36}$$

Hereafter, we are interested in the graded contractions of the de Sitter algebras, with commutation relations given generically by

$$\begin{array}{ll} [\boldsymbol{J},\boldsymbol{J}]=\boldsymbol{J} & \\ [\boldsymbol{J},\boldsymbol{P}]=\boldsymbol{P} & [\boldsymbol{J},\boldsymbol{K}]=\boldsymbol{K} \\ [H,\boldsymbol{P}]=\pm\boldsymbol{K} & [H,\boldsymbol{K}]=\boldsymbol{P} \\ [\boldsymbol{P},\boldsymbol{P}]=\pm\boldsymbol{J} & [P_i,K_j]=\delta_{ij}H \\ [\boldsymbol{K},\boldsymbol{K}]=-\boldsymbol{J} & \end{array} \tag{37}$$

where the upper sign applies to $so(4,1)$ and the lower sign to $so(3,2)$. Following the notation of [17], we let $[\boldsymbol{A},\boldsymbol{B}]=\boldsymbol{C}$ denote any one of the commutators $[A_i,B_j]=\epsilon_{ijk}C_k$ (ϵ_{ijk} is the usual fully antisymmetric tensor), while $[H,\boldsymbol{A}]=\boldsymbol{B}$ denotes $[H,A_i]=B_i$.

The modified commutators of $\overline{so(4,1)}_{\varepsilon,\eta}$ and $\overline{so(3,2)}_{\varepsilon,\eta}$ take the form [16, 17]

$$\begin{array}{ll} [\boldsymbol{J},\boldsymbol{J}]_{\varepsilon,\eta}=\varepsilon_{00,00}\boldsymbol{J}+\eta_{00,00}\alpha_{\boldsymbol{J}}\mathbb{1} & \\ [\boldsymbol{J},\boldsymbol{P}]_{\varepsilon,\eta}=\varepsilon_{00,10}\boldsymbol{P}+\eta_{00,10}\alpha_{\boldsymbol{P}}\mathbb{1} & [\boldsymbol{J},\boldsymbol{K}]_{\varepsilon,\eta}=\varepsilon_{00,11}\boldsymbol{K}+\eta_{00,11}\alpha_{\boldsymbol{K}}\mathbb{1} \\ [H,\boldsymbol{P}]_{\varepsilon,\eta}=\pm\varepsilon_{01,10}\boldsymbol{K}\pm\eta_{01,10}\alpha_{\boldsymbol{K}}\mathbb{1} & [H,\boldsymbol{K}]_{\varepsilon,\eta}=\varepsilon_{01,11}\boldsymbol{P}+\eta_{01,11}\alpha_{\boldsymbol{P}}\mathbb{1} \\ [\boldsymbol{P},\boldsymbol{P}]_{\varepsilon,\eta}=\pm\varepsilon_{10,10}\boldsymbol{J}\pm\eta_{10,10}\alpha_{\boldsymbol{J}}\mathbb{1} & [P_i,K_j]_{\varepsilon,\eta}=\varepsilon_{10,11}\delta_{ij}H+\eta_{10,11}\alpha_H\mathbb{1} \\ [\boldsymbol{K},\boldsymbol{K}]_{\varepsilon,\eta}=-\varepsilon_{11,11}\boldsymbol{J}-\eta_{11,11}\alpha_{\boldsymbol{J}}\mathbb{1}. & \end{array} \tag{38}$$

In all kinematical algebras, the relations that determine the existence of central charges are obtained from (28) by removing the terms that contain $\varepsilon_{00,01}$, $\varepsilon_{01,01}$, $\eta_{00,01}$ and $\eta_{01,01}$ (since $[\boldsymbol{J},H]=[H,H]=0$ always), and by setting $\varepsilon_{00,00}=\varepsilon_{00,10}=\varepsilon_{00,11}=1$:

$$\begin{array}{ll} \eta_{10,10}=\varepsilon_{10,10}\eta_{00,00} & \eta_{11,11}=\varepsilon_{11,11}\eta_{00,00} \\ \eta_{01,10}=\varepsilon_{01,10}\eta_{00,11} & \eta_{01,11}=\varepsilon_{01,11}\eta_{00,10} \\ \varepsilon_{10,10}\eta_{00,11}=\varepsilon_{10,11}\eta_{01,10} & \varepsilon_{11,11}\eta_{00,10}=\varepsilon_{10,11}\eta_{01,11} \\ \varepsilon_{01,10}\eta_{11,11}=\varepsilon_{01,11}\eta_{10,10}. & \end{array} \tag{39}$$

The parameter $\eta_{10,11}$, which occurs in the extension of the commutator $[\boldsymbol{P},\boldsymbol{K}]$, is the only parameter on which there are no conditions, as expected (see, for instance, [17]).

