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Topical Review

Generalized *SU*(2) covariant Wigner functions and some of their applications

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Abstract
We survey some applications of $SU(2)$ covariant maps to the phase space quantum mechanics of systems with fixed or variable spin. A generalization to $SU(3)$ symmetry is also briefly discussed in framework of the axiomatic Stratonovich–Weyl formulation.

Keywords: Wigner, quasidistributions, semiclassical

(Some figures may appear in colour only in the online journal)

1. Introduction

Originally introduced by Wigner [1], phase space methods have blossomed and seeded multiple applications in quantum optics, quantum chemistry, classical optics, signal analysis, speech analysis and other areas [2–11].

In quantum physics, phase space methods have been used for state identification and characterization by plotting symbols of the density matrix as a distribution function on the sphere or in the $q-p$ plane [12] to study the non-classicality of states [13, 14], phase properties of finite quantum systems [105, 106], and for state reconstruction using quantum tomography [15], among others.

Moyal [16] expanded the work of Wigner on the harmonic oscillator by showing how Wigner's approach could be reformulated so that every quantum mechanical operator $\hat{f}$ is in correspondence with a symbol (called the Weyl symbol) $W_f(\Omega)$ on a phase space. This symbol is a c-number function obtained by using an invertible kernel $\hat{w}(\Omega)$, with Ω phase space coordinates. Through this so-called Moyal correspondence, average values of operators are computed as in classical statistical mechanics: by integration over phase space of the Weyl symbol of an operator using the symbol of the density matrix (called quasi-distribution function) as a formal probability distribution.

In this review we deal mostly with maps for spin-like systems, for which the dynamical group is $SU(2)$ and phase space is the 2-sphere [17–22], and their recent generalizations to higher unitary groups [23]. These maps, together with the Heisenberg–Weyl maps, remain the most popular because they allow the visualization of states as distributions in 2-dimensional manifolds (the sphere and the plane, respectively). The maps have relatively simple analytical properties, which are complemented by considerable familiarity with the phase space constructions. Additionally, $SU(2)$ and the Heisenberg–Weyl (HW) group describe an extensive palette of physical situations. The reader interested more exclusively in physical systems having the HW group as a dynamical symmetry, and the flat $q - p$ space as phase space, is referred to [24–40] for a sample of articles that discuss and explore some applications of phase space methods to these widely-used types of systems.

It is Stratonovich in [17] who introduced an axiomatic approach extending beyond the harmonic oscillator and generalizable to quantum systems admitting a dynamical symmetry group which allows the construction of a phase space as an appropriate homogeneous manifold. Indeed, when the observables of a theory are elements (or powers of elements) of a Lie algebra acting irreducibly on the quantum Hilbert space of states, the phase space is a manifold constructed as a quotient space—as described in section 2.1—and is closely related to the set of orbit-type coherent states [41–43] for the corresponding group.

The Stratonovich–Moyal–Weyl correspondence provides a systematic procedure for constructing a family of s-parametrized trace-like maps through a kernel $\hat{w}^{(s)}(\Omega)$ for some common types of dynamical symmetry groups [24–32]. The label s historically specifies the ordering rules for functions of non-commutative operators in the Heisenberg–Weyl case; for higher groups s is used to specify elements of the family so they satisfies certain boundary and duality conditions: the maps labelled by s and $-s$ are mutually dual in the sense of equations (2.4) and (2.5).

Within this approach, operators and density matrices are in one-to-one correspondence with symbols: average values are obtained by convoluting the symbol of an operator with the dual symbol for the density matrix. In applications, the particular form of the map $\hat{w}^{(s)}(\Omega)$ strongly depends on the symmetry group of the Hamiltonian and of the underlying system.

The foundation of the Weyl–Moyal–Stratonovich approach is a specific symbol calculus mapping a product of two operators $\hat{f}$ and $\hat{g}$ onto a (non-commutative) star-product [16, 33] of their symbols $W_f(\Omega) * W_g(\Omega)$. The $*$ operation replaces the standard manipulations of operators in a Hilbert space of states by differential (or integral) operators acting on the product of their symbols in phase space. The axiomatic introduction to the algebra of classical observables of an associative but non-commutative star-product leads to a specific quantization procedure known as deformation quantization [34].

Given the explicit construction of the map, a formal integral representation of the star-product is easily obtained (see e.g. [32, 25, 37]), but rarely useful in practical calculations. Instead, a differential form of the star-product operation, emphasizing the local nature of the phase space approach, is known for systems with Heisenberg–Weyl [16, 38], $E(2)$ [44] and $SU(2)$ [45–47] and some generalizations [48, 49].

Without a doubt one of the most important applications of phase space methods [50] remains the analysis of quantum-classical correspondence during the evolution of a system [51]. The phase space approach may not only substantially simplify the analysis of the dynamical behavior of large-dimensional systems, but also reveals if a physical phenomenon has classical or essentially quantum roots. Using the star-product one can rewrite the Schrödinger equation for the density matrix in the form of an evolution equation (the Moyal equation) for its symbol. The advantage of such a formulation of the evolution problem lies in the possibility of expanding the Moyal equation in powers of one or more small parameters. These so-called semi-classical parameters are related both to the symmetry of the interaction Hamiltonian and the symmetry of the map $\hat{w}^{(s)}(\Omega)$.

For instance, the semi-classical parameter for the HW map is often taken as the inverse number of excitations in the corresponding one-dimensional system, e.g. the average number of photons in a field mode or the energy of a particle in a one-dimensional potential, etc. For spin-like systems, the inverse effective spin length usually plays the role of semi-classical parameter [45, 52].

To lowest order in these parameters, the evolution is described by a first-order partial differential equation, and provides an efficient method for studying quantum dynamics (at least for not very long times) in the semi-classical limit. In this so-called Liouvillian or Truncated Wigner Approximation (TWA), points of the initial distribution are propagated along classical trajectories in phase space [53, 54, 58].

