Unitary transformations for testing Bell inequalities

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It is shown that optical experimental tests of Bell inequality violations can be described by SU(1,1) transformations of the vacuum state, followed by photon coincidence detections. The set of all possible tests are described by various SU(1,1) subgroups of Sp(8,$\mathbb{R}$). In addition to establishing a common formalism for physically distinct Bell inequality tests, the similarities and differences of post-selected tests of Bell inequality violations are also made clear. A consequence of this analysis is that Bell inequality tests are performed on a very general version of SU(1,1) coherent states, and the theoretical violation of the Bell inequality by coincidence detection is calculated and discussed. This group theoretical approach to Bell states is relevant to Bell state measurements, which are performed, for example, in quantum teleportation.

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I. INTRODUCTION

The controversy regarding the completeness of quantum mechanics [1] was presented in the framework of entangled spin-1/2 particles [2]. This context proved to be convenient for Bell's development of an inequality to test the postulates of local realism [3–6]. Recent quantum optics experiments, designed to test Bell inequalities, involve pairs of photons that are produced from the vacuum state, generally by optical parametric downconversion (PDC). PDC offers significant advantages over the earlier atomic cascade approach to generating photon pairs [7]; these advantages include conservation of energy (hence correlation frequencies of the two photons), conservation of linear momentum (hence correlated wavelengths and direction of propagation), and conservation of angular momentum (hence correlated polarizations), as well as near simultaneity of the emission of the two photons in the pair [8]. In addition to PDC acting as a source of correlated pairs of photons, there exists a scheme for which the photon pairs are in a polarization-entangled state [9]. PDC has enabled accurate tests of local realism versus quantum theory to be performed [9–18].

It is common to treat the input state for optical Bell inequality measurements as the singlet state

$$|\psi_{\text{singlet}}\rangle = (|+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle)/\sqrt{2}, \qquad (1.1)$$

corresponding to an entanglement of vertical (+) and horizontal (−) polarized photons in a net zero-angular-momentum state. However, PDC is not a perfect source of pairs of photons; one must account for the higher-order contributions due to more than two photons. Also, the time of emission of the correlated pair is random. These features of PDC will be shown to be accommodated in the group theoretic approach of applying an SU(1,1) transformation to the vacuum state [19,20].

The Bell inequality test is performed first by producing the photon pairs via PDC and then directing the photons through passive optical elements (beam splitters, phase shifters, polarizer rotators). These passive optical elements mix two bosonic fields at each stage and conserve photon number; such transformations are described as SU(2) transformations [19–21]. Thus the input vacuum state is subjected to an overall unitary transformation which can be decomposed into a sequence of SU(1,1) and SU(2) transformations to produce the final output state. This state is then subjected to photon coincidence measurements, and the constraints of local realism impose an upper bound on photon coincidence rates for various parameter choices. A violation of this upper bound corresponds to a violation of Bell's inequality and, hence, a test of local realism.

We shall see that it is natural to characterize Bell inequality experiments in terms of unitary transformations and to identify the Lie algebra which generates these transformations for particular Bell inequality experiments. We show that ideal Bell inequality experiments effect an SU(1,1) transformation, which is distinct from the SU(1,1) transformation that produces the photon pairs. Distinct ideal Bell inequality experiments can be identified with different SU(1,1) subgroups in Sp(8,$\mathbb{R}$).

In Sec. II, we describe tests of Bell's inequality and establish the mathematical framework necessary for studying such tests. The SU(2) transformations for passive optical elements and the SU(1,1) transformations for PDC are discussed. In Sec. III, we treat the ideal test of a Bell inequality by analyzing the experimental arrangement of an SU(1,1) PDC transformation followed by SU(2) passive optical elements. The result is that the ideal Bell inequality test arises as an $\mathrm{SU}(1,1)\subset\mathrm{Sp}(8,\mathbb{R})$ transformation of the vacuum state with some freedom to choose the applicable SU(1,1) transformation. An alternative realization of an ideal Bell inequality test is presented in Sec. III as well as an example of a post-selected form of testing Bell's inequality. Conclusions are presented in Sec. IV and include a brief discussion of the nature of the general SU(1,1) coherent state involved in Bell inequality tests.

II. BACKGROUND

A. Bell inequality test

In the standard Bell inequality test, a source produces a pair of entangled spin-1/2 particles. These two particles propagate in different directions and are detected by spatially separated detectors which can measure the spin state of each of the two particles along specified axes. An example of an entangled state is given by Eq. (1.1). We refer to the two spatially separated components (channels) as *a* and *b*, and the state may be subjected to simultaneous measurements of the spin states of *a* and *b* along preferred axes.

The Clauser-Horne-Shimony-Holt (CHSH) inequality version of the Bell inequality [4] introduces the figure of merit

$$S=\left|C(\theta_a,\theta_b)+C(\theta_a,\theta_b')+C(\theta_a',\theta_b)-C(\theta_a',\theta_b')\right|, \tag{2.1}$$

with θ_a,θ_a' describing measurement axes for system *a*, θ_b,θ_b' for system *b*, and $C(\theta_a,\theta_b)$ the correlation between *a* and *b* (with values in the range $[-1,+1]$). Local realism places a bound of 2 on *S*, giving the CHSH inequality,

$$S\leqslant 2 \quad \text{(for local realism)}, \tag{2.2}$$

and quantum mechanics predicts a violation of this inequality for certain quantum states [22]. For example, using the singlet state (1.1) with the values [23]

$$\theta_a-\theta_b=\theta_a'-\theta_b=\theta_a'-\theta_b'=\tfrac{1}{3}(\theta_a-\theta_b')=\pi/8, \tag{2.3}$$

one obtains a violation of the CHSH inequality of $S=2\sqrt{2}$.

Here we employ the CHSH inequality to investigate Bell inequality tests as unitary transformations. A detailed analysis of Bell inequalities requires consideration of the Clauser-Horne formulation of the inequality [24] and treatment of loopholes in the various experimental tests [25]. However, these issues are not directly relevant to this analysis, and the CHSH inequality suffices to consider an ideal bound on a system which is governed by local realism.

B. Algebra sp(8,$\mathbb{R}$) and its subalgebras

Although the Bell inequality test was devised using two spin-1/2 particles, we may use a boson representation to realize an optical version of the experiment. In this case there are four boson field modes to consider, each with a corresponding annihilation operator: $\hat{a}_+$ corresponding to the vertical polarization for the *a* spatial mode, $\hat{a}_-$ corresponding to the horizontal polarization for the *a* spatial mode, and annihilation operators $\hat{b}_\pm$ for the vertical and horizontal polarizations for the *b* spatial modes. There are thus four mutually commuting boson-operator pairs $\hat{a}_\pm,\hat{b}_\pm$ and their conjugates, which can be presented as

$$\hat{a}_+\to\hat{c}_1,\quad \hat{a}_-\to\hat{c}_2,\quad \hat{b}_+\to\hat{c}_3,\quad \hat{b}_-\to\hat{c}_4, \tag{2.4}$$

$$\hat{a}_+^\dagger\to\hat{c}_1^\dagger,\quad \hat{a}_-^\dagger\to\hat{c}_2^\dagger,\quad \hat{b}_+^\dagger\to\hat{c}_3^\dagger,\quad \hat{b}_-^\dagger\to\hat{c}_4^\dagger. \tag{2.5}$$

These operators obey the usual boson commutation rules

$$[\hat{c}_i,\hat{c}_j^\dagger]=\delta_{ij},\quad [\hat{c}_i,\hat{c}_j]=[\hat{c}_i^\dagger,\hat{c}_j^\dagger]=0. \tag{2.6}$$