The Poincaré algebra is obtained from the de Sitter algebras by using the matrix

$$\varepsilon_{\text{Poin}}=\begin{pmatrix} 1 & \emptyset & 1 & 1 \\ & \emptyset & 0 & 1 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix}$$

which in turns leads to

$$\eta_{\text{Poin}}=\begin{pmatrix} a & \emptyset & b & c \\ & \emptyset & 0 & b \\ & & 0 & d \\ & & & a \end{pmatrix}.$$

Then equation (38) becomes

$$\begin{array}{ll} [\boldsymbol{J}, \boldsymbol{J}]_{\varepsilon,\eta} = \boldsymbol{J} + a\alpha_{\boldsymbol{J}} \mathbb{1} & \\ [\boldsymbol{J}, \boldsymbol{P}]_{\varepsilon,\eta} = \boldsymbol{P} + b\alpha_{\boldsymbol{P}} \mathbb{1} & [\boldsymbol{J}, \boldsymbol{K}]_{\varepsilon,\eta} = \boldsymbol{K} + c\alpha_{\boldsymbol{K}} \mathbb{1} \\ [H, \boldsymbol{P}]_{\varepsilon,\eta} = 0 & [H, \boldsymbol{K}]_{\varepsilon,\eta} = \boldsymbol{P} + b\alpha_{\boldsymbol{P}} \mathbb{1} \\ [\boldsymbol{P}, \boldsymbol{P}]_{\varepsilon,\eta} = 0 & [P_i, K_j]_{\varepsilon,\eta} = \delta_{ij} H + \delta_{ij} d\alpha_H \mathbb{1} \\ [\boldsymbol{K}, \boldsymbol{K}]_{\varepsilon,\eta} = -\boldsymbol{J} - a\alpha_{\boldsymbol{J}} \mathbb{1}. & \end{array} \tag{40}$$

As expected, all the charges are trivial as they can be absorbed by shifting the generators as in (7).

The Galilei, Newton–Hooke and static algebras are more interesting, since they have non-trivial central extensions. For instance, the contraction matrix that corresponds to the Galilei algebra is

$$\varepsilon_{\text{Gal}} = \begin{pmatrix} 1 & \emptyset & 1 & 1 \\ & \emptyset & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

and its η_{Gal} matrix is

$$\eta_{\text{Gal}} = \begin{pmatrix} a & \emptyset & b & c \\ & \emptyset & 0 & b \\ & & 0 & d \\ & & & 0 \end{pmatrix}.$$

In this case, only the commutator $[P_i, K_j]_{\varepsilon,\eta} = 0 + \delta_{ij} d\alpha_H \mathbb{1}$ can be extended in a non-trivial way. This result is in excellent agreement with [17], with the central charge being proportional to the mass operator.

One can also verify that for the Newton–Hooke and static algebras, defined by the matrices

$$\varepsilon_{\text{NH}} = \begin{pmatrix} 1 & \emptyset & 1 & 1 \\ & \emptyset & 1 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \qquad \text{and} \qquad \varepsilon_{\text{Stat}} = \begin{pmatrix} 1 & \emptyset & 1 & 1 \\ & \emptyset & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

the corresponding solutions,

$$\eta_{\text{NH}} = \begin{pmatrix} a & \emptyset & b & c \\ & \emptyset & c & b \\ & & 0 & d \\ & & & 0 \end{pmatrix} \qquad \text{and} \qquad \eta_{\text{Stat}} = \begin{pmatrix} a & \emptyset & b & c \\ & \emptyset & 0 & 0 \\ & & 0 & d \\ & & & 0 \end{pmatrix}$$

also reproduce the extended commutation relations given in [17].

4. $u(n-1)$-bosons limits

4.1. $\mathbb{Z}_3$ gradings and the $su(n) \to u(n-1) \oplus hw(n-1)$ limit

Let us decompose the Lie algebra $\mathcal{L}$ into

$$\mathcal{L} = \boldsymbol{n}_+ \oplus \mathcal{L}_0 \oplus \boldsymbol{n}_- \tag{41}$$

where $\mathcal{L}_0$ is a subalgebra and where $\boldsymbol{n}_\pm$ span, respectively, nilpotent Abelian subalgebras such that

$$\begin{aligned} &[\boldsymbol{n}_+, \boldsymbol{n}_+] = [\boldsymbol{n}_-, \boldsymbol{n}_-] = 0 \\ &[\mathcal{L}_0, \boldsymbol{n}_\pm] \subseteq \pm\boldsymbol{n}_\pm \\ &[\boldsymbol{n}_+, \boldsymbol{n}_-] \subseteq \mathcal{L}_0. \end{aligned} \tag{42}$$

