SU$(n+1)$ coherent states on the n–torus

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Abstract

We obtain a new family of coherent state representations of SU$(n+1)$, in which the coherent states are Wigner functions over a subgroup of SU$(n+1)$. For representations of SU$(n+1)$ of the type $(\lambda, 0, 0, \ldots)$, the basis functions are simple products of n exponential. The corresponding coherent state representations of the algebra su$(n+1)$ are also obtained, and provide a polar decomposition of su$(n+1)$ for any $n+1$. The su$(n+1)$ modules thus obtained are useful in understanding contractions of su$(n+1)$ and su$(n+1)$–phase states of quantum optics.

I. A "POLAR" REALIZATION OF SU(2)

In this contribution, I would like to report on work done in collaboration with Marco Bertola. Our work [1] deals with an new type of coherent state realization of the su$(n+1)$ algebras which appears particularly appropriate for a discussion of "polar decomposition" of su$(n+1)$ operators.

To fix ideas, consider first the algebra su(2) (or, more precisely, the complex extension of su(2)), which is spanned in the usual way by the three operators L_+, L_- and L_z, with non–zero commutation relations given by

$$[L_z, L_\pm] = \pm 2 L_\pm \,, \qquad [L_+, L_-] = L_z \,. \tag{1}$$

Our realization Γ of this algebra is given by

$$\Gamma(L_z) = -i\frac{d}{d\varphi} \,, \tag{2}$$

$$\Gamma(L_+) = e^{2i\varphi} \left[\frac{-1}{2\tan\frac{1}{2}\beta}\right] \left(\lambda + i\frac{d}{d\varphi}\right) \,, \tag{3}$$

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$$\Gamma(L_-) = e^{-2i\varphi}\left[\frac{-\tan\frac{1}{2}\beta}{2}\right]\left(\lambda - i\frac{d}{d\varphi}\right) . \tag{4}$$

The operators of Γ act in a natural on states of the 1–torus, *i.e.* on ordinary exponential functions. In particular, a weight $|m\rangle$ is simply mapped to

$$|m\rangle \mapsto \frac{e^{im\varphi}}{\sqrt{\pi}} , \tag{5}$$

with $\sqrt{\pi}$ a normalization factor.

The space of functions on the circle comes equipped with a natural inner product:

$$\langle\,\psi\,|\,\xi\,\rangle = \int_0^{\pi} d\varphi\, \psi^*(\varphi)\, \xi(\varphi) . \tag{6}$$

The integration over the angle φ can be restricted to the interval $0 \le \varphi < \pi$ as the difference of two weights in an invariant subspace is necessarily an even integer.

One immediately observes that the realization Γ is not hermitian w/r to the above inner product; for instance, $\Gamma(L_+) \neq \Gamma^\dagger(L_-)$. However, this is not a problem, as, whenever the highest weight λ is a positive integer, we are guaranteed by the theorems of representation theory that our realization must be equivalent to a hermitian realization γ, *i.e.* there must exists an intertwining operator $\mathcal{K}$ such that $\gamma = \mathcal{K}^{-1}\,\Gamma\,\mathcal{K}$.

The matrix elements of $\mathcal{K}$ can be inferred from the observation that $\Gamma(L_z)$ is actually hermitian w/r to Eqn.(6), so that $\mathcal{K}$ and L_z are simultaneously diagonal:

$$\mathcal{K}|m\rangle = \mathcal{K}_m|m\rangle . \tag{7}$$

From the condition $\gamma(L_+) = \gamma^\dagger(L_-)$, one then easily finds that, for hermiticity,

$$\frac{\mathcal{K}_{m+2}}{\mathcal{K}_m} = \frac{1}{\tan\frac{1}{2}\beta}\sqrt{\frac{\lambda - m}{\lambda + m + 2}} , \qquad\qquad \lambda \in \mathbb{Z}^+ . \tag{8}$$

Thus, although the discussion will center on Γ, which is not hermitian, we can always transform Γ to the hermitian realization γ.

Before turning our attention to how Γ was obtained, let us look first two simple applications.

A. The su(2)→e(2) contraction

Let $\lambda \to \infty$, and consider a reference state $|m_o\rangle$. If we set

$$\frac{m_o}{\lambda} = \cos\beta , \tag{9}$$

then β represents the classical angle between the $\hat{z}$ axis and a vector of length λ having projection m_o on $\hat{z}$.