TWA describes the dynamics of nonlinear quantum systems drastically better than naive solutions of the Heisenberg equations of motion with partially decoupled correlators (the so-called parametric approximation) [56–57]. This being said, quantum phenomena resulting from self-interference, like appearance of Schrödinger cats, are clearly beyond the scope of TWA [55]. Detailed discussions of TWA with applications to various physical situations can be found in [56–67] (see also [68], where different semi-classical methods for the description of evolution problems are compared).

The phase space approach is also directly generalizable to multipartite systems. The mapping kernel is then just a product of single-particle kernels. Although such a map is not practical for pictorial purposes, the semi-classical ideas can still be employed: the evolution equation in TWA contains only first order derivatives, and so can be solved using the method of characteristics and interpreted as an evolution along 'trajectories' in a direct product of the corresponding classical manifolds [57, 69, 70]. Several physically relevant multi-partite characteristics of the system, such as entanglement and negativity [71], can be described within this framework.

The choice of map used as an interface between the quantum and classical worlds is crucial since it fixes the structure of the phase space manifold. As an archetypal example where different types of phase space mappings with different symmetries can be chosen, we may consider a photon number preserving coupling of two field modes. A first choice is a direct $H(1) \times H(1)$ map into two flat $q-p$ phase spaces, $\mathbb{R}^2 \otimes \mathbb{R}^2$. A second option is to first decompose the initial state over the $SU(2)$- invariant subspaces with fixed photon number, and then map states from each subspace on the two-dimensional sphere $\mathcal{S}^2(\theta, \phi)$. Both options are faithful for the description of the evolution of $su(2)$ observables, such as moments of the Stokes operators. Whereas in the first case there are two semi-classical parameters, given by the inverse of the individual photon numbers in each mode (see for instance [61] where numerous examples of field-field interactions are analyzed), in the second case there is a *single* semi-classical parameter: it is the inverse *sum* of excitations in both modes, i.e. the inverse effective spin length. Thus, the $SU(2)$ mapping is more appropriate for the analysis of the semi-classical dynamics of number-preserving exchange of excitations between the modes than the $H(1) \times H(1)$ map. As an empiric observation, it seems that the higher the symmetry of the map, the better the performance of the TWA [72].

In spite of its broad successes, the standard Stratonovich–Weyl mapping is not always adequate for the description of a variety of physical situations. This occurs when the symmetry group of the map does not act irreducibly on the density matrix of the system. This is especially important for the analysis of the dynamics and occurs for instance, when the Hamiltonian and/or an observable or its evolution mixes and/or contains contribution from irreducible subspaces of the symmetry group used for the mapping from the Hilbert space into the classical phase space. In the previous example this would correspond to a situation where a non-number-preserving external pumping field or dissipation is present, or when analyzing the evolution of a *single* field mode under some mode-coupling Hamiltonian.

In order to characterize quantum systems, their semi-classical limits and in particular their semi-classical dynamics in situations where the standard approach is not directly applicable, one can relax the requirement of mapping into a 'true' phase space, keep only the covariance condition (under an appropriate transformation group) and maintain a one-to-one relation between operators and their symbols. It results that, at least in the case of the $SU(2)$ symmetry, this program can be accomplished [49] and a generalized $SU(2)$ covariant mapping endowed with the correct contraction limits and differential form of the star-product can be constructed. In addition, in the semi-classical limit an approximation corresponding to TWA in the general Stratonovich–Weyl framework can also be established [73] and describes an effective dynamics on the four-dimensional cotangent bundle $T^*\mathcal{S}^2$ proper to quantum systems with $E(3)$ as dynamical group. This approach is especially useful for the description of physical systems with a variable number of excitations (or a variable spin) and allows to naturally extend the concept of phase operators introduced for fixed spin-like systems.

There are also many important physical systems for which the Stratonovich construction is not directly applicable: for instance optical angular momentum systems with $E(2)$ symmetry, and rigid rotors and heavy tops, for which $E(3)$ is appropriate. In these cases, different constructions of Wigner-like representations have been proposed in [44, 74].

Mappings into meta phase space, related to the co-adjoint representation of the symmetry group, with a map defined as a Fourier-like transform of the group element in the polar parametrization, were analyzed in [75] and applied to representations of polarization states of light in [76].

Finally, one should not overlook various other types of maps introduced and discussed in the recent literature. This includes the approaches of [77, 78] applicable to physical systems with curved configuration space and related to the group manifold based on the 'midpoint' approach to the Wigner–Weyl mapping [78, 79].

Beyond $SU(2)$, applications of the higher symmetry $SU(n)$ maps may pave the way to investigation of the semi-classical dynamics of phenomena which can be realized in more than one way and have different time scales. Broadly speaking, this is a consequence of the weight space being no longer one-dimensional: as a result there is typically more than one path in weight space connecting the initial and final states, and carefully tailored Hamiltonians can enhance or lessen correlations or other quantum features along different paths. An example of this was presented in [80].

In spite of this possibility, applications of the higher symmetry $SU(n)$ maps are limited and not well discussed in literature [23, 48, 81]. The general construction of phase space mappings is currently available for the symmetric representations of the $SU(n)$ group, and presented in section 7. Some results for $SU(3)$ maps-the next simplest case after $SU(2)$- are presented in sections 7.2 and 7.4, but technical challenges impede the development of the star-product machinery, resulting in a significant reduction of the systematic use of such mappings beyond the semi-classical limit [82].

2. Axiomatic formulation: Stratonovich–Weyl scheme

In this section we briefly outline the main ingredients of the axiomatic Stratonovich–Weyl construction of phase space mapping (see for an extended discussion e.g. [35, 37]). We suppose for now a quantum system with states elements a Hilbert space $\mathbb{H}$ which carries a unitary irreducible representation λ of a compact Lie group $\mathfrak{G}$.