Such a decomposition, which is a $\mathbb{Z}_3$ grading with $\mathcal{L}_0$ the zero grade subspace and $\mathcal{L}_\pm = \boldsymbol{n}_\pm$, occurs, for instance, when we consider the subalgebra chain $su(n) \supset u(n-1)$ with $\mathcal{L}$ and $\mathcal{L}_0$ the algebras $su(n)$ and $u(n-1)$, respectively. In this section, we will deal in detail with this example. An analysis of the $sp(2n-2) \to u(n-1) \oplus hw\left(\frac{1}{2}n(n-1)\right)$ and $so(2n-2) \to u(n-1) \oplus hw\left(\frac{1}{2}(n-1)(n-2)\right)$ contractions is identical to the $su(n)$ case if we choose in all instances $\mathcal{L}_0$ to be $u(n-1)$: the subspaces $\boldsymbol{n}_\pm$ then contain, respectively, raising and lowering operators with commutation relations having the structure given in equation (42). The difference between $su(n)$, $so(2n-2)$ and $sp(2n-2)$ lies in the dimension of the $\boldsymbol{n}_\pm$ subspaces. One should also note that, in all cases, the operators in $\boldsymbol{n}_\pm$ are the components of an irreducible tensor under the subalgebra $\mathcal{L}_0$.

Returning to $su(n)$, we first give a basis for the Lie algebra $u(n)$ in terms of n^2 operators

$$\{E_{ij}, i, j = 1, \ldots, n\} \tag{43}$$

with commutation relations

$$[E_{ij}, E_{kl}] = E_{il}\delta_{jk} - E_{kj}\delta_{il}. \tag{44}$$

A basis for $su(n)$ is extracted by selecting the subset of generators

$$\begin{array}{lll} E_{ij} \quad i > j = 1, \ldots, n & & \text{lowering operators} \\ h_i = E_{ii} - E_{i+1,i+1} & i = 1, \ldots, n-2 & n-2 \text{ Cartan operators} \\ W = (n-1)E_{nn} - \displaystyle\sum_{i=1}^{n-1} E_{ii} & & \text{last Cartan operator} \\ E_{ij} \quad j > i = 1, \ldots, n & & \text{raising operators.} \end{array} \tag{45}$$

The contents of each of the $\mathbb{Z}_3$-graded subspaces can now be given explicitly. $\mathcal{L}_0$ consists of a $u(n-1) \sim su(n-1) \oplus u(1)$ subalgebra with ladder operators $\{C_{ij} = E_{ij}, i \neq j = 1, \ldots, n-1\}$, together with the Cartan subalgebra of the initial $su(n)$, the first $n-2$ elements of which form the Cartan subalgebra of $su(n-1)$. The Cartan operator of $u(n-1)$ not in $su(n-1)$ is W. The $\mathcal{L}_\pm$ subspaces consist of the (commuting) raising and lowering operators $\{A_j = E_{jn}, j = 1, \ldots, n-1\}$ and $\{B_j = E_{nj}, j = 1, \ldots, n-1\}$, respectively.

We now consider the contraction of $su(n)$ where only the commutation relations between elements of the subalgebra $\mathcal{L}_0 \sim u(n-1)$ remain unchanged, whereas everything else is forced to commute. In terms of ε, this amounts to setting $\varepsilon_{01} = \varepsilon_{0,-1} = \varepsilon_{1,-1} = 0$, but keeping $\varepsilon_{00} = 1$. The parameters ε_{11} and $\varepsilon_{-1,-1}$ are irrelevant. One can check that these values are solutions of (12) once the irrelevant parameters have been removed. The corresponding ε matrix is given by

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ & \emptyset & 0 \\ & & \emptyset \end{pmatrix} \tag{46}$$

where the lines and columns are ordered according to the $\mathbb{Z}_3$ grading labels $0, 1, -1$.

To investigate the possible central charges associated with this contraction, we must solve for the η's. Observing that only ε_{00} is non-zero, we find that the only non-trivial relations between the ε and the η are obtained only when two in the triple of indices (μ, ν, σ) are 0. Choosing $\mu = \nu = 0$ therefore yields

$$\varepsilon_{00}\eta_{0\sigma} = \varepsilon_{0\sigma}\eta_{0\sigma} = 0 \to \eta_{0\sigma} \qquad \sigma \neq 0 \tag{47}$$

with no conditions on η_{00} or $\eta_{1,-1}$. The parameters η_{11} and $\eta_{-1,-1}$ are irrelevant since the commutators $[A_i, A_j] \sim [\mathcal{L}_1, \mathcal{L}_1]$ and $[B_i, B_j] \sim [\mathcal{L}_{-1}, \mathcal{L}_{-1}]$ are zero before the contraction. Thus we have the solution matrix

$$\eta = \begin{pmatrix} x & 0 & 0 \\ & \emptyset & y \\ & & \emptyset \end{pmatrix} \tag{48}$$

where x and y are arbitrary parameters.