It follows from this that, for two states "near" the reference state $|m_o\rangle$, we find

$$\frac{\mathcal{K}_{m_o+p}}{\mathcal{K}_{m_o+q}} \sim 1 + \mathcal{O}(1/\lambda)\,. \tag{10}$$

In other words, the transformation that makes Γ hermitian is constant to leading order in $1/\lambda$, which, in turns, indicates that Γ itself is hermitian near the reference state. With this, and always to leading order in $1/\lambda$, we can compute from $\Gamma(L_+)$ and $\Gamma(L_-)$ the matrix element of L_y, the generator of rotations about the $\hat{y}$ axis. This gives

$$\langle m_o + p|\Gamma(L_y)|m_o + q\rangle \sim \tfrac{1}{2}\lambda(\sin\beta\sin 2\varphi + \mathcal{O}(1/\lambda))\,, \tag{11}$$

where $(m_o + p)/\lambda \sim \cos\beta$ has been used. Hence, the exponential

$$\begin{aligned} d^{\frac{1}{2}\lambda}_{\frac{1}{2}(m_o+p),\frac{1}{2}(m_o+q)}(\theta) &= \langle m_o + p|e^{i\theta\Gamma(L_y)}|m_o + q\rangle\,, \\ &\sim \int_0^\pi d\varphi\, e^{i(p-q)\varphi} e^{i\theta\frac{1}{2}\lambda\sin\beta\sin 2\varphi}\,, \\ &= J_{\frac{1}{2}(p-q)}(-\tfrac{1}{2}\lambda\sin\beta\,\theta) + \mathcal{O}(1/\lambda)\,, \end{aligned} \tag{12}$$

where we have used a well–known integral representation for the Bessel function. This is a classic result, albeit without the extra $\sin\beta$, which provides a correction that allows one to extend the range of validity of the approximation. An illustration of this approximation is given in figure 1. More details can be found in [4]

B. su(2) phase states

The subject of phase states and, in particular, of su(2) phase states, has been studied in depth before [2,3]. Our objective here is not to obtain new results but rather to present another example of application of our realization for later comparison with the results for su(3).

We start from the hermitian representation γ. Selecting one of the ladder operators, say, L_+, we have

$$\gamma(L_+) = \mathcal{K}^{-1}\left[e^{2i\varphi}\left(\frac{1}{2\tan\frac{1}{2}\beta}\right)\left(\lambda + i\frac{d}{d\varphi}\right)\right]\mathcal{K} \;= e^{2i\varphi}\,\mathcal{K}^{-1}\left[\left(\frac{-1}{2\tan\frac{1}{2}\beta}\right)\left(\lambda + i\frac{d}{d\varphi}\right)\right]\mathcal{K} \tag{13}$$

We note that the operator $\mathcal{K}$ plays *two* important roles: its serves as a projector by selecting from the set of functions on the torus a subset that span the irreducible subspace with highest weight λ ($\mathcal{K}_m = 0$ if $|m| > \lambda$), and it simultaneously adjusts the matrix elements of $\Gamma(L_+)$ so that as to make $\gamma(L_+)$ the hermitian adjoint of $\gamma(L_-)$.

From Eqn.(13), we can clearly identify the expression for $\gamma(L_+)$ as the product of a radial part and a "phase part" $\hat{E}_\varphi$, defined by

$$\hat{E}_\varphi = e^{2i\varphi}\,. \tag{14}$$

The matrix elements of the operator $\hat{E}_\varphi$ are easy to find in the basis of states on the circle. The matrix representation of $\hat{E}_\varphi$ takes the form

$$
\hat{E}_\varphi = \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{15}
$$

This matrix is not unitary, and therefore cannot be the exponential of a hermitian operator. It is possible to transform this matrix into a unitary matrix without affecting the "polar decomposition" by changing the entry in the first column of the last line from 0 to $e^{i\xi}$. The phase ξ cannot be determined simply on mathematical grounds: the rank of the matrix $\gamma(L_+)$ is less than the dimension of the matrix, which implies that the polar decomposition of $\gamma(L_+)$ is not unique. Let us define

$$
E_{\hat{\varphi}(\xi)} = \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ e^{i\xi} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{16}
$$

The notation is chosen to suggest that $E_{\hat{\varphi}(\xi)}$ is the exponential of a hermitian operator $\hat{\varphi}(\xi)$. Of primary interest with the matrix $E_{\hat{\varphi}(\xi)}$ are its eigenvectors, which are the so–called phase states. It is an easy matter to show that the phase factor $e^{i\xi}$ in $E_{\hat{\varphi}(\xi)}$ simply shifts the eigenvalues and eigenvectors of $E_{\hat{\varphi}(\xi)}$ is an inessential way, so it will be convenient to fix the SU(2) phase operator once and for all by defining its exponential as the matrix $E_{\hat{\varphi}} \equiv E_{\hat{\varphi}(0)}$, so that

$$
E_{\hat{\varphi}} = \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{17}
$$

The eigenvalues and eigenvectors of this matrix are those found by Vourdas in reference [2].