2.1. The coset space

We write $T^{\lambda}(\omega)$ for the matrix realization of the element $\omega \in \mathfrak{G}$ in the irreducible representation (irrep) λ. Since λ is usually fixed, we generally omit it and use $T(\omega)$ to lighten the notation. $T(\omega)$ act by linear transformations on $\mathbb{H}$.

The Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ can be organized in the usual way in a set of commuting operators spanning the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a set of raising operators $\mathfrak{n}^{+}$ and a set of lowering operators $\mathfrak{n}^{-}$.

As $\mathbb{H}$ carries the irrep λ, a basis for $\mathbb{H}$ is also a basis for the irrep λ. Within the basis set there is a unique (up to a phase) *highest weight state* $|\,\lambda; \text{h.w.}\rangle$ with the property that

$$h_k|\,\lambda; \text{h.w.}\rangle = \nu_k|\lambda; \text{h.w.}\rangle, \qquad n_i^{+}|\,\lambda; \text{h.w.}\rangle = 0, \tag{2.1}$$

for any $h_k \in \mathfrak{h}$ and $n_i^{+} \in \mathfrak{n}^{+}$.

Let $\mathfrak{H}$ be the largest subgroup of $\mathfrak{G}$ that leaves $|\,\lambda; \text{h.w.}\rangle$ invariant (up to a phase). A classic result [83] shows the coset $\mathcal{M} = \mathfrak{G}/\mathfrak{H}$ is isomorphic to the classical phase space for the system. The group $\mathfrak{G}$ acts transitively on $\mathcal{M}$; points in $\mathcal{M}$ are denoted by Ω, and an arbitrary element $\omega \in \mathfrak{G}$ can be decomposed as

$$\omega = \Omega \circ \eta, \qquad \Omega \in \mathfrak{G}/\mathfrak{H}, \quad \eta \in \mathfrak{H}. \tag{2.2}$$

2.2. The s-parameterized kernel $\hat{w}^{(s)}(\Omega)$

Following Stratonovich [17], a one-to-one mapping from the space of operators $L(\hat{f}\,)$ acting in $\mathbb{H}$ to the family of functions on $\mathcal{M}$ (labelled by an index s)

$$\hat{f} \leftrightarrow W_f^{(s)}(\Omega) \in \mathcal{M}. \tag{2.3}$$

is implemented though a trace operation with a kernel $\hat{w}^{(s)}(\Omega)$

$$W_f^{(s)}(\Omega) = \text{Tr}(\hat{f}\;\hat{w}^{(s)}(\Omega)). \tag{2.4}$$

The inverse mapping is an integral transform

$$\hat{f} = \int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\hat{w}^{(-s)}(\Omega)W_f^{(s)}(\Omega), \tag{2.5}$$

where $\mathrm{d}\mu(\Omega)$ is the invariant measure on the coset $\mathfrak{G}/\mathfrak{H}$.

The mathematical constructions of the kernel $\hat{w}^{(s)}(\Omega)$ is constrained to satisfy the following familiar rules:

1. To guarantee that Hermitian operators $\hat{f}$ are mapped to real functions $W_f^{(s)}(\Omega)$ on $\mathcal{M}$, we require that $\hat{w}^{(s)}(\Omega)$ be Hermitian: $\hat{w}^{(s)}(\Omega) = \hat{w}^{(s)}(\Omega)^{\dagger}$;
2. The trace of an operator $\hat{f}$ becomes a phase space integration

$$\text{Tr}(\hat{f}\,) \mapsto \int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\, W_f^{(s)}(\Omega) \tag{2.6}$$

   by imposing the normalization conditions

$$\text{Tr}(\hat{w}^{(s)}(\Omega)) = 1, \qquad \int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\, \hat{w}^{(s)}(\Omega) = \mathbb{1}. \tag{2.7}$$

3. The covariance of map of equation (2.3) is ensured by the requirement

$$T(\omega)\hat{w}^{(s)}(\Omega)T^{\dagger}(\omega) = \hat{w}^{(s)}(\omega \circ \Omega), \quad \omega \in \mathfrak{G}. \tag{2.8}$$

This covariance property implies $W^{(s)}_{f_\omega}(\Omega) = W^{(s)}_{f}(\omega^{-1}\circ\Omega)$, if $\hat{f}_\omega := T(\omega)\hat{f}\,T^\dagger(\omega)$.

4. Finally, the trace of a product of operators is a convolution of phase space symbols

$$\mathrm{Tr}(\hat{f}\;\hat{g}) = \int_{\mathcal{M}} \mathrm{d}\mu(\Omega) W^{(s)}_{f}(\Omega) W^{(-s)}_{g}(\Omega), \tag{2.9}$$

when we require the kernel to have a traciality property:

$$\mathrm{Tr}(\hat{w}^{(s)}(\Omega)\hat{w}^{(-s)}(\Omega')) = \Delta(\Omega,\Omega') \tag{2.10}$$

where $\Delta(\Omega,\Omega')$ satisfies the self-reproducing condition

$$\int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\Delta(\Omega,\Omega')f(\Omega) = f(\Omega'), \tag{2.11}$$

for any arbitrary $f(\Omega)$ on $\mathcal{M}$. This ensures, when $\hat{f} = \hat{\rho}$, that the quantum mechanical average of an operator $\hat{g}$ is the phase space integration of the corresponding symbol weighted by a probability (quasi)-distribution.

Because the kernel is constructed to be invariant under $\mathfrak{H}$

$$T(\eta)\hat{w}^{(s)}(0)T^\dagger(\eta) = \hat{w}^{(s)}(0), \quad \eta\in\mathfrak{H}, \tag{2.12}$$

a convenient representation of the kernel is the explicitly covariant form

$$\hat{w}^{(s)}(\Omega) = T(\omega)\hat{w}^{(s)}(0)T^\dagger(\omega), \tag{2.13}$$

where $T(\omega) = T(\Omega)T(\eta)$ with $\omega = \Omega\circ\eta$, $\eta\in\mathfrak{H}$ and $\Omega\in\mathfrak{G}/\mathfrak{H}$.