The commutators in $\overline{\mathcal{L}}_{\varepsilon,\eta}$ are now given by

$$\begin{aligned}
\left[A_j, B_i\right]_{\varepsilon,\eta} &= y\beta_{(+,j),(-,i)} && (i, j \neq n) \\
\left[A_j, C_{ik}\right]_{\varepsilon,\eta} &= 0 && (i, j, k \neq n) \\
\left[B_i, C_{kj}\right]_{\varepsilon,\eta} &= 0 && (i, j, k \neq n) \\
\left[C_{jk}, C_{li}\right]_{\varepsilon,\eta} &= \delta_{kl}C_{ji} - \delta_{ij}C_{lk} + x\beta_{(0,ji),(0,li)} && (i, j, k, l \neq n).
\end{aligned} \tag{49}$$

To complete the algorithm of section 2, we consider first the $u(n-1)$ commutators of the type $[C_{jk}, C_{li}]_{\varepsilon,\eta}$. Choose an $su(n-1) \oplus u(1)$ basis in this subspace, with W as the $u(1)$ generator. If $[C_{jk}, C_{li}]_{\varepsilon,\eta}$ is a commutator of two $su(n-1)$ elements, then the corresponding $\beta_{(0,jk),(0,li)}$ is either zero or equivalent to zero because $su(n-1)$ is semisimple. If we have a commutator of the type $[W, C_{jk}]$, then the corresponding $\beta_{(0,jk),(li)}$ is necessarily zero by equation (9) because W commutes with every element in $su(n-1)$. Thus, since x is arbitrary,

$$\beta_{(0,jk),(0,li)} \sim 0 \qquad \forall\ (jk), (li) \tag{50}$$

and we have

$$\left[C_{jk}, C_{li}\right]_{\varepsilon,\eta} = \left[C_{jk}, C_{li}\right]. \tag{51}$$

Consider now the commutators $\left[A_j, B_i\right]_{\varepsilon,\eta}$. Suppose first that $i \neq j$. If $\overline{\mathcal{L}}'$ is considered by itself we then have no reason to eliminate the charge $y\beta_{(+,j),(-,i)}$. However, because we consider $\overline{\mathcal{L}}'$ as a contraction of $\mathcal{L}$, there are further constraints (of the type found in equation (21)) on $\beta_{(+,j),(-,i)}$; we will show that $\beta_{(+,j),(-,i)}$ is, in fact, equivalent to zero. Indeed, using (15), we have $\beta_{(+,j),(-,i)} = \alpha_{(0,ji)}$, the shift of an $su(n-1) \subset \mathcal{L}_0$ ladder operator, which in turn is equal to $\beta_{(0,h_k),(0,ji)}$, which has just been shown in equation (50) to be trivial. On the other hand, when $i = j$ we have

$$\beta_{(+,i),(-,i)} = \alpha_{(0,ii)} - \alpha_{(0,nn)} \tag{52}$$

which can be expressed as a combination of shifts of Cartan operators from the semisimple algebra $su(n-1) \subset \mathcal{L}_0$, all of which are equivalent to zero, plus a shift for the operator W, which is *not* in $su(n-1)$. In fact, there is no commutator $[E_{ij}, E_{kl}]_{\varepsilon,\eta}$ in $\mathcal{L}'$ that will yield something proportional to W since it is in the centre of $u(n-1)$: the structure constants

$\varepsilon_{\mu,\nu} c^{(0,W)}_{(\mu,i),(\nu,j)}$ are all zero, so that the charge $\beta_{(+,i),(-,i)} \sim \alpha_{(0,WW)}$ cannot be eliminated by a transformation of the type found in (9) and is therefore non-trivial. Thus, we can write

$$\left[A_j, B_i\right]_{\varepsilon,\eta} = \delta_{ij}\, y\alpha_{(0,WW)} \tag{53}$$

which are the commutation relations of $hw(n-1)$. Combining (51) and (53), we see that, when $su(n)$ is contracted using the contraction matrix of equation (46), the only possible central charge appears so as to deform $su(n)$ into the direct sum $u(n-1) \oplus hw(n-1)$.