An optical test of Bell's inequality can employ PDC, polarization rotation (where the spin-1/2 state corresponds to a polarization state of the photon), beam splitters, phase shifters, and mirrors as stages of the processing of the quantum state. Each of these stages can be represented mathematically as a unitary transformation provided that losses are neglected. The infinitesimal generators of these transformations consist of quadratic combinations of the operators (2.4) and (2.5), of the form $\hat{c}_i\hat{c}_j$, $\hat{c}_i^\dagger\hat{c}_j$, and $\hat{c}_i^\dagger\hat{c}_j^\dagger$. These quadratic operators span the complexification of the algebra sp(8,$\mathbb{R}$), with the standard basis ($i,j\in\{1,2,3,4\}$)

$$\hat{A}_{ij}=\hat{c}_i^\dagger\hat{c}_j^\dagger, \tag{2.7}$$

$$\hat{C}_{ij}=\tfrac{1}{2}(\hat{c}_i^\dagger\hat{c}_j+\hat{c}_j\hat{c}_i^\dagger), \tag{2.8}$$

$$\hat{B}_{ij}=\hat{c}_i\hat{c}_j. \tag{2.9}$$

These operators obey the (complexified) sp(8,$\mathbb{R}$) commutation relations

$$\begin{aligned}
&[\hat{A}_{ij},\hat{A}_{kl}]=0=[\hat{B}_{ij},\hat{B}_{kl}],\\
&[\hat{C}_{ij},\hat{C}_{kl}]=\delta_{jk}\hat{C}_{il}-\delta_{il}\hat{C}_{kj},\\
&[\hat{C}_{ij},\hat{A}_{kl}]=\delta_{jk}\hat{A}_{il}+\delta_{jl}\hat{A}_{ik},\\
&[\hat{C}_{ij},\hat{B}_{kl}]=-\delta_{il}\hat{B}_{jk}-\delta_{ik}\hat{B}_{jl},\\
&[\hat{A}_{ij},\hat{B}_{kl}]=-\delta_{ki}\hat{C}_{jl}-\delta_{kj}\hat{C}_{il}-\delta_{il}\hat{C}_{jk}-\delta_{jl}\hat{C}_{ik}.
\end{aligned} \tag{2.10}$$

Note that the generators $\{\hat{C}_{ij}\}$ span a complex u(4) subalgebra. This four-boson realization of the algebra sp(8,$\mathbb{R}$) provides the language with which to describe Bell inequality experiments (and many other optical experiments as well).

In optical versions of Bell inequality tests, the measurement of the coincidence rate $C(\theta_a,\theta_b)$ used in Eq. (2.1) for photons is to record simultaneous photodetections in spatial modes a and b. A convenient expression for the correlation function in terms of this four-boson realization is [26]

$$C(\theta_a,\theta_b)=\frac{\langle(\hat{a}_+^\dagger\hat{a}_+-\hat{a}_-^\dagger\hat{a}_-)(\hat{b}_+^\dagger\hat{b}_+-\hat{b}_-^\dagger\hat{b}_-)\rangle}{\langle(\hat{a}_+^\dagger\hat{a}_++\hat{a}_-^\dagger\hat{a}_-)(\hat{b}_+^\dagger\hat{b}_++\hat{b}_-^\dagger\hat{b}_-)\rangle}. \tag{2.11}$$

Strictly speaking, this expression is applicable to the CHSH inequality when the photon pair flux is sufficiently low that the probability of more than one pair of photons arriving at the detectors is negligible. The spontaneous generation of pairs by PDC permits a sufficiently short interval to be chosen, in principle, to ensure that higher-order terms (beyond the vacuum and photon pairs) can be neglected. The vacuum produces no coincidences and the coincidence rate is set to zero in this case. The normalization is trivial for the case of a single pair, with photons arriving at a and b detectors. The coincidence rate represented by Eq. (2.11) is appropriate for quantum optics experiments. We show in Sec. III A that the flux rate of photon pairs cancels via the denominator and, therefore, the flux rate does not appear in calculations of Bell's inequality.

The algebra sp(8,$\mathbb{R}$) contains many subalgebras that have physical significance in terms of quantum optics and Bell inequality tests. In the following, we identify certain subalgebras with optical transformations induced by beam splitters, phase shifters, polarization rotations, and PDC's.

C. Realizations of su(2) subalgebras

Many passive (i.e., photon number conserving) optical transformations can be described by various su(2) subalgebras in sp(8,$\mathbb{R}$). For example, many useful su(2) subalgebras can be realized as a two-boson realization for any $i\neq j$, given by

$$\hat{J}_x^{(ij)}=\tfrac{1}{2}(\hat{c}_i^\dagger\hat{c}_j+\hat{c}_i\hat{c}_j^\dagger),$$

$$\hat{J}_y^{(ij)}=\frac{1}{2i}(\hat{c}_i^\dagger\hat{c}_j-\hat{c}_i\hat{c}_j^\dagger),$$

$$\hat{J}_z^{(ij)}=\tfrac{1}{2}(\hat{c}_i^\dagger\hat{c}_i-\hat{c}_j^\dagger\hat{c}_j), \tag{2.12}$$

and satisfying $[\hat{J}_x^{(ij)},\hat{J}_y^{(ij)}]=i\hat{J}_z^{(ij)}$ with x,y,z cyclic.

Some of the realizations of these su(2) subalgebras correspond to (i) mixing of two modes (interactions of the type $\hat{a}_+^\dagger\hat{b}_++\hat{a}_+\hat{b}_+^\dagger$) via a beam splitter, (ii) mixing polarizations in one mode ($\hat{a}_+^\dagger\hat{a}_-+\hat{a}_+\hat{a}_-^\dagger$), and (iii) mixing both spatial modes and polarization modes ($\hat{a}_\pm^\dagger\hat{b}_\mp$).

Consider, for example, the polarization-independent beam splitter [21]. The generator associated to this optical device is

$$\hat{J}_{\rm BS}=\hat{J}_x^{(13)}+\hat{J}_x^{(24)}=\tfrac{1}{2}(\hat{a}_+\hat{b}_+^\dagger+\hat{a}_+^\dagger\hat{b}_++\hat{a}_-\hat{b}_-^\dagger+\hat{a}_-^\dagger\hat{b}_-). \tag{2.13}$$

The associated unitary transformation of a 50/50 polarization-independent beam splitter is

$$U_{\rm BS}=\exp[i(\pi/4)\hat{J}_{\rm BS}], \tag{2.14}$$

which is an element of the SU(2) subgroup corresponding to polarization-independent channel mixing [27]. As another example, the operator

$$\hat{J}_{\rm PS}=\hat{J}_z^{(13)}+\hat{J}_z^{(24)}=\tfrac{1}{2}(\hat{a}_+^\dagger\hat{a}_+-\hat{b}_+^\dagger\hat{b}_++\hat{a}_-^\dagger\hat{a}_--\hat{b}_-^\dagger\hat{b}_-) \tag{2.15}$$

describes a polarization-independent phase shifter and also generates a transformation in this same SU(2) subgroup.

As an example of mixing polarizations in one spatial mode, consider the operator

$$\hat{J}_a=\hat{J}_x^{(12)}=\tfrac{1}{2}(\hat{a}_+^\dagger\hat{a}_-+\hat{a}_+\hat{a}_-^\dagger). \tag{2.16}$$

This operator generates the unitary transformation

$$U_a(\theta_a)=\exp(i\theta_a\hat{J}_a), \tag{2.17}$$

which rotates the polarization in channel a by an angle θ_a and does not affect channel b; i.e., this transformation describes a polarization rotator of angle θ_a in the a channel.

The above are just some of the su(2) subalgebras used to describe lossless, passive optical elements: elements for which the total number of input quanta equals the total number of output quanta.

D. Realizations of su(1,1) subalgebras

The transformations associated with parametric downconversion are active; they create or annihilate pairs of photons. The Lie algebra su(1,1) has been shown [19,20] to describe these transformations.