C. Obtaining Γ

We now turn to the question of obtaining Γ. Consider the highest weight state $|\chi_\lambda\rangle$ of an irrep of the complex extension of su(2). This state satisfies

$$L_+|\chi_\lambda\rangle = 0\,, \qquad L_z|\chi_\lambda\rangle = \lambda|\chi_\lambda\rangle\,. \tag{18}$$

To every state $|\psi\rangle$ we associate a "coherent–state" wave function through the map

$$|\psi\rangle \mapsto \psi_\beta(\varphi) \equiv \langle\chi_\lambda|R_y(\beta)e^{i\varphi L_z}|\psi\rangle\,. \tag{19}$$

Using this, the coherent state representation $\Gamma(\hat{X})$ of an operator $\hat{X}$ is defined by

$$\hat{X}|\psi\rangle \mapsto \left[\Gamma(\hat{X})\psi\right]_\beta(\varphi) \equiv \langle\chi_\lambda|R_y(\beta)e^{i\varphi L_z}\,\hat{X}\,|\psi\rangle\,. \tag{20}$$

Note that the fiducial vector $|\chi_\lambda\rangle$ is not translated by the most general SU(2) element. However, a more general translation would simply produce extra phases in the matrix elements.

It is now an easy matter to obtain the Γ–map of the su(2) generators. For instance,

$$\hat{L}_z|\psi\rangle \mapsto \langle\chi_\lambda|R_y(\beta)e^{i\varphi L_z}\,\hat{L}_z\,|\psi\rangle = -i\frac{d}{d\varphi}\psi_\beta(\varphi)\,. \tag{21}$$

Since $|\psi\rangle$ is arbitrary, we immediately recover Eqn.(2). To obtain $\Gamma(L_+)$, we start with

$$\begin{aligned}
[\Gamma(L_+)\psi]_\beta(\varphi) &= \langle\chi_\lambda|R_y(\beta)e^{i\varphi L_z}\,L_+\,|\psi\rangle\,,\\
&= \langle\chi_\lambda|R_y(\beta)\,e^{i\varphi L_z}\,L_+\left[e^{-i\varphi L_z}\,e^{i\varphi L_z}\right]\,|\psi\rangle\,,\\
&= e^{2i\varphi}\langle\chi_\lambda|R_y(\beta)\,L_+\,e^{i\varphi L_z}|\psi\rangle\,,\\
&= e^{2i\varphi}\langle\chi_\lambda|R_y(\beta)\,L_+\left[R_y^{-1}(\beta)\,R_y(\beta)\right]\,e^{i\varphi L_z}|\psi\rangle\,.
\end{aligned} \tag{22}$$

The key step is to realize that one can write

$$\left[R_y(\beta)\,L_+R_y^{-1}(\beta)\right] = \eta L_- + \zeta L_z + \kappa R_y(\beta)L_zR_y^{-1}(\beta) \tag{23}$$

where the expansion coefficients η, ζ and κ cannot depend on the particular choice of representation: if they did, the map Γ would hold only for that particular irrep with highest weight λ. Thus, we can choose to evaluate η, ζ and κ in the 2×2 representation, where

$$L_+ \mapsto \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\,,\quad L_- \mapsto \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}\,,\quad L_z \mapsto \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\,,\quad R_y(\beta) \mapsto \begin{pmatrix} \cos\frac{1}{2}\beta & -\sin\frac{1}{2}\beta\\ \sin\frac{1}{2}\beta & \cos\frac{1}{2}\beta\end{pmatrix}\,, \tag{24}$$

so that

$$\begin{aligned}
R_y(\beta)L_+R_y^{-1}(\beta) &= \begin{pmatrix} -\cos\frac{1}{2}\beta\sin\frac{1}{2}\beta & \cos^2\frac{1}{2}\beta\\ -\sin^2\frac{1}{2}\beta & \cos\frac{1}{2}\beta\sin\frac{1}{2}\beta\end{pmatrix}\\
&= \eta\begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix} + \zeta\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix} + \kappa\begin{pmatrix} \cos^2\frac{1}{2}\beta - \sin^2\frac{1}{2}\beta & 2\cos\frac{1}{2}\beta\sin\frac{1}{2}\beta\\ 2\cos\frac{1}{2}\beta\sin\frac{1}{2}\beta & -\cos^2\frac{1}{2}\beta + \sin^2\frac{1}{2}\beta\end{pmatrix}\,.
\end{aligned} \tag{25}$$

This leads to a system of three linearly independent equations with three unknowns, which can be solved to find

$$\kappa = \frac{1}{2\tan\frac{1}{2}\beta}\,, \qquad \zeta = \frac{-1}{2\tan\frac{1}{2}\beta}\,. \tag{26}$$

Note that $\langle\chi_\lambda|L_- = 0$ by construction, so that there is no need to solve for η. The final result is given by Eqn.(3). The expression for $\Gamma(L_-)$ is found in the same way.