The same reasoning can be repeated for the $sp(2n) \to u(n) \oplus hw(\frac{1}{2}n(n+1))$ and $so(n) \to u(n) \oplus hw(\frac{1}{2}n(n-1))$ contractions: since the graded structure of these is identical to that of $su(n)$, and since the structure of the ε matrix governing the contraction is the same as in equation (48), the equations linking the η's and β's will be identical, as will be the final solutions.

4.2. The $so(2n+1) \to u(n) \oplus hw(n) \oplus hw(\frac{1}{2}n(n-1))$ contractions as $\mathbb{Z}_2 \otimes \mathbb{Z}_3$ graded contractions

Suppose again that

$$\mathcal{L} = \boldsymbol{n}_+ \oplus \mathcal{L}_0 \oplus \boldsymbol{n}_- \qquad \mathcal{L}_0 \subset \mathcal{L}$$

but that, this time

$$\begin{aligned} &[\boldsymbol{n}_+, \boldsymbol{n}_+] \subseteq \boldsymbol{n}_+ && [\boldsymbol{n}_+, [\boldsymbol{n}_+, \boldsymbol{n}_+]] = 0 \\ &[\boldsymbol{n}_-, \boldsymbol{n}_-] \subseteq \boldsymbol{n}_- && [\boldsymbol{n}_-, [\boldsymbol{n}_-, \boldsymbol{n}_-]] = 0 \\ &[\boldsymbol{n}_-, \boldsymbol{n}_+] \subseteq \mathcal{L}_0. \end{aligned} \tag{54}$$

Such a decomposition occurs for $so(2n+1)$ with $\mathcal{L}_0 = u(n)$. Because the subalgebras $\boldsymbol{n}_\pm$ are now nilpotent of order three, we will require a refinement of the grading found in the previous section.

The $so(2n+1)$ algebra is naturally $\mathbb{Z}_2$ graded, the zero-grade subspace being the $so(2n)$ subalgebra spanned by generalized angular momentum operators $\{L_{ij}, i = 1, \ldots, 2n\}$, antisymmetric in ij, and the grade-one subspace spanned by the extra generators

$$L_{2n+1,2i-1} \qquad L_{2n+1,2i} \qquad i = 1, \ldots, n \tag{55}$$

which are again antisymmetric under exchange of indices. The $\mathbb{Z}_2$ grading property (23) can be verified from the commutation relations

$$\left[L_{ij}, L_{kl}\right] = -\mathrm{i}(\delta_{jk} L_{il} - \delta_{jl} L_{ik} + \delta_{il} L_{jk} - \delta_{ik} L_{jl}). \tag{56}$$

This grading is well known; whereas operators in the zero-grade subspace can be realized as fermion pair operators (i.e. bosons),

$$A_{ij} \sim b_i^\dagger b_j^\dagger \qquad C_{ij} \sim \tfrac{1}{2}(b_i^\dagger b_j - b_j b_i^\dagger) \qquad B_{ij} \sim A_{ij}^\dagger \tag{57}$$

the operators in the grade-one subspace are realized in terms of single fermions,

$$\mathcal{A}_i \sim \frac{1}{\sqrt{2}} b_i^\dagger \qquad \mathcal{B}_i \sim \mathcal{A}_i^\dagger \tag{58}$$

provided that the operators b_i and $b_j^\dagger$ satisfy the usual fermionic anticommutation relations:

$$\{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0 \qquad \{b_i, b_j^\dagger\} = \delta_{ij}. \tag{59}$$

This $\mathbb{Z}_2$ grading of $so(2n+1)$ was explicitly exploited in [19] (see equation (3.31) therein). We now need to refine this grading so as to decompose further the zero-grade subspace, which contains an $so(2n)$ subalgebra. The $\mathbb{Z}_3$ grading appropriate for the analysis of $so(2n)$ is similar to the one that we used in the $su(n)$ example of the previous section. Thus, the grading underlying the analysis of $so(2n+1)$ is a $\mathbb{Z}_2 \otimes \mathbb{Z}_3$ grading:

$$\mathcal{L} = \mathcal{L}_{00} \oplus \mathcal{L}_{01} \oplus \mathcal{L}_{0,-1} \oplus \mathcal{L}_{1,-1} \oplus \mathcal{L}_{11} \tag{60}$$

with the subspace $\mathcal{L}_{10}$ empty.