In PDC, a crystal with a $\chi^{(2)}$ nonlinearity is pumped by a coherent field, wherein each pump photon spontaneously decays into a pair of photons. In *degenerate* PDC, the two photons in the pair are identical; in *nondegenerate* PDC, the pump photon decays into two nonidentical photons. For $(\mathbf{k}_i,\omega_i)$ the wave vector and angular frequency of the ith field, with $i=0$ for the pump field and $i=1,2$ for the two output fields, energy conservation yields $\omega_0=\omega_1+\omega_2$ and momentum conservation yields $\mathbf{k}_0=\mathbf{k}_1+\mathbf{k}_2$. For degenerate PDC, $\omega_1=\omega_2$ and $\mathbf{k}_1=\mathbf{k}_2$ [17].

For below-threshold operation, the pump field may be considered to be a classical field. Treating the pump field as classical allows the annihilation and creation operators for the pump field photon to be treated as c numbers and not as operators.

1. PDC and the algebra su(1,1)

By analogy with the beam splitter, which is described by an SU(2) transformation, PDC is described by an SU(1,1) transformation. A basis for the su(1,1) algebra is given by the set of operators $\{\hat{K}_x,\hat{K}_y,\hat{K}_z\}$, with commutation relations

$$[\hat{K}_x,\hat{K}_y]=-i\hat{K}_z,\quad [\hat{K}_y,\hat{K}_z]=i\hat{K}_x,\quad [\hat{K}_z,\hat{K}_x]=i\hat{K}_y. \tag{2.18}$$

For degenerate PDC, the appropriate realizations of su(1,1) are one-boson realizations given by the generators

$$\hat{K}_x^{(i)}=\tfrac{1}{4}(\hat{c}_i^\dagger\hat{c}_i^\dagger+\hat{c}_i\hat{c}_i),$$

$$\hat{K}_y^{(i)}=\frac{1}{4i}(\hat{c}_i^\dagger\hat{c}_i^\dagger-\hat{c}_i\hat{c}_i),$$

$$\hat{K}_z^{(i)}=\tfrac{1}{4}(\hat{c}_i^\dagger\hat{c}_i+\hat{c}_i\hat{c}_i^\dagger), \tag{2.19}$$

Here the annihilation operator $\hat{c}_i$ can refer to any of $\hat{a}_+$, $\hat{a}_-$, $\hat{b}_+$, or $\hat{b}_-$.

For the nondegenerate case, where PDC generates two nonidentical photons, the appropriate realizations of su(1,1) are two-boson realizations given by the generators

$$\hat{K}_x^{(ij)}=\tfrac{1}{2}(\hat{c}_i^\dagger\hat{c}_j^\dagger+\hat{c}_i\hat{c}_j),$$

$$\hat{K}_y^{(ij)}=\frac{1}{2i}(\hat{c}_i^\dagger\hat{c}_j^\dagger-\hat{c}_i\hat{c}_j),$$

$$\hat{K}_z^{(ij)}=\tfrac{1}{2}(\hat{c}_i^\dagger\hat{c}_i+\hat{c}_j\hat{c}_j^\dagger). \tag{2.20}$$

A type-I PDC is one for which, typically, $\hat{c}_i=\hat{a}_+$ and $\hat{c}_j=\hat{b}_+$, whereas, in a type-II PDC, $\hat{c}_i=\hat{a}_+$ and $\hat{c}_i=\hat{b}_-$. That is, a pair of photons is created in the same polarization in type-I PDC and a pair of photons is created in opposite polarizations in type-II downconversion.

It is also possible to design PDC's which generate entangled pairs [9]. Such a setup involves a type-II PDC, where the emission directions of the *a* and *b* channel photons are made to overlap, and is described by a four-boson realization of su(1,1) [28]. There are several such realizations, each of which describes the generation of a different entangled state. One example is given by the generators

$$\hat{K}_x=\tfrac{1}{2}(\hat{a}_+^\dagger\hat{b}_-^\dagger-\hat{a}_-^\dagger\hat{b}_+^\dagger+\hat{a}_+\hat{b}_--\hat{a}_-\hat{b}_+),$$

$$\hat{K}_y=\frac{1}{2i}(\hat{a}_+^\dagger\hat{b}_-^\dagger-\hat{a}_-^\dagger\hat{b}_+^\dagger-\hat{a}_+\hat{b}_-+\hat{a}_-\hat{b}_+),$$

$$\hat{K}_z=\tfrac{1}{2}(\hat{a}_+^\dagger\hat{a}_++\hat{b}_-\hat{b}_-^\dagger+\hat{a}_-^\dagger\hat{a}_-+\hat{b}_+\hat{b}_+^\dagger). \tag{2.21}$$

It is interesting to note that the above four-boson realization is a direct sum of two of the two-boson realizations in Eq. (2.20), with a sign change, as follows:

$$\hat{K}_x=\hat{K}_x^{(14)}-\hat{K}_x^{(23)},$$

$$\hat{K}_y=\hat{K}_y^{(14)}-\hat{K}_y^{(23)},$$

$$\hat{K}_z=\hat{K}_z^{(14)}+\hat{K}_z^{(23)}. \tag{2.22}$$

One can easily check that these generators also satisfy the commutation relations of su(1,1). Adjusting the parameters of the PDC (such as the relative phase) can lead to other similar four-boson realizations. It will be shown in the following that this particular realization generates an SU(1,1) transformation which describes the generation of the singlet state.

It is possible to design a different PDC that is also described by this four-boson realization [17] and which also generates entangled pairs. This setup, however, entangles the photons in wave number rather than polarization. Pairs of photons are selected by four pinholes in a diaphragm placed downstream from the PDC to produce four channels, labeled 1–4, with wave vectors $\{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4\}$. These wave vectors satisfy

$$|\mathbf{k}_1|=|\mathbf{k}_4|,$$

$$|\mathbf{k}_2|=|\mathbf{k}_3|,\quad \text{but}\quad |\mathbf{k}_1|\neq|\mathbf{k}_2|, \tag{2.23}$$

and

$$\mathbf{k}_1+\mathbf{k}_3=\mathbf{k}_2+\mathbf{k}_4=\mathbf{k}, \tag{2.24}$$

where $\mathbf{k}$ is the wave vector of the beam incident on the crystal.

Let the annihilation operators for these four wave numbers correspond to the ordered set $\{\hat{a}_+,\hat{b}_+,\hat{b}_-,\hat{a}_-\}$. Thus we are able to employ the earlier notation although the physical system is entirely different. The su(1,1) algebra describing this PDC is also given by the four-boson realization of Eq. (2.21).

2. Pair generation using PDC

The rate of pair creation using PDC is proportional to the nonlinearity $\chi^{(2)}$, the strength of the (classical) pump field, and the interaction time. In the following, we develop a one-parameter transformation which describes pair generation from the vacuum state for PDC.

Consider the action of an SU(1,1) transformation, generated by the realization corresponding to either the degenerate PDC algebra of Eq. (2.19) or the nondegenerate PDC algebra of Eq. (2.20), on the vacuum state $|0\rangle$. This state is an eigenstate of $\hat{K}_z$ and is annihilated by $\hat{K}_-=\hat{K}_x-i\hat{K}_y$, and thus by using a normal-ordered form, it is sufficient to express a general SU(1,1) transformation of the vacuum state as the one-parameter transformation

$$\Upsilon(\gamma)|0\rangle=\exp(i\gamma\hat{K}_x)|0\rangle,\quad \gamma\in\mathbb{R}. \tag{2.25}$$

The resultant state is not simply a pair of photons, but a superposition of photon number states which also includes the vacuum, pairs-of-pairs, and higher order contributions. For γ small, the resulting state can be approximated as

$$\Upsilon(\gamma)|0\rangle\approx|0\rangle+i\gamma\hat{K}_x|0\rangle. \tag{2.26}$$

The role of the vacuum in the superposition (2.26) is to include in the state the feature that the creation of the desired photon pair occurs at a random time. That is, the photon pair cannot be created ''on demand.'' Note that photon counting

does not detect the vacuum, so the inclusion of this state does not alter the final measurement process.