II. "POLAR" DECOMPOSITION OF SU(3)

We now generalize the construction of Eqns(2-4) to the su(3). First, we start with u(3) and note that, as usual, it is spanned by the set of 9 operators $\{\hat{C}_{ij}, i,j=1,2,3\}$ satisfying the commutation relations

$$[\hat{C}_{ij}, \hat{C}_{k\ell}] = \hat{C}_{i\ell}\delta_{jk} - \hat{C}_{jk}\delta_{i\ell}\,. \tag{27}$$

The su(3) algebra is obtained by selecting from the above set the operators $\hat{C}_{ij}, i \neq j$ and the two diagonal operators $\hat{h}_1 = \hat{C}_{11} - \hat{C}_{22}$, $\hat{h}_2 = \hat{C}_{22} - \hat{C}_{33}$. The root diagram for A_2, the complex extension of su(3), is presented in figure 2.

We want to construct the "polar" decomposition of su(3), given explicitly by

$$\begin{aligned}
\Gamma(\hat{h}_1) &= -i\frac{\partial}{\partial\varphi_1}\,, \qquad \Gamma(\hat{h}_2) = -i\frac{\partial}{\partial\varphi_2}\,,\\
\Gamma(\hat{C}_{12}) &= \frac{-1}{3\cos\frac{1}{2}\beta_3\,\tan\frac{1}{2}\beta_2}\,\mathrm{e}^{i(2\varphi_1-\varphi_2)}\left[\lambda + i\frac{\partial}{\partial\varphi_1} - i\frac{\partial}{\partial\varphi_2}\right],\\
\Gamma(\hat{C}_{21}) &= \frac{-\cos\frac{1}{2}\beta_3\,\tan\frac{1}{2}\beta_2}{3}\,\mathrm{e}^{-i(2\varphi_1-\varphi_2)}\left[\lambda - 2i\frac{\partial}{\partial\varphi_1} - \frac{\partial}{\partial\varphi_2}\right],\\
\Gamma(\hat{C}_{13}) &= \frac{-1}{3\sin\frac{1}{2}\beta_3\,\tan\frac{1}{2}\beta_2}\,\mathrm{e}^{i(\varphi_1+\varphi_2)}\left[\lambda + i\frac{\partial}{\partial\varphi_1} + 2i\frac{\partial}{\partial\varphi_2}\right],\\
\Gamma(\hat{C}_{31}) &= \frac{-\sin\frac{1}{2}\beta_3\,\tan\frac{1}{2}\beta_2}{3}\,\mathrm{e}^{-i(\varphi_1+\varphi_2)}\left[\lambda - 2i\frac{\partial}{\partial\varphi_1} - i\frac{\partial}{\partial\varphi_2}\right],\\
\Gamma(\hat{C}_{23}) &= \frac{1}{3\tan\frac{1}{2}\beta_3}\,\mathrm{e}^{i(-\varphi_1+2\varphi_2)}\left[\lambda + i\frac{\partial}{\partial\varphi_1} + 2i\frac{\partial}{\partial\varphi_2}\right],\\
\Gamma(\hat{C}_{32}) &= \frac{\tan\frac{1}{2}\beta_3}{3}\,\mathrm{e}^{i(\varphi_1-2\varphi_2)}\left[\lambda + i\frac{\partial}{\partial\varphi_1} - i\frac{\partial}{\partial\varphi_2}\right]
\end{aligned} \tag{28}$$

T his realization acts naturally on functions over the 2–torus, with basis states

$$|\nu_1\nu_2\nu_3\rangle \mapsto \frac{e^{i(\nu_1-\nu_2)\varphi_1 + i(\nu_2-\nu_3)\varphi_2}}{2\pi}\,, \qquad \nu_1+\nu_2+\nu_3 = \lambda\,. \tag{29}$$

As an example of this labeling scheme, the weight diagram for the irrep $(3,0)$ can be found in figure 2, with some of the states explicitly labeled in terms of the three integers ν_1, ν_2, ν_3.