These subspaces are spanned by

$$\begin{aligned}
&\mathcal{L}_{00} = \{C_{ij}, 1 \leqslant i, j \leqslant n\} && C_{ij} = \tfrac{1}{2}(L_{2i-1,2j} + L_{2i,2j-1} + \mathrm{i}L_{2i,2j} + \mathrm{i}L_{2i-1,2j-1}) \\
&\mathcal{L}_{01} = \{A_{ij}, 1 \leqslant i, j \leqslant n\} && A_{ij} = \tfrac{1}{2}(L_{2i-1,2j} + L_{2i,2j-1} + \mathrm{i}L_{2i,2j} - \mathrm{i}L_{2i-1,2j-1}) \\
&\mathcal{L}_{0,-1} = \{B_{ij}, 1 \leqslant i, j \leqslant n\} && B_{ij} = \tfrac{1}{2}(L_{2i-1,2j} + L_{2i,2j-1} - \mathrm{i}L_{2i,2j} + \mathrm{i}L_{2i-1,2j-1}) \\
&\mathcal{L}_{11} = \{\mathcal{B}_i, 1 \leqslant i \leqslant n\} && \mathcal{B}_i = \frac{1}{\sqrt{2}}\left(L_{2n+1,2i} + \mathrm{i}L_{2n+1,2i-1}\right) \\
&\mathcal{L}_{1,-1} = \{\mathcal{A}_i, 1 \leqslant i \leqslant n\} && \mathcal{A}_i = \frac{1}{\sqrt{2}}\left(L_{2n+1,2i} - \mathrm{i}L_{2n+1,2i-1}\right).
\end{aligned} \tag{61}$$

This decomposition is presented explicitly for $so(5)$ in figure 1. It is a non-generic grading, the parameters $\varepsilon_{k,(10)}$, $\varepsilon_{(01),(01)}$, $\varepsilon_{(0,-1),(0,-1)}$, $\varepsilon_{(01),(1,-1)}$ and $\varepsilon_{(0,-1),(11)}$ being irrelevant. The subspaces $\mathcal{L}_{01}$ and $\mathcal{L}_{1,-1}$ form a set of raising operators where

$$[\mathcal{L}_{1,-1}, \mathcal{L}_{1,-1}] \subset \mathcal{L}_{01} \tag{62}$$

by virtue of the cyclicity modulo two and three, respectively, of the addition of the grading indices in $\mathbb{Z}_2$ and $\mathbb{Z}_3$. $\mathcal{L}_{01}$ forms an Abelian nilpotent subalgebra. The parallel observations hold for the set of lowering operators spanned by elements in the $\mathcal{L}_{0,-1}$ and $\mathcal{L}_{1,1}$ subspaces.

The subspace $\mathcal{L}_{00}$ spans a $u(n)$ subalgebra of $so(2n+1)$. It contains a semisimple part, the $su(n)$ subalgebra with ladder operators $\{C_{ij}, i \neq j = 1, \ldots, n\}$, together with the Cartan subalgebra of $so(2n+1)$, the first $n-1$ elements of which span the Cartan subalgebra of the

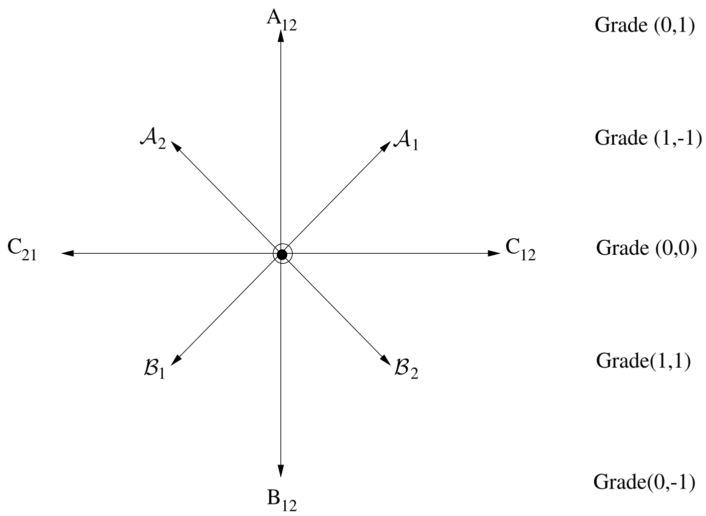

Figure 1. The root decomposition of $so(5)$ along with the $\mathbb{Z}_2 \otimes \mathbb{Z}_3$ grading labels.

aforementioned $su(n)$ semisimple algebra. Again, the Cartan operator of $\mathcal{L}_{00}$ that is not in $su(n)$ is the operator $W = \sum_{i=1}^{n} C_{ii}$ of equation (45).