Consider, for example, the case of a type-I nondegenerate PDC, described by $\hat{K}_x^{(13)}=\frac{1}{2}(\hat{a}_+^\dagger \hat{b}_+^\dagger+\hat{a}_+\hat{b}_+)$. The resulting (approximate) state is

$$\Upsilon_{\text{type-I}}(\gamma)|0\rangle \approx |0\rangle+\frac{i}{2}\gamma|1,0,1,0\rangle, \tag{2.27}$$

where the Fock notation $|i,j,k,l\rangle$ describes a state with i,j,k,l photons in the a_+, a_-, b_+, b_- modes, respectively.

Provided that observations are conditioned on actually detecting photons, the vacuum state plays no role. Thus ''conditioning'' the state can be described by a projection $\hat{\pi}$, defined by

$$\hat{\pi}=|1,0,1,0\rangle\langle 1,0,1,0|+|1,0,0,1\rangle\langle 1,0,0,1| +|0,1,1,0\rangle\langle 0,1,1,0|+|0,1,0,1\rangle\langle 0,1,0,1|, \tag{2.28}$$

which projects any state in the Fock space onto the subspace of states with exactly one photon in channel a and one in channel b. Such a projection relies on photodetectors that can discriminate between one and more than one photon [29]. As the photon-pair flux rate is assumed to be small, contributions due to higher-order terms are negligible, and thus current photodetectors which do not discriminate between one and more photons are adequate.

By projecting the state in Eq. (2.27), we obtain the ''conditioned photon pair state'' $|1,0,1,0\rangle$. We thus have the requisite pair of correlated particles, but not an entangled state.

A relevant basis for the subspace of degenerate eigenstates of $\hat{\pi}$ is the so-called ''Bell state basis,'' given by

$$|\psi_\pm\rangle=\frac{1}{\sqrt{2}}(|1,0,0,1\rangle\pm|0,1,1,0\rangle),$$

$$|\phi_\pm\rangle=\frac{1}{\sqrt{2}}(|1,0,1,0\rangle\pm|0,1,0,1\rangle). \tag{2.29}$$

The state $|\psi_-\rangle$ is the singlet state $|\psi_{\text{singlet}}\rangle$ of Eq. (1.1). The other Bell states are equally suitable entangled states for testing the Bell inequality. It is desirable, in tests of the Bell inequality, to be able to generate entangled states such as these.

As an example of a realization of SU(1,1) that will generate an entangled (Bell) state, consider the four-boson realization given by Eq. (2.21). To lowest order in γ, we have

$$\Upsilon_{\text{singlet}}(\gamma)|0\rangle=\exp(i\gamma\hat{K}_x)|0\rangle \approx|0\rangle+\frac{i}{2}\gamma(|1,0,0,1\rangle-|0,1,1,0\rangle). \tag{2.30}$$

By applying the projection $\hat{\pi}$, the state reduces, ''conditioned'' on photons being present, to the singlet state $|\psi_-\rangle$. In the experimental setup of Kwiat *et al.* [9], a PDC described by this transformation has been shown to generate the singlet state. By adjusting the parameters of the PDC and performing local unitary transformations, any of the Bell states of Eq. (2.29) can be produced; the generators correspond to different four-boson realizations of su(1,1) similar to that of Eq. (2.21). It is also possible to use PDC to generate an entangled state in wave number, using the method described above and described by the same SU(1,1) transformation.

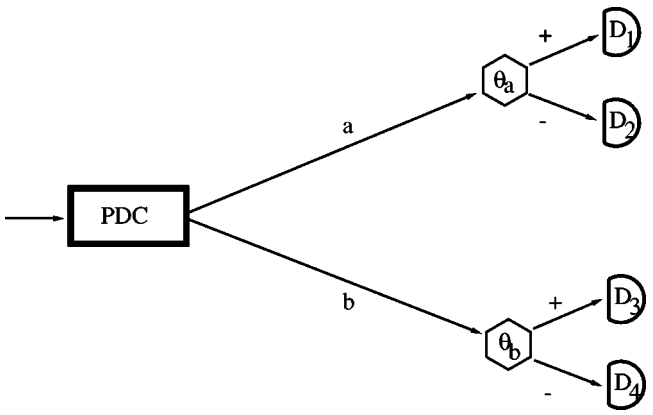

FIG. 1. Diagrammatic representation of the ideal Bell inequality experiment. The PDC which produces the singlet state is used. Channel a is passed through a polarization analyzer at angle θ_a and channel b through one at angle θ_b. Photodetectors D_1, D_2, D_3, and D_4 measure the corresponding photocounts.

III. REALIZATIONS OF THE BELL INEQUALITY TEST

A. Ideal Bell inequality test

In this section we construct simple transformations on the vacuum state which correspond to an ideal Bell inequality experiment. We establish the algebra which generates these transformations to be su(1,1) and calculate the quantum-mechanical correlation functions $C(\theta_a,\theta_b)$ for the corresponding state.

The ideal Bell inequality experiment is depicted in Fig. 1. This experiment has been performed by Kwiat *et al.* [9]. The first requirement is a device which produces the entangled photon pairs, thereby providing the necessary Bell state. The singlet state of Eq. (2.30) is obtained by using the projector $\hat{\pi}$ on a state produced via PDC described by the four-boson realization of Eq. (2.21). The presence of the vacuum state in the superposition signifies that the ''location,'' or ''creation time,'' of the singlet is indeterminate; singlet states are not created ''on demand.'' The SU(1,1) transformation $\Upsilon_{\text{singlet}}(\gamma)$ is generated by the operator

$$\hat{K}=\tfrac{1}{2}(\hat{a}_+^\dagger\hat{b}_-^\dagger-\hat{a}_-^\dagger\hat{b}_+^\dagger+\hat{a}_+\hat{b}_--\hat{a}_-\hat{b}_+). \tag{3.1}$$

Note that the Bell inequality test can be performed with any one of the four Bell states in Eq. (2.29); our choice of the singlet state is simply for convention. Thus this choice of $\hat{K}$ as the Bell state generator is not unique.

The PDC output is directed to local polarization rotators, one for the a mode and one for the b mode, each followed by a polarizing beam splitter. The polarizing beam splitter sepa-

rates the two orthogonal polarization components of the field and directs them to two photodetectors, which can count the photons in each of the two polarizations. We refer to the combination of the polarizer rotator, with an adjustable parameter $\theta_{a,b}$, and the polarizing beam splitter, which separates the two polarizations into distinct spatial modes, as a polarization analyzer. This polarization analyzer is depicted as the hexagon in Fig. 1.

Bell's inequality establishes an upper bound to the measurable photon coincidence rate allowed by local realistic assumptions for various choices of θ_a and θ_b of the two polarization analyzers. The polarizations are transformed independently by a $\mathrm{U}(1)_a \otimes \mathrm{U}(1)_b$ rotation, with two independent, local parameters θ_a and θ_b, with the following two mutually commuting generators. For polarization rotation of the a mode, the generator $\hat{J}_a$ of Eq. (2.16) is required; similarly, for the b mode, we require

$$\hat{J}_b = \hat{J}_x^{(34)} = \tfrac{1}{2}(\hat{b}_+^\dagger \hat{b}_- + \hat{b}_+ \hat{b}_-^\dagger). \tag{3.2}$$

Equal polarization rotations for modes a and b leave the singlet state invariant, as

$$[\hat{K}, \hat{J}_a + \hat{J}_b] = 0. \tag{3.3}$$

Thus it is only necessary [30] to consider a difference transformation $U_-(\theta_-)$ generated by $\hat{J} = \hat{J}_a - \hat{J}_b$, given by

$$U_-(\theta_-) = e^{i\theta_- \hat{J}_a} e^{-i\theta_- \hat{J}_b} = e^{i\theta_- \hat{J}}. \tag{3.4}$$