A. How to get Γ

Again we start from a highest weight state $|\chi_\lambda\rangle$ for an irrep $(\lambda, 0)$, defined by

$$\begin{aligned}
\hat{C}_{12}|\chi_\lambda\rangle = \hat{C}_{23}|\chi_\lambda\rangle = \hat{C}_{32}|\chi_\lambda\rangle &= \hat{C}_{13}|\chi_\lambda\rangle = 0\,,\\
\hat{h}_1|\chi_\lambda\rangle = \lambda|\chi_\lambda\rangle\,, \quad \hat{h}_2|\chi_\lambda\rangle &= 0\,.
\end{aligned} \tag{30}$$

An arbitrary state $|\psi\rangle$ is mapped to the coherent–state wave function

$$|\psi\rangle \mapsto \psi_\beta(\varphi_1,\varphi_2) \equiv \langle\chi_\lambda|M(\beta)e^{i\varphi_1\hat{h}_1}\, e^{i\varphi_2\hat{h}_2}\,|\psi\rangle\,, \tag{31}$$

where $M(\beta)$ is the SU(3) matrix

$$M(\beta) \equiv R_{12}(\beta_2)R_{23}(\beta_3) = \begin{pmatrix} \cos\frac{1}{2}\beta_2 & -\sin\frac{1}{2}\beta_2 & 0 \\ \sin\frac{1}{2}\beta_2 & \cos\frac{1}{2}\beta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{1}{2}\beta_3 & -\sin\frac{1}{2}\beta_3 \\ 0 & \sin\frac{1}{2}\beta_3 & \cos\frac{1}{2}\beta_3 \end{pmatrix}. \tag{32}$$

Note that the matrix M is not the most general SU(3) matrix. However, a fully general SU(3) transformation simply leads to extra phases in the expression of the generators.

The coherent-state representation of an operator $\hat{X}$ is defined as before by

$$\hat{X}|\psi\rangle \mapsto \left[\Gamma(\hat{X})\psi\right]_\beta(\varphi_1,\varphi_2) \equiv \psi_\beta(\varphi_1,\varphi_2) \equiv \langle\chi_\lambda|M(\beta)e^{i\varphi_1\hat{h}_1}\, e^{i\varphi_2\hat{h}_2}\,\hat{X}\,|\psi\rangle\,, \tag{33}$$

from which it follows immediately that

$$\Gamma(\hat{h}_1) = -i\frac{\partial}{\partial\varphi_1}\,,\quad \Gamma(\hat{h}_2) = -i\frac{\partial}{\partial\varphi_2}\,. \tag{34}$$

To find the Γ–realization of, say, $\hat{C}_{12}$, we note that, by definition,

$$\begin{aligned} \Gamma(\hat{C}_{12})\psi_\beta(\varphi) &= \langle\chi_\lambda|M(\beta)e^{i\varphi_1\hat{h}_1}\, e^{i\varphi_2\hat{h}_2}\,\hat{C}_{12}|\psi\rangle \\ &= \langle\chi_\lambda|M(\beta)e^{i\varphi_1\hat{h}_1}\, e^{i\varphi_2\hat{h}_2}\,\hat{C}_{12}\left[e^{-i\varphi_1\hat{h}_1}\, e^{-i\varphi_2\hat{h}_2}e^{i\varphi_1\hat{h}_1}\, e^{i\varphi_2\hat{h}_2}\right]\,|\psi\rangle \\ &= e^{i(2\varphi_1-\varphi_2)}\langle\chi_\lambda|M(\beta)\hat{C}_{12}\,e^{i\varphi_1\hat{h}_1}\, e^{i\varphi_2\hat{h}_2}\,|\psi\rangle \\ &= e^{i(2\varphi_1-\varphi_2)}\langle\chi_\lambda|M(\beta)\hat{C}_{12}\,[M(\beta)^{-1}\,M(\beta)]\,e^{i\varphi_1\hat{h}_1}\, e^{i\varphi_2\hat{h}_2}\,|\psi\rangle \end{aligned} \tag{35}$$

Now, write

$$\begin{aligned} M(\beta)\hat{C}_{12}M^{-1}(\beta) &= x^{12}\hat{C}_{12} + x^{23}\hat{C}_{23} + x^{32}\hat{C}_{32} + x^{13}\hat{C}_{13} \\ &\quad + y^1\hat{h}_1 + y^2\hat{h}_2 + M(\beta)\left(z^1\hat{h}_1 + z^2\hat{h}_2\right)M^{-1}(\beta)\,. \end{aligned} \tag{36}$$