The Stratonovich s-parametrized kernel for $SU(2)$ is given in equations (3.8). For $SU(n)$ the general form is given in equation (7.8); for $SU(3)$ the various factors required to specialize equation (7.8) are found in equations (7.20), (7.24) and (7.26).

2.3. s-parameterized quasi-distributions

When $\hat{f}$ is the density matrix $\hat{\rho}$, its symbol $W^{(s)}_{\rho}(\Omega)$ is usually called the *quasi-distribution function* of the system.

The kernel is constructed so as to satisfy requirements stemming from early work on quasi-distribution functions. By tradition the Husimi-Berezin Q-function [18, 19] corresponds to the value $s = -1$:

$$\hat{\rho}\to Q_\rho(\Omega) := W^{(s=-1)}_{\rho}(\Omega) = \langle\lambda;\Omega|\hat{\rho}|\,\lambda;\Omega\rangle, \tag{2.14}$$

where

$$|\lambda;\Omega\rangle := T(\Omega)|\,\lambda;\mathrm{h.w.}\rangle, \qquad \Omega\in\mathfrak{G}/\mathfrak{H}, \tag{2.15}$$

is the Perelomov coherent state [41, 42] for the irrep λ of $\mathfrak{G}$.

By construction the Q-function is a positive distribution (since $\hat{\rho}$ is a positive operator). The Q-function is typically used for representation of quantum states when interference effects are not of major interest. From equation (2.5) it follows immediately that the kernel $\hat{w}^{(s)}(\Omega)$ must satisfy, for $s = -1$, the 'boundary condition'

$$\hat{w}^{(-1)}(\Omega) = T(\Omega)|\lambda;\mathrm{h.w.}\rangle\langle\lambda;\mathrm{h.w.}|T^\dagger(\Omega). \tag{2.16}$$

The value $s = 1$ is used for the so-called P-function:

$$\hat{\rho} \rightarrow P_\rho(\Omega) := W_\rho^{(s=1)}(\Omega) = \int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\, P_\rho(\Omega)\, |\lambda;\Omega\rangle\langle\lambda;\Omega|. \tag{2.17}$$

The P-distribution frequently has a quasi-singular behaviour: for instance, the P-symbol of a coherent state (2.15) is a δ-function on the classical manifold.

The condition of equation (2.9) shows that, in general, the trace of two operators will require the s and $-s$ symbols of these operators. For this reason, the $s = 1$ and $s = -1$ symbols of an operator $\hat{f}$ are *dual* to each other and known as the contravariant and covariant symbols, respectively. The duality between the P-function and the Q-function can be used as an alternative 'boundary condition' for the kernel.

By interpolating between $s = -1$ and $s = 1$, we obtain the self-dual $s = 0$ solutions [35], leading to the so-called *Wigner symbol* for which we omit the s index when $s = 0$:

$$W_f(\Omega) := W_f^{(s=0)}(\Omega). \tag{2.18}$$

When $\hat{f} = \hat{\rho}$, the Wigner quasi-distribution can be negative and is appropriate for highlighting and detecting quantum interference effects especially in systems with HW symmetry. This negativity property has also been proposed [36] for the detection of 'quantumness' of states. A drawback of the Wigner function is its extremely noisy structure for nontrivial linear combinations of basis states, which does not make it very useful for the identification of complex states.

The assignment of the values s has a direct interpretation in harmonic oscillator systems, where s refers to an ordering of operators. Although this interpretation is not formally possible in $SU(n)$ systems, the values $s = 1,0$ and -1 are still traditionally associated with the Q, Wigner and P-functions respectively.

The s-parametrized maps can be also used for the phase space description of compound systems if the density matrix can be decomposed into a direct sum of operators acting in subspaces invariant under the action of the symmetry group. Such subspaces are typically determined by as a set $\{I\}$ of integrals of motion; a one-to-one mapping can be established for every component

$$\hat{\rho}_I \Leftrightarrow W_\rho^{(s)}(\Omega; I) \tag{2.19}$$

in the decomposition

$$\hat{\rho} = \sum_I p_I \hat{\rho}_I. \tag{2.20}$$

The use of integrals of motion is especially convenient if the system, once reduced to a submanifold with fixed constants of motion, retains $SU(n)$ as a dynamical symmetry group so that the action of $SU(n)$ does not mix the subspaces.

The standard phase space approach cannot be applied for the construction of maps (with a required symmetry properties) if the density matrix cannot be decomposed on the irreducible representations of the dynamical group.

To deal with such cases we may relax the requirement of mapping operators into distributions in a classical phase space but still try to fulfill the Stratonovich–Weyl conditions. Neither will the invariance property of equation (2.12) be necessarily satisfied, nor will interpretation of the Q- and P-distribution in terms of Perelomov-like coherent states. Details of this kind of generalized map for $SU(2)$ are presented in section 4.

It follows from equation (2.5) that the density matrix can be expressed in the so-called *tomographic form*, i.e. in terms of probability of its projection on the coherent states [15, 86]:

$$\hat{\rho} = \int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\hat{w}^{(1)}(\Omega)\langle\lambda;\mathrm{h.w.}|\hat{\rho}(\Omega)|\lambda;\mathrm{h.w.}\rangle, \tag{2.21}$$

$$= \int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\hat{w}^{(1)}(\Omega)\langle\lambda;\Omega|\,\hat{\rho}\,|\lambda;\Omega\rangle, \tag{2.22}$$

where $\hat{\rho}(\Omega) = T^{\dagger}(\Omega)\hat{\rho}\,T(\Omega)$ is the transformed density matrix. In practice, equation (2.21) often leads to substantial errors due to a singular nature of the mapping kernel $\hat{w}^{(1)}(\Omega)$, including for compact groups in limit of large dimensions.