Note that the operators $\hat{J}$, $\hat{K}$, and

$$\hat{L} = \frac{1}{2i}(\hat{a}_-^\dagger \hat{b}_-^\dagger - \hat{a}_+^\dagger \hat{b}_+^\dagger - \hat{a}_- \hat{b}_- + \hat{a}_+ \hat{b}_+) \tag{3.5}$$

close under commutation to form a realization of su(1,1):

$$[\hat{J}, \hat{K}] = i\hat{L}, \quad [\hat{L}, \hat{J}] = i\hat{K}, \quad [\hat{K}, \hat{L}] = -i\hat{J}. \tag{3.6}$$

This realization of su(1,1) is distinct from any of the realizations describing PDC. This algebra generates the Lie group SU(1,1), which can be applied to the ground state to generate the state

$$|\gamma, \theta_-\rangle = U_-(\theta_-)\Upsilon_{\text{singlet}}(\gamma)|0\rangle = e^{i\theta_- \hat{J}} e^{i\gamma \hat{K}}|0\rangle. \tag{3.7}$$

The transformation of the ground state $|0\rangle$ consists of a PDC transformation to generate an entangled state, followed by local polarization rotations on the a and b modes by angles θ_- and $-\theta_-$, respectively. To lowest order, the state (3.7) is a superposition of a vacuum state and a two-photon state. Neglecting the vacuum state, the effective state is then a Bell state (2.29) if $\theta_- = 0$. However, for general θ_-, the two-photon contribution to the superposition is an entanglement of nonorthogonal SU(2) coherent states [31].

The correlation function $C(\theta_a, \theta_b)$ of Eq. (2.11) for the state $|\gamma, \theta_a - \theta_b\rangle$ is given by

$$C(\theta_a, \theta_b) = \frac{\langle \gamma, \theta_a - \theta_b | (\hat{\sigma}_z)_a (\hat{\sigma}_z)_b | \gamma, \theta_a - \theta_b \rangle}{\langle \gamma, \theta_a - \theta_b | (\hat{\sigma}_0)_a (\hat{\sigma}_0)_b | \gamma, \theta_a - \theta_b \rangle}, \tag{3.8}$$

where $(\hat{\sigma}_z)_a = \hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-$, $(\hat{\sigma}_0)_a = \hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_-$, and likewise for $(\hat{\sigma}_z)_b$ and $(\hat{\sigma}_0)_b$.

Note that, by using the approximation for γ small of Eq. (2.30) and ''conditioning'' the state on photons being present (i.e., excluding the vacuum state), the PDC generates the singlet state $|\psi_-\rangle$ of Eq. (2.29). Calculating the correlation function for this state, one obtains the familiar result

$$\begin{aligned} C(\theta_a, \theta_b) &= \langle \psi_- | U_-^{-1}(\theta_a - \theta_b)[(\hat{\sigma}_z)_a (\hat{\sigma}_z)_b] \\ &\quad \times U_-(\theta_a - \theta_b)|\psi_-\rangle \\ &= -\cos 2(\theta_a - \theta_b). \end{aligned} \tag{3.9}$$

The singlet state can lead to a violation $S = 2\sqrt{2}$ for the parameter choices (2.3).

There is an interesting su(1,1) structure to the correlation function $C(\theta_a, \theta_b)$, which we detail as follows. Beginning with the numerator, we first obtain the result

$$\begin{aligned} &U_-^{-1}(\theta_a - \theta_b)[(\hat{\sigma}_z)_a (\hat{\sigma}_z)_b] U_-(\theta_a - \theta_b) \\ &\quad = [\cos(\theta_a - \theta_b)(\hat{\sigma}_z)_a - \sin(\theta_a - \theta_b)(\hat{\sigma}_y)_a] \\ &\qquad \times [\cos(\theta_a - \theta_b)(\hat{\sigma}_z)_b + \sin(\theta_a - \theta_b)(\hat{\sigma}_y)_b]. \end{aligned} \tag{3.10}$$

Then consider the following change of basis:

$$\begin{aligned} &\hat{J}_z^+ = (\hat{\sigma}_z)_a + (\hat{\sigma}_z)_b, \quad \hat{J}_z^- = (\hat{\sigma}_z)_a - (\hat{\sigma}_z)_b, \\ &\hat{J}_y^+ = (\hat{\sigma}_y)_a + (\hat{\sigma}_y)_b, \quad \hat{J}_y^- = (\hat{\sigma}_y)_a - (\hat{\sigma}_y)_b. \end{aligned} \tag{3.11}$$

This basis transforms simply under the action of $\Upsilon_{\text{singlet}}(\gamma)$ as follows:

$$\begin{aligned} &\Upsilon_{\text{singlet}}^{-1}(\gamma)\hat{J}_z^+ \Upsilon_{\text{singlet}}(\gamma) = \hat{J}_z^+, \\ &\Upsilon_{\text{singlet}}^{-1}(\gamma)\hat{J}_y^+ \Upsilon_{\text{singlet}}(\gamma) = \hat{J}_y^+, \\ &\Upsilon_{\text{singlet}}^{-1}(\gamma)\hat{J}_z^- \Upsilon_{\text{singlet}}(\gamma) = \cosh(\gamma)\hat{J}_z^- + \sinh(\gamma)\hat{L}_z, \\ &\Upsilon_{\text{singlet}}^{-1}(\gamma)\hat{J}_y^- \Upsilon_{\text{singlet}}(\gamma) = \cosh(\gamma)\hat{J}_y^- + \sinh(\gamma)\hat{L}_y, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \hat{L}_z &= \frac{1}{2i}(\hat{a}_+^\dagger \hat{b}_-^\dagger + \hat{a}_-^\dagger \hat{b}_+^\dagger - \hat{a}_+ \hat{b}_- + \hat{a}_- \hat{b}_+), \\ \hat{L}_y &= \tfrac{1}{2}(\hat{a}_-^\dagger \hat{b}_-^\dagger + \hat{a}_+^\dagger \hat{b}_+^\dagger + \hat{a}_- \hat{b}_- + \hat{a}_+ \hat{b}_+). \end{aligned} \tag{3.13}$$

Evaluating the numerator of Eq. (3.8) gives

$$\langle \gamma, \theta_a - \theta_b | (\hat{\sigma}_z)_a (\hat{\sigma}_z)_b | \gamma, \theta_a - \theta_b \rangle$$
$$= \sinh^2(\gamma)[-\cos^2(\theta_a - \theta_b)\langle 0|\hat{L}_z^2|0\rangle$$
$$+ \sin^2(\theta_a - \theta_b)\langle 0|\hat{L}_y^2|0\rangle]$$
$$= -\tfrac{1}{2}\sinh^2(\gamma)\cos 2(\theta_a - \theta_b). \quad (3.14)$$

We have utilized the fact that the mixed terms such as $\sin(\theta_a-\theta_b)\cos(\theta_a-\theta_b)(\hat{\sigma}_z)_a(\hat{\sigma}_y)_b$, etc., in Eq. (3.10) do not contribute and also that the vacuum expectation values for the operators of the ''*J* type'' (of the form $\hat{c}_i^\dagger \hat{c}_j$) vanish.