Again, we note that the coefficients z^1, z^2, y^1, y^2 and x^{ij} do not depend on the particular representation, and can therefore be evaluated in the 3×3 representation. This provides us with a system of eight linearly independent equations for eight unknowns, which can always be solved. Furthermore,

$$\langle\chi_\lambda|\left(x^{12}\hat{C}_{12} + x^{23}\hat{C}_{23} + x^{32}\hat{C}_{32} + x^{13}\hat{C}_{13} + y^2\hat{h}_2\right) = 0\,. \tag{37}$$

Thus, using the solutions for z^1, z^2 and y^2, we find

$$\begin{aligned} \Gamma(\hat{C}_{12})\psi_\beta(\varphi) &= e^{i(2\varphi_1-\varphi_2)}\left(y^1\lambda + iz^1\frac{\partial}{\partial\varphi_1} + iz^2\frac{\partial}{\partial\varphi_2}\right) \\ &= e^{i(2\varphi_1-\varphi_2)}\left(\frac{-1}{3\cos\frac{1}{2}\beta_3\tan\frac{1}{2}\beta_2}\right)\left[\lambda - i\frac{\partial}{\partial\varphi_1} - i\frac{\partial}{\partial\varphi_2}\right] \end{aligned} \tag{38}$$

The Γ realization of the other operators $\hat{C}_{ij}, i\neq j$ are found in the same way.

B. The $\lambda \to \infty$ limit

To obtain some insight into the large λ limit of the realization Γ, we set

$$\bar{\nu}_1 = \lambda \left(\cos \tfrac{1}{2}\beta_2\right)^2 , \quad \bar{\nu}_2 = \lambda \left(\sin \tfrac{1}{2}\beta_2 \cos \tfrac{1}{2}\beta_3\right)^2 , \quad \bar{\nu}_3 = \lambda \left(\sin \tfrac{1}{2}\beta_2 \sin \tfrac{1}{2}\beta_2\right)^2 . \qquad (39)$$

Then, to order $\mathcal{O}(1/\lambda)$, matrix elements of the ladder operators are approximately constants for states near some reference state $|\bar{\nu}_1\bar{\nu}_2\bar{\nu}_3\rangle$. For instance,

$$\begin{aligned}
\langle \nu_1'\nu_2'\nu_3'|\Gamma(\hat{C}_{12})|\nu_1\nu_2\nu_3\rangle &\sim \frac{-1}{3\cos\frac{1}{2}\beta_3 \tan\frac{1}{2}\beta_2} e^{i(2\varphi_1-\varphi_2)} \left[\lambda - (\bar{\nu}_1 - \bar{\nu}_2) + (\bar{\nu}_2 - \bar{\nu}_3)\right] \\
&= \frac{-1}{3\cos\frac{1}{2}\beta_3 \tan\frac{1}{2}\beta_2} e^{i(2\varphi_1-\varphi_2)} \lambda \left[3\bar{\nu}_2\right] , \qquad (\bar{\nu}_1 + \bar{\nu}_2 + \bar{\nu}_3 = \lambda) \qquad (40) \\
&= -2\lambda e^{i(2\varphi_1-\varphi_2)} \sin\beta_2 \cos\tfrac{1}{2}\beta_3 .
\end{aligned}$$

if $|\nu_i - \bar{\nu}_i| \ll \lambda$ and $|\nu_i' - \bar{\nu}_i| \ll \lambda$. Similarly:

$$\begin{aligned}
\Gamma(\hat{C}_{21}) &\to -2\,\lambda\, \mathrm{e}^{-i(2\varphi_1-\varphi_2)} \sin\beta_2 \cos\tfrac{1}{2}\beta_3 , \\
\Gamma(\hat{C}_{13}) &\to -2\,\lambda\, \mathrm{e}^{i(\varphi_1+\varphi_2)} \sin\beta_2 \sin\tfrac{1}{2}\beta_3 , \\
\Gamma(\hat{C}_{31}) &\to -2\,\lambda\, \mathrm{e}^{-i(\varphi_1+\varphi_2)} \sin\beta_2 \cos\tfrac{1}{2}\beta_3 , \qquad (41) \\
\Gamma(\hat{C}_{23}) &\to 2\,\lambda\, \mathrm{e}^{i(-\varphi_1+2\varphi_2)} (\sin\tfrac{1}{2}\beta_2)^2 \sin\beta_3 , \\
\Gamma(\hat{C}_{32}) &\to 2\,\lambda \mathrm{e}^{i(\varphi_1-2\varphi_2)} (\sin\tfrac{1}{2}\beta_2)^2 \sin\beta_3 ,
\end{aligned}$$

with $\Gamma(\hat{h}_1)$ and $\Gamma(\hat{h}_2)$ unchanged. From this, it is clear that, in the $\lambda \to \infty$ limit, the realization Γ of su(3) contracts to a realization of the algebra $[\mathbb{R}^6]U(1)^2$. The angles β_2 and β_3 of the SU(3) transformation can be seen to parameterize the distribution of λ photons in 3 channels.