As a result it may be more convenient to reconstruct directly from the experimental data the Wigner function of the state, using

$$W_{\rho}(\Omega') = \int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\langle\lambda;\mathrm{h.w.}|\hat{\rho}(\Omega)|\lambda;\mathrm{h.w.}\rangle\mathrm{Tr}(\hat{w}^{(1)}(\Omega)\hat{w}^{(0)}(\Omega')). \tag{2.23}$$

In section 5.1 we discuss applications of phase space methods to the tomography of a density matrix $\hat{\rho}$ for spin-like systems.

2.4. Dynamics, semi-classical limit and TWA

The star-product takes into account the non-commutative features of quantum mechanical operators. It is formally defined through

$$W_f^{(s)}(\Omega) * W_g^{(s)}(\Omega) = W_{fg}^{(s)}(\Omega) := \mathbf{L}_{f,g}\left[W_f^{(s)}(\Omega)W_g^{(s)}(\Omega)\right], \tag{2.24}$$

where $\mathbf{L}_{f,g}$ is a differential or integral operator acting on the ordered product of symbols: $\mathbf{L}_{f,g} \neq \mathbf{L}_{g,f}$ in general. In the particular case of the Wigner mapping, the star-product operation satisfies the useful property

$$\int_{\mathcal{M}} \mathrm{d}\mu(\Omega)W_f^{(0)}(\Omega) * W_g^{(0)}(\Omega) = \int_{\mathcal{M}} \mathrm{d}\mu(\Omega)W_f^{(0)}(\Omega)W_g^{(0)}(\Omega), \tag{2.25}$$

which follows immediately from the integral representation of $L_{f,g}$,

$$W_{fg}^{(0)}(\Omega) = \int_{\mathcal{M}} \mathrm{d}\mu(\Omega')\mathrm{d}\mu(\Omega'')\mathcal{K}(\Omega,\Omega',\Omega'')W_f^{(0)}(\Omega')W_g^{(0)}(\Omega''), \tag{2.26}$$

$$\mathcal{K}(\Omega,\Omega',\Omega'') = \mathrm{Tr}(\hat{w}^{(0)}(\Omega)\hat{w}^{(0)}(\Omega')\hat{w}^{(0)}(\Omega'')). \tag{2.27}$$

The star-product enables the 'translation' of the operator algebra into operations with phase space functions.

Using the star-product one can represent the Schrödinger equation for the density matrix

$$\mathrm{i}\partial_t\hat{\rho} = [\hat{H},\hat{\rho}], \tag{2.28}$$

as a Liouville-type evolution for s-parametrized symbols, called the *Moyal equation*:

$$\mathrm{i}\partial_t W_{\rho}^{(s)}(\Omega) = \{W_H^{(s)}(\Omega), W_{\rho}^{(s)}(\Omega)\}_M, \tag{2.29}$$

where $W_H^{(s)}(\Omega)$ is the symbol of the Hamiltonian $\hat{H}$ of the system and

$$\{W_f^{(s)}(\Omega), W_g^{(s)}(\Omega)\}_M := W_f^{(s)}(\Omega) * W_g^{(s)}(\Omega) - W_g^{(s)}(\Omega) * W_f^{(s)}(\Omega), \tag{2.30}$$

is the so-called Moyal bracket with $*$ denoting the star-product.

To obtain the Moyal equation of equation (2.29) for a wide class of physically meaningful Hamiltonians, one can successfully use instead of the full star-product machinery *correspondence rules* describing the action group generators on the density matrix. The formal expressions of group generators acting on the density matrix are sometimes called the Bopp operators [87, 88, 90] or D-operator algebras [42, 89].

In practice, the local (differential) form of the star-product operator $\mathbf{L}_{f,g}$ is fairly complicated even for systems having $H(1)$ or $SU(2)$ as symmetry groups. In applications, the star-product is often limited to the zeroth and first order terms of the expansion of $\mathbf{L}_{f,g}$ in a semi-classical parameter ε that captures the strength of quantum fluctuations through some physical property frequently expressed in terms of the inverse of the total energy, the number of photons, the spin size, the inverse dimension of the irrep λ of the symmetry group, etc. Specialized forms of the semi-classical parameter will be considered in sections 3.2, 4.4 and 7.4.

The expansion of the star-product immediately yields a very convenient feature of a phase space representation of the quantum evolution: the possibility of expanding the Moyal brackets in series in the semi-classical parameter ε to eventually arrive at a first-order Liouville-type equation [53] describing the time evolution of the initial phase space distribution via trajectories defined through a set of first-order ordinary differential equations.

In light of the correspondence principle, the leading order terms of the Moyal equation can be represented in the form of Poisson brackets of the symbols of the functions f and g on the classical manifold, so that the s-parametrized Wigner function dynamics is governed by

$$\partial_t W_\rho^{(s)}(\Omega) = \varepsilon\{W_\rho^{(s)}(\Omega), W_H^{(s)}(\Omega)\}_P + \text{correction terms}, \tag{2.31}$$

where $\{f(\Omega), g(\Omega)\}_P$ is the Poisson bracket on the manifold. Dropping the correction terms on the rhs of equation (2.31) gives the so-called *truncated Wigner approximation*, or TWA. This scheme has been widely used in numerous applications (for a recent review of applications to HW case see e.g. [63]).

The approximate evolution equation of equation (2.31) can be solved using the method of characteristics, resulting in an evolution of the Wigner function of the form

$$W_\rho^{(s)}(\Omega|t) = W_\rho^{(s)}(\Omega(-t)|t=0), \tag{2.32}$$

where $\Omega(t)$ denotes classical trajectories. These trajectories are solutions of the classical Hamiltonian equations, i.e. each point of the initial quantum distribution evolves along the corresponding classical trajectory.