Next, evaluating the denominator in a similar fashion, we first observe that

$$U_-^{-1}(\theta_a - \theta_b)[(\hat{\sigma}_0)_a(\hat{\sigma}_0)_b]U_-(\theta_a - \theta_b) = (\hat{\sigma}_0)_a(\hat{\sigma}_0)_b. \quad (3.15)$$

Again, consider the change of basis

$$\hat{N}_0^+ = (\hat{\sigma}_0)_a + (\hat{\sigma}_0)_b, \quad \hat{N}_0^- = (\hat{\sigma}_0)_a - (\hat{\sigma}_0)_b. \quad (3.16)$$

This basis transforms simply under the action of $\Upsilon_{\text{singlet}}(\gamma)$ as follows:

$$\Upsilon_{\text{singlet}}^{-1}(\gamma)\hat{N}_0^+\Upsilon_{\text{singlet}}(\gamma) = \cosh(\gamma)\hat{N}_0^+ + \sinh(\gamma)\hat{L}_0,$$
$$\Upsilon_{\text{singlet}}^{-1}(\gamma)\hat{N}_0^-\Upsilon_{\text{singlet}}(\gamma) = \hat{N}_0^-, \quad (3.17)$$

where

$$\hat{L}_0 = -\frac{1}{2i}(\hat{a}_+^\dagger \hat{b}_-^\dagger - \hat{a}_-^\dagger \hat{b}_+^\dagger - \hat{a}_+\hat{b}_- + \hat{a}_-\hat{b}_+). \quad (3.18)$$

Evaluating the denominator gives

$$\langle \gamma, \theta_a - \theta_b | (\hat{\sigma}_0)_a (\hat{\sigma}_0)_b | \gamma, \theta_a - \theta_b \rangle$$
$$= \sinh^2(\gamma)\langle 0|\hat{L}_0^2|0\rangle = \tfrac{1}{2}\sinh^2(\gamma). \quad (3.19)$$

Thus we find that the correlation $C(\theta_a, \theta_b)$ is given by

$$C(\theta_a, \theta_b) = -\cos 2(\theta_a - \theta_b). \quad (3.20)$$

The result is identical to the correlation function of the singlet state, given by Eq. (3.9), and is independent of the flux rate term γ. The cancellation of γ occurs because of the normalization with respect to the cross correlation of total number of photons at *a* and *b*. Although the dependence on γ vanishes in the expression, it is assumed that γ is sufficiently small to ensure that the probability of more than one pair arriving at the detectors is negligible over the detector integration time per event.

This simple formulation of the Bell inequality test reveals a basic su(1,1) structure to the experiment. In the following, this result is shown to be general for several realized experiments. There is considerable choice of the $\text{su}(1,1) \subset \text{sp}(8,\mathbb{R})$ subalgebra that can be used, depending on the type of Bell state generated and the corresponding optical transformations performed on it. It will be shown in the following how existing experiments use other su(1,1) subalgebras distinct from the *JKL* algebra to test the Bell inequality.

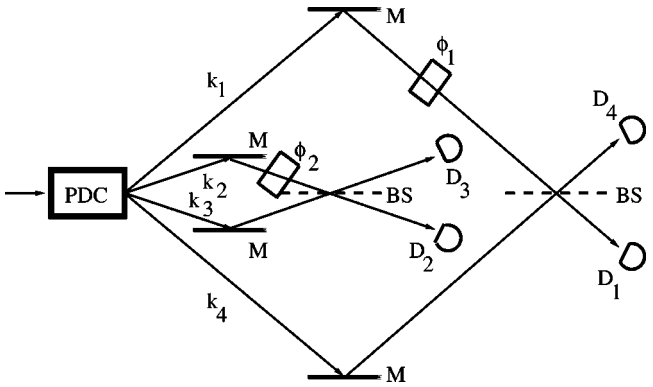

FIG. 2. Schematic of the experiment of Horne *et al.* for testing the Bell inequality.

B. Alternative ideal Bell inequality test

Although the ideal Bell inequality test has been presented in terms of entangled photons with respect to polarization, an alternative test was suggested by Horne *et al.* [32], depicted in Fig. 2, and realized experimentally by Rarity and Tapster [12]. This realization employs a PDC which creates entanglement in wave number rather than polarization, as described by a four-boson realization of su(1,1) similar to that of Eq. (2.21).

Using the description of photon pairs entangled in wave number given in Sec. II D 1, the appropriate generator for producing entangled pairs is

$$\hat{K}' = \tfrac{1}{2}(\hat{a}_+^\dagger \hat{b}_-^\dagger + \hat{a}_-^\dagger \hat{b}_+^\dagger + \hat{a}_+\hat{b}_- + \hat{a}_-\hat{b}_+), \quad (3.21)$$

which produces a pair of photons with wave numbers k_1 and k_3, entangled with a pair of photons with wave numbers k_2 and k_4. [The use of the prime on $\hat{K}'$ is meant to distinguish this generator from that of Eq. (3.1).] Employing the approximation that only one photon pair is created, $\hat{K}'$ generates the Bell state $|\psi_+\rangle$ of Eq. (2.29). Rather than subjecting these fields to polarization rotation, phase shifts (ϕ_1, ϕ_2) are applied, and the corresponding generators, following Eq. (2.15) and the notation of Sec. II C, are

$$\hat{J}_{\text{PS}}^a = \hat{J}_z^{(12)} = \tfrac{1}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-) \quad (3.22)$$

and

$$\hat{J}_{\text{PS}}^b = \hat{J}_z^{(34)} = \tfrac{1}{2}(\hat{b}_+^\dagger \hat{b}_+ - \hat{b}_-^\dagger \hat{b}_-). \quad (3.23)$$

Similar to the ideal case, only the phase shift *difference* between the two channels will actually transform the entangled state, and thus we apply the generator

$$\hat{J}' = \hat{J}_{\text{PS}}^a - \hat{J}_{\text{PS}}^b = \tfrac{1}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_- - \hat{b}_+^\dagger \hat{b}_+ + \hat{b}_-^\dagger \hat{b}_-) \quad (3.24)$$

in the form of the unitary operator

$$U_{\rm PS}(\phi_-)=\exp(i\phi_-\hat{J}'),\quad \phi_-=\phi_1-\phi_2. \tag{3.25}$$

Note that the entangled state generator $\hat{K}'$, the phase shift operator $\hat{J}'$, and the operator

$$\hat{L}'=\frac{1}{2i}(\hat{a}_-^\dagger\hat{b}_-^\dagger-\hat{a}_-^\dagger\hat{b}_+^\dagger-\hat{a}_+\hat{b}_-+\hat{a}_-\hat{b}_+), \tag{3.26}$$

close to an su(1,1) algebra with commutation relations

$$[\hat{J}',\hat{K}']=i\hat{L}',\quad [\hat{L}',\hat{J}']=i\hat{K}',\quad [\hat{K}',\hat{L}']=-i\hat{J}'. \tag{3.27}$$

The experimental scheme involves an interferometric arrangement for the phase shifts to be meaningful; the fields must be mixed by a wavelength-independent 50/50 beam splitter (BS), as described by $U_{\rm BS}$ of Eq. (2.14).

The apparatus performs a transformation on the vacuum state to give the entangled state

$$|\gamma,\theta_-\rangle=U_{\rm BS}e^{i\phi_-\hat{J}'}e^{i\gamma\hat{K}'}|0\rangle, \tag{3.28}$$

followed by photon coincidence detection in each of the four output modes (detectors D_i, $i=1,2,3,4$). Since the vacuum is invariant under the transformation $U_{\rm BS}$, we can express this transformation as

$$\begin{aligned}|\gamma,\gamma_-\rangle&=U_{\rm BS}e^{i\phi_-\hat{J}'}e^{i\gamma\hat{K}'}U_{\rm BS}^{-1}|0\rangle\\&=e^{i\phi_-(U_{\rm BS}\hat{J}'U_{\rm BS}^{-1})}e^{i\gamma(U_{\rm BS}\hat{K}'U_{\rm BS}^{-1})}|0\rangle.\end{aligned} \tag{3.29}$$

Thus the transformation on the vacuum can be expressed as an SU(1,1) transformation generated by the algebra (3.27), *conjugated* by $U_{\rm BS}$. The relevant su(1,1) subalgebra for this alternative Bell inequality test is spanned by the operators $U_{\rm BS}\hat{J}'U_{\rm BS}^{-1}$, $U_{\rm BS}\hat{K}'U_{\rm BS}^{-1}$, and $U_{\rm BS}\hat{L}'U_{\rm BS}^{-1}$.