C. SU(3) phase states

We now seek to generalize the SU(2) phase operator $\hat{\varphi}$ to SU(3). From Γ, we start with the two obvious phase–like operators:

$$\hat{E}_{12} = e^{i(2\varphi_1-\varphi_2)} , \qquad \hat{E}_{23} = e^{i(-\varphi_1+2\varphi_2)} . \qquad (42)$$

One can easily verify that

$$\left[\tfrac{1}{2}\Gamma(\hat{h}_1), \hat{E}_{12}\right] = \hat{E}_{12} \qquad \left[\tfrac{1}{2}\Gamma(\hat{h}_2), \hat{E}_{23}\right] = \hat{E}_{23} , \qquad (43)$$

thus showing that, if $\hat{E}_{12}$ and $\hat{E}_{23}$ can be related to the exponentials of hermitian phase operators $\hat{\varphi}_{12}$ and $\hat{\varphi}_{23}$, the phase operators would be conjugate to the relative number operators $\hat{h}_1$ and $\hat{h}_2$, respectively. However, we immediately notice that

$$\left[\tfrac{1}{2}\Gamma(\hat{h}_1), \hat{E}_{23}\right] \neq 0\,, \qquad\qquad \left[\tfrac{1}{2}\Gamma(\hat{h}_2), \hat{E}_{12}\right] \neq 0 \tag{44}$$

One way to understand the situation is to concentrate on a particular case, say the irrep $(3,0)$. The weight diagram for this irrep is given in figure 2. The operators $\hat{E}_{12}$ and $\hat{E}_{23}$ have matrix representations given explicitly by

$$\hat{E}_{12} = \begin{pmatrix} 0 & 1 & & & 0 & & & 0 & & 0 \\ 0 & 0 & 1 & & 0 & & & 0 & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & 0 & 1 & & 0 & & 0 \\ 0 & & & & 0 & & 1 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & 0 & & & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{E}_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 0 & & 1 & & & & \\ 0 & 0 & 0 & 0 & & & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & & 1 & & \\ 0 & 0 & 0 & 0 & & & & & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & & & & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

One immediately observes that $[\hat{E}_{12}, \hat{E}_{23}] \neq 0$, a surprising result if we consider Eqn.(42) and the classical behavior of phases. More precisely:

$$\left[\hat{E}_{12}, \hat{E}_{23}\right] = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & & & & & 0 \\ 0 & & & & & & & & & 0 \\ 0 & & & & & & & & & 0 \\ 0 & & & & & & 0 & 1 & 0 & 0 \\ 0 & & & & & & & & & 0 \\ 0 & & & & & & & & & 0 \\ 0 & & & & & & & & 0 & 1 \\ 0 & & & & & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{46}$$

It is interesting to note that there are $\lambda = 3$ entries which are 1 rather than zero in this commutator. The "faulty" non–zero matrix elements appear in positions corresponding to matrix elements of the type $\langle \nu_1+1, 0, \nu_3-1|\hat{E}_{13}|\nu_1, 0, \nu_3\rangle$, *i.e.* matrix elements involving vacuum states in mode 2. The treatment of the vacuum in the formalism of phases states is always problematic, as the vacuum does not have a well–defined phase. This problem appear here in the form of non–commuting operators.

From the special case of the $(3,0)$ representation, we can infer that, for a general representation λ, the number of "faulty" non–zero matrix elements in commutators of the type $[\hat{E}_{ij}, \hat{E}_{jk}]$ will grow like λ, since the number of states having the vacuum as one mode grows like λ. On the other hand, the number of states in the irrep $(\lambda, 0)$ grows like λ^2, so that the classical limit where $\lambda \to \infty$ corresponds to the limit where the phases commute to the extent that we ignore the relatively small number of "faulty" non–zero matrix elements compared to the number of "correct" zero matrix elements. This relative number grows like $1/\lambda$.

It is worth mentioning that, in the classical $\lambda \to \infty$ limit, the realization Γ becomes singular when we consider states near the vacuum state, as β_1 or β_2 can be zero in the interpretation of section II B.