To obtain the contraction $so(2n+1) \to u(n) \oplus hw(n) \oplus hw(\frac{1}{2}n(n-1))$, we must leave only the commutators $[\mathcal{L}_{00}, \mathcal{L}_{00}]$ unchanged, while all the others become zero. This can be realized in terms of a $\mathbb{Z}_2 \otimes \mathbb{Z}_3$ contraction in two steps, the first of which is to set any $\mathbb{Z}_2$ commutators of the type $[\mathcal{L}_{0i}, \mathcal{L}_{1j}]$ and $[\mathcal{L}_{1i}, \mathcal{L}_{1j}]$ to zero while leaving $[\mathcal{L}_{0i}, \mathcal{L}_{0j}]$ unchanged. This corresponds to the contraction matrix ε^V of (24). The second step appropriately contracts the $so(2n)$ subalgebra of the $\mathbb{Z}_2$ subspace labelled by 0. The contraction matrix was given in (46) of the previous section. The desired $\mathbb{Z}_2 \otimes \mathbb{Z}_3$ contraction is obtained by tensoring the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ solutions and corresponds to the solution matrix

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ & \emptyset & 0 \\ & & \emptyset \end{pmatrix}. \tag{63}$$

In this matrix, the lines and columns are ordered as $(00), (01), (0,-1), (1,0), (1,-1), (1,1)$. The fourth line and fourth column, corresponding to the subspace (10), should be removed from ε as the $(1,0)$ subspace is empty in the grading decomposition; this makes ε into a 5×5 matrix. Using the properties of tensor product of matrices, one can verify that a solution to (13) is given by the tensor product of the appropriate η matrices:

$$\eta = \begin{pmatrix} a & 0 \\ & b \end{pmatrix} \otimes \begin{pmatrix} x & 0 & 0 \\ & \emptyset & y \\ & & \emptyset \end{pmatrix} = \begin{pmatrix} ax & 0 & 0 & 0 & 0 & 0 \\ & \emptyset & ay & 0 & \emptyset & 0 \\ & & \emptyset & 0 & 0 & \emptyset \\ & & & bx & 0 & 0 \\ & & & & \emptyset & yb \\ & & & & & \emptyset \end{pmatrix} \tag{64}$$

where a, b, x, y are arbitrary parameters, provided that we again remove from this matrix the fourth line and fourth column corresponding to the subspace (10). Thus, we have

$$\eta_{(00),(00)} = ax \qquad \eta_{(01),(0,-1)} = ay \qquad \eta_{(1,1),(1,-1)} = yb. \tag{65}$$

The commutation relations now read

$$\begin{aligned} [A_{ij}, B_{kl}]_{\varepsilon,\eta} &= ay\beta_{(01,ij),(0-1,kl)} \\ [\mathcal{A}_i, \mathcal{B}_j]_{\varepsilon,\eta} &= yb\beta_{(1-1,i),(11,j)} \\ [C_{ij}, C_{kl}]_{\varepsilon,\eta} &= [C_{ij}, C_{kl}] + ax\beta_{(00,ij),(00,kl)}. \end{aligned} \tag{66}$$

The subspace $\mathcal{L}_{00}$ spans a $u(n)$ subalgebra. We can then repeat the argument that led to (51) to show that all the charges $\beta_{(00,ij),(00,kl)}$ are equivalent to zero. Furthermore, by repeating the argument that led to (53), we see that the only non-trivial charges occur when the commutators $[A_{ij}, B_{kl}]_{\varepsilon,\eta}$ and $[\mathcal{A}_i, \mathcal{A}_j]_{\varepsilon,\eta}$ are proportional to elements in the Cartan subalgebra, i.e.

$$[A_{ij}, B_{kl}]_{\varepsilon,\eta} = \delta_{ik}\delta_{jl}\, ay\beta_{(01,ij),(0-1,ij)} \qquad [\mathcal{A}_i, \mathcal{B}_j]_{\varepsilon,\eta} = \delta_{ij}\, yb\beta_{(1-1,i),(11,i)} \tag{67}$$

since otherwise the appropriate β's are proportional to shifts of generators in the semisimple subalgebra $su(n-1) \subset \mathcal{L}_0$, and therefore equivalent to zero.

Now consider

$$[A_{ij}, B_{ij}]_{\varepsilon,\eta} = ay\beta_{(01,ij),(0-1,ij)} = -ay(\alpha_{(00,ii)} + \alpha_{(00,jj)}). \tag{68}$$

This shift is a linear combination of a shift of Cartan operators in the semisimple part of $\mathcal{L}_0$ and a shift $\alpha_{(00,WW)}$ of the operator W not in this semisimple part. Eliminating the $su(n)$ shifts as they are equivalent to zero, we are left, in general, with

$$[A_{ij}, B_{kl}]_{\varepsilon,\eta} = -\delta_{ik}\delta_{jl}\, ay\alpha_{(00,WW)} \tag{69}$$

which are the commutation relations of the $hw(\frac{1}{2}n(n-1))$ algebra.