The evolution distorts the initial distribution but cannot convert positive regions of the Wigner function into negative regions (and vice versa); this follows from conservation of local Poincaré invariants under the action of Poisson bracket. In this sense, the phase space distribution in the TWA behaves as a incompressible fluid, which greatly facilitates an intuitive interpretation of the dynamics of the system. In the case of compound systems with the density matrix of the form (2.20) the evolution of the whole system is described by a weighted sum of the dynamics in each submanifold, with the weight p_I for each submanifold obtained from equations (2.19) and (2.20):

$$W_\rho^{(s)}(\Omega|t) = \sum_I p_I W^{(s)}(\Omega_I(-t); I|t=0), \tag{2.33}$$

where the classical trajectories $\Omega_I(t)$ in different manifolds satisfy the same Hamiltonian equations and differ only by the value of the integral of motion.

The correction terms are known for the Heisenberg–Weyl and $SU(2)$ systems; they are generically expanded in terms of the semi-classical parameter as follows:

$$\text{correction terms} = s\mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^3). \tag{2.34}$$

The net size of the correction terms containing higher-order derivatives depends not only on the exponent of the semi-classical parameter but also on a 'dynamical part' which is closely related to the degree of non-linearity of the Hamiltonian. In addition, in systems for which equation (2.20) holds, the semi-classical parameter ε is usually a function of integrals of motion related to the number of excitations.

Clearly, the $\mathcal{O}(\varepsilon^2)$ terms disappear for the $s = 0$ (Wigner) mapping. The absence of this second order contributions $\sim \varepsilon^2$ is especially important in several physically interesting situations where, due to non-linearity of the Hamiltonian, the value of first correction does not tend to zero in the semi-classical limit. This explains why the Wigner function dynamics is usually used for the semi-classical description of quantum dynamics.

The TWA of equation (2.32) describes well the initial stage of the nonlinear dynamics—when one can neglect self-interference—for a class of initial so-called semi-classical states [2, 23, 56–58, 61, 62]. In applications, these semi-classical states are represented as localized distributions (as for instance, are coherent states) and the 'classical domain' of their evolution, i.e. the timescale over which they faithfully follow the classical trajectories, is related to their transformation properties under the action of the invariance group of the Hamiltonian [91].

The semi-classical solution of equation (2.32) allows the calculation of mean values of observables $\{\hat{f}_j\}$ leading to considerably better results than obtained in the parametric approximation, which consists in decoupling correlators in the Heisenberg equations:

$$\partial_t \langle \hat{f}_j(t) \rangle = \sum_k \beta_{jk} F(\langle \hat{f}_k(t) \rangle), \tag{2.35}$$

where the functions $F_{(\hat{f}_k)}$ (which need not be linear) are defined through

$$\sum_k \beta_{jk} F(\hat{f}_k) = i[\hat{f}_j, \hat{H}]. \tag{2.36}$$

The TWA leads to the evolution equation for average values of f_j in the form

$$\partial_t \langle \hat{f}_j(t) \rangle = \sum_k \alpha_{jk} \langle F(\hat{f}_k)(t) \rangle, \tag{2.37}$$

where

$$\alpha_{jk} = \beta_{jk} + s\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^2), \tag{2.38}$$

in accordance with (2.34). This allows the use the semi-classical solution (2.32) for the description of short-time physical phenomena - for instance, squeezing—attributed to the deformation of the initial distribution.

One indicator of deviation from the semi-classical evolution could be taken to be higher moments of the Wigner distribution

$$m_k(t) = \left(\frac{2S+1}{4\pi} \right)^k \int_{\mathcal{M}} \mathrm{d}\,\Omega\, W_\rho^k(\Omega|-t), \qquad k > 2. \tag{2.39}$$

In the semi-classical picture, where the evolution of phase space coordinates is generated by canonical transformations, these higher moments are time-independent. Thus, the deviations from their initial values describe a spread of the initial distribution in phase space due to purely quantum effects in the evolution. More precisely, since $\partial_t m_k(t=0) = 0$, the widths of the $m_k(t)$ at $t = 0$, given by $\delta_k \sim \partial_t^2 m_k(t = 0)$ define the timescales over which the semi-classical approximation allows a description of the dynamics of different sets of quantum observables. In practice, the fourth-order moment is most commonly used [92] at it is positive and captures well the evolution of quantum interference effects, and the so-called semiclassical time, τ_{sem} can be defined as $\tau_{\text{sem}} \sim \delta_4^{-2}$.

3. The *SU*(2) Stratonovich–Weyl mapping

The Moyal-Stratonovich quantization program encapsulated in equations (2.4) and (2.5) can be carried out in a systematic way for symmetric representations of compact semi-simple groups. The group $SU(2)$ is the simplest case where a mapping kernel $\hat{w}_S^{(s)}(\Omega)$ can be constructed in this program for a system with a single value of S. Owing to the simplicity of the kernel, a number of analytical results can be obtained, as exemplified below. In particular, a local form of the star-product can be explicitly found, so that the semi-classical limit of the Stratonovich–Weyl mapping becomes easily accessible.

3.1. The kernel and some properties of the resulting mappings

We consider a $2S + 1$-dimensional Hilbert space $\mathbb{H}$ of that carries a unitary irreducible representation of the $SU(2)$ group, with group element $\omega := (\phi, \theta, \psi)$ expressed in terms of three Euler angles leading to a $(2S + 1) \times (2S + 1)$ matrix representation $T(\omega)$ of $\omega \in SU(2)$ realized by the sequence of exponentiations

$$T(\omega) = \mathrm{e}^{-\mathrm{i}\phi\hat{S}_z}\mathrm{e}^{-\mathrm{i}\theta\hat{S}_y}\mathrm{e}^{-\mathrm{i}\psi\hat{S}_z}, \quad 0 \leqslant \phi < 2\pi, \quad 0 \leqslant \theta < \pi, \quad 0 \leqslant \psi < 4\pi. \tag{3.1}$$

Here $\hat{S}_{x,y,z} \in su(2)$ are generators of the $SU(2)$ group and satisfy the usual angular momentum commutation relations.