Note that the generator $U_{\rm BS}\hat{K}'U_{\rm BS}^{-1}$ can be calculated to be

$$U_{\rm BS}\hat{K}'U_{\rm BS}^{-1}=-\tfrac{1}{2}(\hat{a}_+^\dagger\hat{b}_+^\dagger-\hat{a}_-^\dagger\hat{b}_-^\dagger+\hat{a}_+\hat{b}_+-\hat{a}_-\hat{b}_-), \tag{3.30}$$

and thus the approximate Bell state generated by this operator is $|\phi_-\rangle$ of Eq. (2.29). Thus the experiment proposed by Horne *et al.* is equivalent to an ideal Bell inequality test using the entangled Bell state $|\phi_-\rangle$.

C. Postselected Bell inequality test

We have seen that the ideal Bell inequality experiment can be described as an appropriate SU(1,1) transformation on the ground state. However, not all Bell inequality experiments are equivalent to the ideal test given in Sec. III A, yet nonetheless test the Bell inequality. A particularly salient example is the postselected Bell inequality test of Ou and Mandel [10]. Although the experiment was designed to test the Clauser-Horne version of the Bell inequality [24], a simplified version of the experimental arrangement, depicted in Fig. 3, would test the CHSH inequality and suffices for this analysis.

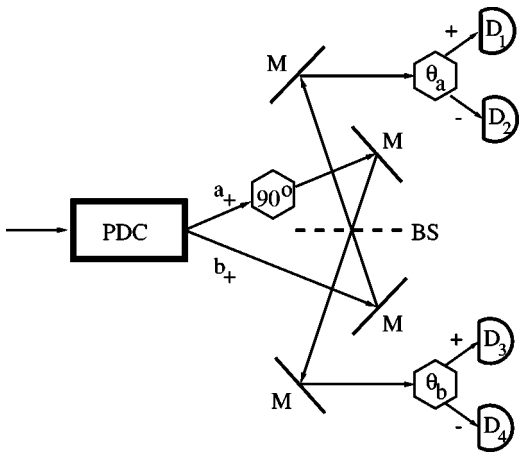

FIG. 3. Schematic of the experiment of Ou and Mandel for testing the Bell inequality.

An important difference between this arrangement and those depicted in Figs. 1 and 2 is that, in this scheme, it is as likely for both photons to go to analyzer *a* or *b* as having one photon going to *a* and one photon to *b*. The projection of the state produced in the scheme depicted in Fig. 3 is a postselection process whereby the vacuum contribution is removed (no detections occur), higher-pair contributions are neglected (such events are rare), and the case that two photons go to the same region, *a* or *b*, is detected with photon counting detectors that discriminate between one and two photons arriving.

In the absence of a photodetector which discriminates between the arrival of one and two photons, the cases where both photons go to one detector is registered as a single-photon detection. This single-photon detection is not distinguishable from a background of single-photon events that arise due to detector inefficiencies, and, therefore, photon-pair events arriving at one detector introduce a loophole [33]. This problem may be rectified with new detectors that do discriminate between one and two photons being detected [29], and these detectors are being used for Bell inequality tests in the Ou-Mandel scheme [34].

In the Ou-Mandel experiment, correlated photon pairs are generated by a type-I PDC, described by a transformation of the form of Eq. (2.26), i.e., the transformation

$$\Upsilon_{\rm OM}(\gamma)=\exp(i\gamma\hat{K}_{\rm OM}), \tag{3.31}$$

where

$$\hat{K}_{\rm OM}=\hat{K}_x^{(13)}=\tfrac{1}{2}(\hat{a}_+^\dagger\hat{b}_+^\dagger+\hat{a}_+\hat{b}_+). \tag{3.32}$$

This transformation produces correlated photons, but does not produce an entangled pair. To do so, the polarization of the *a* port is rotated by 90°, which is described by the transformation $U_a(\theta_a)$ of Eq. (2.17) with $\theta_a=\pi/2$, and entanglement is then produced by a polarization-independent 50/50 BS, described by the transformation $U_{\rm BS}$ of Eq. (2.14). The result of all these transformations on the vacuum state is to produce the state

$$|\psi(\gamma)\rangle = U_{\rm BS}U_a(\pi/2)\Upsilon_{\rm OM}(\gamma)|0\rangle$$
$$= [U_{\rm BS}U_a(\pi/2)]\Upsilon_{\rm OM}(\gamma)[U_{\rm BS}U_a(\pi/2)]^{-1}|0\rangle$$
$$= \Upsilon'_{\rm OM}(\gamma)|0\rangle, \tag{3.33}$$

where we define $\Upsilon'_{\rm OM}(\gamma)$ to be the conjugated transformation

$$\Upsilon'_{\rm OM}(\gamma) = [U_{\rm BS}U_a(\pi/2)]\Upsilon_{\rm OM}(\gamma)[U_{\rm BS}U_a(\pi/2)]^{-1}$$
$$= \exp(i\gamma\hat{K}'_{\rm OM}), \tag{3.34}$$

with

$$\hat{K}'_{\rm OM} = \tfrac{1}{4}[(\hat{a}^\dagger_- + \hat{b}^\dagger_-)(\hat{b}^\dagger_+ - \hat{a}^\dagger_+) + (\hat{a}_- + \hat{b}_-)(\hat{b}_+ - \hat{a}_+)]. \tag{3.35}$$

As mentioned above, the generator for entangled pair production includes the possibility that both photons may go to polarizer *a*, with none at *b*, and vice versa. By expressing $\hat{K}'_{\rm OM}$ as the sum $\hat{K}'_{\rm OM} = \hat{K}^1_{\rm OM} + \hat{K}^2_{\rm OM}$, with

$$\hat{K}^1_{\rm OM} = \tfrac{1}{4}(\hat{a}^\dagger_-\hat{b}^\dagger_+ - \hat{a}^\dagger_+\hat{b}^\dagger_- + \hat{a}_-\hat{b}_+ - \hat{a}_+\hat{b}_-),$$

$$\hat{K}^2_{\rm OM} = \tfrac{1}{4}(\hat{b}^\dagger_+\hat{b}^\dagger_- - \hat{a}^\dagger_+\hat{a}^\dagger_- + \hat{b}_+\hat{b}_- - \hat{a}_+\hat{a}_-), \tag{3.36}$$

it is clear that $\hat{K}^1_{\rm OM}$ generates an entangled pair (specifically, the singlet state), whereas $\hat{K}^2_{\rm OM}$ generates photon pairs both traveling either to channel *a* or channel *b*. The latter events cannot enable tests of local realism. By post-selecting, the experiment essentially disregards the component of the state generated by $\hat{K}^2_{\rm OM}$ and considers only the singlet component generated by $\hat{K}^1_{\rm OM}$.

We can state the idea of post-selection formally using the projection $\hat{\pi}$ of Eq. (2.28). Projecting the state $|\psi(\gamma)\rangle$ of Eq. (3.33) gives

$$\hat{\pi}(|\psi(\gamma)\rangle) \rightarrow |\psi_-\rangle, \tag{3.37}$$

where $|\psi_-\rangle$ of Eq. (2.29) is the singlet state.

The operator $\hat{J}_a$ of Eq. (2.16) describes the final polarization rotation for the *a* mode prior to photodetection, and the corresponding *b* mode operator is $\hat{J}_b$ of Eq. (3.2). The transformations that these operators perform on the state generated by $\Upsilon'_{\rm OM}(\gamma)$ of Eq. (3.34) are not trivial. However, if post-selection is performed (by applying the projection $\hat{\pi}$), then the resulting transformations become identical to that of the ideal Bell inequality test.

Thus the Bell inequality test of Ou and Mandel is distinct from the ideal test presented in Sec. III A. However, if post-selection is given by the projection $\hat{\pi}$ of Eq. (2.28), then the test becomes equivalent to the ideal test. It should be noted again that the realization of this projection relies on photodetectors which can distinguish between different multiple photon events, such as two photons in channel *a* and zero in channel *b*.

Whereas the disadvantage of the Ou-Mandel scheme is the need for post-selection, an advantage is the relatively high flux of photon pairs from type-I PDC compared to the production of entangled-polarization pairs via PDC [9]. For applications of Bell state measurements to quantum teleportation and other schemes, higher pair flux is an advantage.