Finally, consider

$$[\mathcal{A}_i, \mathcal{B}_i]_{\varepsilon,\eta} = yb\beta_{(1-1,i),(11,i)} = yb\alpha_{(00,ii)}. \tag{70}$$

Once again, $\alpha_{(00,ii)}$ is a linear combination of shifts from the Cartan subalgebra of the semisimple $su(n)$ algebra and a shift from the trace W, which is not in the semisimple part of $\mathcal{L}_0$. After eliminating the former shifts as equivalent to zero, we find, in general,

$$[\mathcal{A}_i, \mathcal{B}_j]_{\varepsilon,\eta} = \delta_{ij}\, yb\alpha_{(00,WW)} \tag{71}$$

which are the commutation relations for the algebra $hw(n)$. Combining all of these, we find $\overline{\mathcal{L}}' \sim u(n) \oplus hw(n) \oplus hw(\frac{1}{2}n(n-1))$, as expected.

The solution η of equation (64) is *not* the most general solution to equations (6) coupling η and ε (as was the case in [1] for the ε's). The most general solution would have $\eta_{(01),(11)}$ and $\eta_{(0,-1),(1,-1)}$ non-zero. Our interpretation of our solution is that it corresponds to a sequence of deformations: first a $\mathbb{Z}_2$, then a $\mathbb{Z}_3$ contraction.

5. Discussion and conclusion

In this paper, we have introduced a way to generalize the theory of graded contractions in order to include central charges, and therefore generate central extensions, which have one more dimension than the original algebra. The method has been applied to two different physical settings. In the first example, we examined kinematical algebras as continuous (in the sense of [21]) contractions of the de Sitter algebras, whereas, in the second example, we considered $u(n)$-bosons limits as contractions, discontinuous in the sense of [21], of the classical algebras. The location of the central charges in $\overline{\mathcal{L}}_{\varepsilon,\eta}$ can be inferred from the grading decomposition of $\mathcal{L}$, which reflects the tensorial nature of the subspaces decomposing $\mathcal{L}$, the original uncontracted algebra. For the kinematical algebras, each $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ subspace carries a representation of the group $\Pi \otimes \Theta$ of space and time inversion. In the second case, each subspace carries an irreducible representation of the $u(n-1)$ subalgebra contained in the $\mathcal{L}_0$ or $\mathcal{L}_{00}$ subspace.

Non-trivial charges always occur, by construction, in the commutator of two commuting Abelian subalgebras in $\overline{\mathcal{L}}_{\varepsilon,\eta}$. Furthermore, we have $[l_{\mu,i}, l_{\nu,j}]_{\varepsilon,\eta} = \eta_{\mu,\nu}\,\beta_{(\mu,i),(\nu,j)} \neq 0$ if and only if (a) $l_{\mu+\nu,k}$ commutes with every element in $\mathcal{L}_0$ or $\mathcal{L}_{00}$, and (b) $[l_{\mu,i}, l_{\nu,j}] \neq 0$ in $\mathcal{L}$. Consider, for instance, the (2 + 1)-dimensional de Sitter algebra, with $\mathcal{L}_{00} = \{J\}$. This subspace trivially commutes with itself, and we can have $[K_1, K_2]_{\varepsilon,\eta} = -k\alpha_J\, \mathbb{1}$ in the Galilei algebra. In the (3 + 1)-dimensional case, however, $\mathcal{L}_{00} = \{\boldsymbol{J}\}$ no longer contains Abelian generators, and it is impossible to extend $[\boldsymbol{K}, \boldsymbol{K}]_{\varepsilon,\eta}$ in the (3 + 1)-dimensional Galilei algebra. A similar line of reasoning can be applied to the $u(n)$-boson limits: the only non-trivial central parameter occurs when a commutator in $\mathcal{L}$ is a linear combination of terms containing the operator W, which commutes with everything in the $u(n-1)$ subalgebra of either $\mathcal{L}_0$ or $\mathcal{L}_{00}$.

We believe that our formalism is obviously not limited to the examples presented in this paper. For instance, despite the fact that the interest in central extensions was originally related to representations of algebras, we have not considered such representations at all.

Also the method could be used to investigate the infinite-dimensional Lie algebras and Sugawara construction, as is done in [22] by using standard Wigner–Inönü contractions. An obvious continuation of this work is to study the deformations with central extensions at the group level. Other possibilities include the extensions by spaces of dimension larger than one.

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