The Hilbert space $\mathbb{H}$ is spanned by the orthonormal basis $\{|S, m\rangle,\ m = -S, ..., S\}$. The basis elements are (as usual) chosen to be eigenstates of $\hat{S}_z$ and $\hat{\mathbf{S}}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$,

$$\hat{S}_z|S, m\rangle = m|S, m\rangle, \qquad \hat{\mathbf{S}}^2|S, m\rangle = S(S + 1)|S, m\rangle. \tag{3.2}$$

The usual $SU(2)$ coherent states $|\Omega; S\rangle$ are defined (up to a global phase) by action of the displacement operator

$$T(\Omega) = \exp\left[-\tfrac{1}{2}\theta(\hat{S}_+\,\mathrm{e}^{-\mathrm{i}\phi} - \hat{S}_-\,\mathrm{e}^{\mathrm{i}\phi})\right], \quad \Omega := (\theta, \phi), \tag{3.3}$$

on the highest weight state, with explicit expression in terms of (θ, ϕ) given by:

$$\begin{aligned}|\Omega; S\rangle &= T(\Omega)|S, S\rangle \\ &= \sum_{m=-S}^{S}\sqrt{\frac{(2S)\,!}{(S+m)\,!\,(S-m)\,!}}\,\mathrm{e}^{-\mathrm{i}m\,\phi}\left(\cos\tfrac{1}{2}\theta\right)^{S+m}\left(\sin\tfrac{1}{2}\theta\right)^{S-m}|S, m\rangle.\end{aligned} \tag{3.4}$$

The corresponding classical phase space, like the coherent states, depends only on the two coordinates (θ, ϕ), and is isomorphic to the 2-sphere $\mathcal{S}^2(\theta, \phi) = SU(2)/U(1)$, where $U(1)$ corresponds to the transformation $\mathrm{e}^{-\mathrm{i}\psi\hat{S}_z}$ which leaves the highest weight state $|S, S\rangle$ invariant up to a phase.

The s-parametrized kernel $\hat{w}_S^{(s)}(\Omega)$ satisfying the Stratonovich–Weyl conditions is well-known [17, 32, 35]. It is constructed from a set of tensor operators [93]

$$\hat{T}^{S}_{LM} = \sqrt{\frac{2L+1}{2S+1}} \sum_{m,m'=-S}^{S} \left\langle \begin{matrix} S \\ m \end{matrix} ; \begin{matrix} L \\ M \end{matrix} \middle| \begin{matrix} S \\ m' \end{matrix} \right\rangle |S, m'\rangle \, \langle S, m|, \tag{3.5}$$

where $L = 0, 1, \ldots, 2S$ and $M = -L, \ldots, L$ and where $\left\langle \begin{smallmatrix} j_1 \\ m_1 \end{smallmatrix} ; \begin{smallmatrix} j_2 \\ m_2 \end{smallmatrix} \middle| \begin{smallmatrix} J \\ M \end{smallmatrix} \right\rangle$ is an $su(2)$ Clebsch–Gordan coefficient. The tensors of equation (3.5) form an orthogonal basis of operators in the space of $(2S+1) \times (2S+1)$ matrices acting on the system and are transformed under the $SU(2)$ group action of equation (3.1) as

$$T(\omega)\hat{T}^{S}_{LM}T^{\dagger}(\omega) = \sum_{M'=-L}^{L} D^{L}_{M'M}(\omega)\hat{T}^{S}_{LM'}, \tag{3.6}$$

where

$$D^{L}_{M'M}(\omega) = \langle L, M'|\, T\,(\omega)|L, M\rangle, \tag{3.7}$$

is the Wigner D-function.

In terms of these tensors the kernel $\hat{w}^{(s)}_{S}(\Omega)$ is written in the expanded form

$$\hat{w}^{(s)}_{S}(\Omega) = \sqrt{\frac{4\pi}{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} \left\langle \begin{matrix} S \\ S \end{matrix} ; \begin{matrix} L \\ 0 \end{matrix} \middle| \begin{matrix} S \\ S \end{matrix} \right\rangle^{-s} Y^{*}_{LM}(\Omega)\, \hat{T}^{S}_{LM}, \tag{3.8}$$

with $Y_{LM}(\Omega) := Y_{LM}(\theta, \phi)$ the usual spherical harmonics. In this way the kernel automatically satisfies the required transformation and normalization properties given in equations (2.7)–(2.10), where integration is over $\mathcal{S}^2$ with the measure

$$\frac{2S+1}{4\pi} \int \mathrm{d}\,\Omega = \frac{2S+1}{4\pi} \int_{0}^{\pi} \mathrm{d}\theta \sin\theta \int_{0}^{2\pi} \mathrm{d}\phi. \tag{3.9}$$

The kernel $\hat{w}^{(s)}_{S}(\Omega)$ can also be given in the explicitly covariant form

$$\hat{w}^{(s)}_{S}(\Omega) = T(\Omega)\, \hat{w}^{(s)}_{S}(0)\, T^{\dagger}(\Omega), \tag{3.10}$$

$$\hat{w}^{(s)}_{S}(0) = \sum_{L=0}^{2S} \left\langle \begin{matrix} S \\ S \end{matrix} ; \begin{matrix} L \\ 0 \end{matrix} \middle| \begin{matrix} S \\ S \end{matrix} \right\rangle^{-s} \sqrt{\frac{2L+1}{2S+1}}\, \hat{T}^{S}_{L0}, \tag{3.11}$$

with $\hat{w}^{(s)}_{S}(0)$ explicitly invariant under z-rotations in the basis of equation (3.2):

$$\mathrm{e}^{-\mathrm{i}\psi\hat{S}_z}\hat{w}^{(s)}_{S}(0)\mathrm{e}^{\mathrm{i}\psi\hat{S}_z} = \hat{w}^{(s)}_{S}(0). \tag{3.12}$$

The particular case $s = -1$ is just a projection into the highest state:

$$\hat{w}^{(s=-1)}_{S}(0) = |S, S\rangle\langle S, S|, \tag{3.13}$$

as discussed in equation (2.16).