IV. CONCLUSIONS

In studies of Bell inequalities, it is common to assume from the outset that one is supplied with one of the four Bell states (2.29). In quantum optical experiments, such states are generated from the vacuum state by an SU(1,1) transformation corresponding to parametric downconversion. Local manipulations of the output state from the PDC are described by SU(2) transformations. Using these basic facts, we establish that Bell inequality experiments, which manipulate four bosonic fields, are $\mathrm{SU}(1,1)\subset\mathrm{Sp}(8,\mathbb{R})$ transformations and that distinct four-boson realizations of SU(1,1) correspond to different experiments. For the post-selected Bell inequality, a projection operator is necessary to recover the SU(1,1) transformation equivalent to the ideal Bell inequality test.

This analysis is useful for a number of reasons. It is useful to know that an optical realization of the ideal Bell inequality test, which begins with a vacuum state as a source, is described by a four-boson realization of SU(1,1) to enable classification and comparison between differing tests of Bell inequalities as well as to consider new tests. According to the formalism we establish, new optical tests of Bell inequalities would arise as distinct realizations of $\mathrm{SU}(1,1)\subset\mathrm{Sp}(8,\mathbb{R})$. The question of various distinct tests of local realism can thus be related to the mathematical question of distinct realizations of the subgroup SU(1,1) in $\mathrm{Sp}(8,\mathbb{R})$ and the transformations which relate these subgroups. This question may be relevant to continuous-variable approaches to tests of Bell inequalities [35] where degenerate PDC and the one-boson realization (2.19) are used.

The employment of a unitary description of Bell inequality tests is useful as it includes higher-order photon number contributions and incorporates the nondeterministic creation time for pairs of photons. It also establishes a scheme for classifying existing Bell inequality tests and proposing new tests. In addition to the importance of Bell inequalities, not only for testing local realism, but also for their relevance to quantum cryptography [36], the approach employed in this paper can be extended to studying quantum teleportation [37], quantum dense coding [38], and entanglement swapping [39]. These concepts and experiments in quantum information apply the Bell states (2.29) and their measurements to larger systems. The application of group theoretical methods to such systems follows from the analysis in this paper and is under investigation.

Finally, the group theoretic approach establishes that the Bell inequality apparatus, described as a unitary transformation, produces an output state which can be regarded as a generalized coherent state [40]; these coherent states are distinct from the bi-pair (four-boson) coherent states investigated by Bambah and Agarwal [28], as the relevant group is not a direct product $\mathrm{SU}(1,1)\otimes\mathrm{SU}(1,1)$. The output coherent state is the transformed vacuum state. However, the vacuum state $|0\rangle$ is not a lowest weight state for the relevant realiza-

tions of SU(1,1) to describe Bell inequality tests. The representation containing the vacuum state is certainly reducible. However, the description of the state as a generalized coherent state does provide a useful method for thinking about the state which arrives at the photodetectors. In this way of thinking, the state may be characterized by probability distributions for measurements, and the correlation function (3.8) can be regarded as being related to a covariance of a joint probability distribution for $(\hat{\sigma}_z)_a$ and $(\hat{\sigma}_z)_b$ for this generalized coherent state. The elegance of the calculations in Sec. III A of the Bell inequality violations suggests that there is something natural about considering these generalized coherent states in such studies.

ACKNOWLEDGMENTS

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **41**, 777 (1935); N. Bohr, *ibid.* **48**, 696 (1935).
[2] D. Bohm, *Quantum Theory* (Prentice Hall, Englewood Cliffs, NJ, 1951), pp. 614–622.
[3] J. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964); Rev. Mod. Phys. **38**, 447 (1966).
[4] J. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[5] J. Clauser and A. Shimony, Rep. Prog. Phys. **41**, 1881 (1978).
[6] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).
[7] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982); A. Aspect, P. Grangier, and G. Roger, *ibid.* **49**, 91 (1982).
[8] C. K. Hong and L. Mandel, Phys. Rev. A **31**, 2409 (1985); S. Friberg, C. K. Hong, and L. Mandel, Phys. Rev. Lett. **54**, 2011 (1985).
[9] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. Shih, Phys. Rev. Lett. **75**, 4337 (1995).
[10] Z. Y. Ou and L. Mandel, Phys. Rev. Lett. **61**, 50 (1988).
[11] Y. H. Shih and C. O. Alley, Phys. Rev. Lett. **61**, 2921 (1988).
[12] J. G. Rarity and P. R. Tapster, Phys. Rev. Lett. **64**, 2495 (1990).
[13] J. D. Franson, Phys. Rev. Lett. **67**, 290 (1991).
[14] J. Brendel, E. Mohler, and W. Martienssen, Europhys. Lett. **20**, 575 (1992).
[15] P. G. Kwait, A. M. Steinberg, and R. Y. Chiao, Phys. Rev. A **47**, R2472 (1993).
[16] Y. H. Shih, A. V. Sergienko, and M. H. Rubin, Phys. Rev. A **47**, 1288 (1993).
[17] A review of Bell inequality experiments can be found in P. Hariharan and B. C. Sanders, Prog. Opt. **36**, 49 (1996).
[18] W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, Phys. Rev. Lett. **81**, 3563 (1998).
[19] D. R. Truax, Phys. Rev. D **31**, 1988 (1985); C. C. Gerry, Phys. Rev. A **31**, 2721 (1985); K. Wódkiewicz and J. H. Eberly, J. Opt. Soc. Am. B **2**, 458 (1985).
[20] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).
[21] R. A. Campos, B. E. A. Saleh, and M. C. Teich, Phys. Rev. A **40**, 1371 (1989).
[22] S. L. Braunstein, A. Mann, and M. Revzen, Phys. Rev. Lett. **68**, 3259 (1992).
[23] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).
[24] J. F. Clauser and M. A. Horne, Phys. Rev. D **10**, 526 (1974).
[25] E. Santos, Phys. Rev. Lett. **66**, 1388 (1991); **68**, 2702 (1992).
[26] M. D. Reid and D. F. Walls, Phys. Rev. A **34**, 1260 (1986).
[27] We follow the convention of using a carat on operators (e.g., $\hat{J}$) and the symbol U to represent a unitary transformation (group element) generated by such an operator.
[28] B. A. Bambah and G. S. Agarwal, Phys. Rev. A **51**, 4918 (1995).
[29] J. Kim, S. Takeuchi, Y. Yamamoto, and H. H. Hogue, Appl. Phys. Lett. **74**, 902 (1999).
[30] This transformation is not local. If one prefers to work with only local transformations, the group $U(1)_a \otimes U(1)_b$ can be applied, describing separate local transformations, and leads to the same results.
[31] X. Wang, B. C. Sanders, and S.-H. Pan, J. Phys. A **33**, 7451 (2000).
[32] M. A. Horne, A. Shimony, and A. Zeilinger, Phys. Rev. Lett. **62**, 2209 (1989).
[33] E. Santos, Phys. Lett. A **212**, 10 (1996).
[34] S. Takeuchi (unpublished).
[35] T. C. Ralph, W. J. Munro, and R. E. S. Polkinghorne, Phys. Rev. Lett. **85**, 2035 (2000).
[36] A. Ekert, Phys. Rev. Lett. **67**, 661 (1991); I. Cirac and N. Gisin, Phys. Lett. A **229**, 1 (1997); C. A. Fuchs, N. Gisin, R. B. Griffiths, C.-S. Niu, and A. Peres, Phys. Rev. A **56**, 1163 (1997).
[37] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993); D. Bouwmeester, J. W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature (London) **390**, 575 (1997); A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science **282**, 706 (1998).
[38] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[39] M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, Phys. Rev. Lett. **71**, 4287 (1993); J. W. Pan, D. Bouwmeester, H. Weinfurter, and A. Zeilinger, *ibid.* **80**, 3891 (1998).
[40] A. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin, 1986).