There is a second problem in trying to construct su(3) phase operators: there is no unique way of going from, say, the operator $\hat{E}_{12}$ to the unitary operator $E_{\hat{\varphi}_{12}}$ (or $\hat{E}_{23} \to E_{\hat{\varphi}_{23}}$). $\hat{E}_{12}$ contains $\lambda + 1 = 4$ null lines, which must be modified in order to transform $\hat{E}_{12}$ into $E_{\hat{\varphi}_{12}}$. Unlike the su(2) case, there is no unique or mathematically preferred way of making the appropriate modification.

The simplest choice is to note that $\hat{E}_{12}$ has non–zero matrix elements only within irreducible $SU(2)_{12}$ subspaces, invariant under the action of $\hat{C}_{12}, \hat{C}_{21}$ and $\hat{h}_1$. If we require this to be preserved for $E_{\hat{\varphi}_{12}}$, then we can (up to phases) uniquely transform $\hat{E}_{12}$ into $E_{\varphi_{12}}$. Quite clearly, this choice is not unique although justifiable on physical grounds. A similar approach can be used to construct the unitary matrix $E_{\hat{\varphi}_{23}}$. The problem of non–commutativity, however, $[E_{\hat{\varphi}_{12}}, E_{\hat{\varphi}_{23}}] \neq 0$, remains.

III. CONCLUSION

We have presented explicitly a method of constructing realizations of the su(2) and su(3) algebras which are easily interpreted in terms of a "polar decomposition" of the operators in the respective algebras. The algorithm is sufficiently systematic to be generalized to any $su(n+1)$ algebra. More results on the general case of an irrep $(\lambda, 0, 0, \ldots)$ of $su(n+1)$ can be found in reference [1].

Possible applications of this type of realization include contraction limits of $su(n+1)$. However, the realizations are particularly well suited for an analysis of $su(n+1)$ phase states. Already, we have found, for su(3), that there is no unique way of defining phase operators, and that these phase operators do not commute. These problems are expected to persist for all the $su(n+1)$ algebras with $n \geq 2$.

The possibility of generalizing this construction to other Lie algebras has also been discussed briefly in [1]. A simple counting argument shows that our method is limited to those algebras having a complex extension of the type A_n. In particular, the construction also works for su(1,1)$\sim$sp(2,$\mathbb{R}$), a Lie algebra of importance in quantum optics, but fails for the real symplectic algebras of higher ranks.

The method presented here can also be generalized to obtain realizations having highest weights of the type $(\lambda, \mu, 0, \ldots)$. In particular, the construction of such realizations for $su(n+1)$ could be useful in studying classical limits of polarized optical beams.

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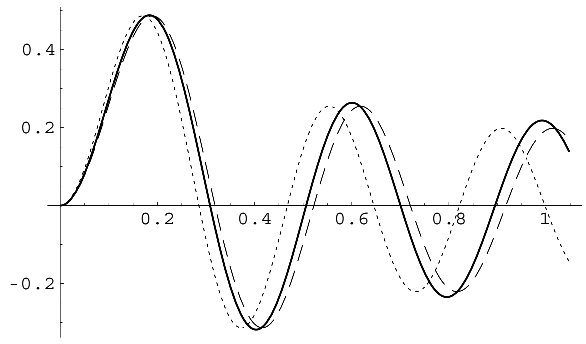

FIG. 1. The exact Wigner function $d^{18}_{9,7}(\theta)$ (full line) and its approximations by Bessel functions using the correction factor $\lambda \sin\beta$ (dotted line) or simply λ (dashed line).

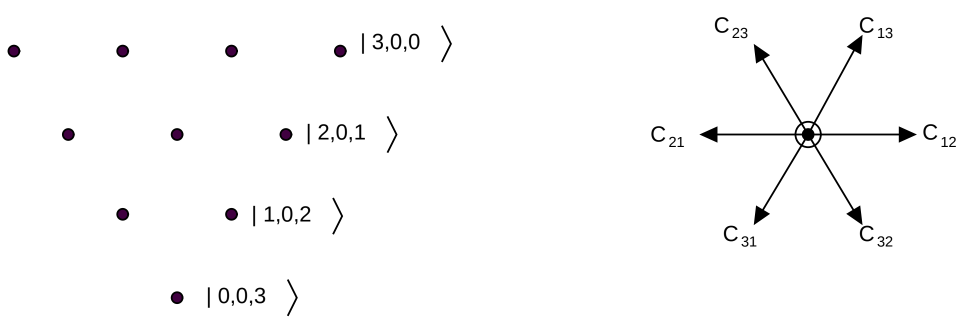

FIG. 2. The location of the weights for the irrep (3,0) of su(3), the explicit expression of some weights in terms of the integer ν_1, ν_2 and ν_3, and the root diagram of su(3).