The symbol of an operator $\hat{f}$ transformed by a group element T as given in equation (3.1), $\hat{f}\,(\omega) := T^{\dagger}(\omega)\hat{f}\;T(\omega)$ is recovered from that of $\hat{f}$ using the covariance condition equation (2.8)

$$W^{(s)}_{f(\omega)}(\mathbf{n}) = W^{(s)}_{f}(\omega^{-1}\circ\mathbf{n}), \tag{3.14}$$

where it is convenient to consider the argument of the symbol as a unit vector

$$\mathbf{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \tag{3.15}$$

Table 1. Some simple operators and their symbols. Here: $\{\hat{A},\hat{B}\}_+$ is the anticommutator, $A_S^{(s)} = [S(2S-1)]^{(1-s)/2}\,[(2S+3)(S+1)]^{(1+s)/2}$ and $\mathbf{n}$ is given in equation (3.15).

Operator	Symbol
$\hat{S}_i$	$W^{(s)}_{\hat{S}_i}(\Omega) = \left(\frac{S}{S+1}\right)^{-s/2}\sqrt{S(S+1)}\,n_i$
$\{\hat{S}_i,\hat{S}_k\}_+$	$W^{(s)}_{\{\hat{S}_i,\hat{S}_k\}_+}(\Omega) = A_S^{(s)} n_i n_k, \quad (i \neq k)$
$\hat{S}_j^2$	$W^{(s)}_{\hat{S}_j^2}(\Omega) = \frac{1}{2}A_S^{(s)}\left(n_i^2 - \frac{1}{3}\right) + \frac{S(S+1)}{3}$

pointing in the direction θ,ϕ on the sphere, so that the group element acts on it as a 3×3 matrix [93].

An operator acting in the $2S+1$ Hilbert space can be decomposed on the basis of the irreducible tensor operators equation (3.5) as

$$\hat{f} = \sum_{L=0}^{2S}\sum_{M=-L}^{L} f_{LM}\,\hat{T}^S_{LM}, \qquad f_{LM} = \mathrm{Tr}((\hat{T}^S_{LM})^\dagger \hat{f}). \tag{3.16}$$

By linearity its symbol is the sum of symbols of the tensor components:

$$W_f^{(s)}(\Omega) = \frac{2\sqrt{\pi}}{\sqrt{2S+1}}\sum_{L=0}^{2S}\sum_{M=-L}^{L}\left\langle \begin{matrix} S \\ S \end{matrix};\begin{matrix} L \\ 0 \end{matrix}\middle|\begin{matrix} S \\ S \end{matrix}\right\rangle^{-s} f_{LM}\,Y_{LM}(\Omega). \tag{3.17}$$

Some simple examples are provided in table 1.

In a manner reminiscent of Heisenberg–Weyl systems, the symbols $W_f^{(\pm 1)}(\Omega)$ are related through a simple transformation:

$$W_f^{(-s)}(\Omega) = \sum_{L=0}^{2S}\left\langle \begin{matrix} S \\ S \end{matrix};\begin{matrix} L \\ 0 \end{matrix}\middle|\begin{matrix} S \\ S \end{matrix}\right\rangle^{2s}\int_{\mathcal{S}^2}\mathrm{d}\,\Omega'\; W_f^{(s)}(\Omega')P_L(\cos\zeta), \tag{3.18}$$

where $s=\pm 1$, $P_L(z)$ is the Legendre polynomial, and

$$\cos\zeta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'). \tag{3.19}$$

The symbol $W_f^{(-1)}(\Omega)$ is obtained from $W_f^{(1)}(\Omega)$ by a smoothing transformation, as becomes clear for $S \gg 1$ since

$$\left\langle \begin{matrix} S \\ S \end{matrix};\begin{matrix} L \\ 0 \end{matrix}\middle|\begin{matrix} S \\ S \end{matrix}\right\rangle^{2} \simeq \frac{2L+1}{2S+1}\exp\left[-\frac{L(L+1)}{2S+1}\right], \tag{3.20}$$

while the inverse transformation becomes singular in this limit.

The kernel for a multipartite system is a product of single particle kernels given in equation (3.8), and the corresponding classical manifold is $\mathcal{S}^2\times ... \times\mathcal{S}^2$. It follows that

$$W_\rho^{(s)}(\Omega_1,...,\Omega_N) = \mathrm{Tr}(\rho\,\hat{\omega}^{(s)}_{S;1}(\Omega_1)\otimes\ldots\otimes\hat{\omega}^{(s)}_{S;N}(\Omega_N)). \tag{3.21}$$

The differential form of star-product $*$ that depends on the local coordinates Ω,

$$W_f^{(s)}(\Omega) * W_g^{(s)}(\Omega) := \mathbf{L}_{f,g}(\Omega)(W_f^{(s)}W_g^{(s)}) \tag{3.22}$$

was found in [45, 46] (see also [47]) and has a somewhat involved form given in equation (B.1) of the appendix. In what follows we examine its approximate form in the asymptotic limit $S \gg 1$.

3.2. Semi-classical limit

The semi-classical limit in spin-like systems is related to large value of spin or, alternatively, large dimension of $SU(2)$ representations. It is natural to choose the semi-classical parameter as $\varepsilon = (2S+1)^{-1} \gg 1$. The semi-classical states are usually associated with states having a smooth and localized distributions with extension $\sim \sqrt{S}$. Algebraically, the density matrix of semi-classical states is decomposed only on low rank tensors of equation (3.16). For such states, one can provide a quite detailed description of the kinematic and dynamic.

3.2.1. The kernel $\hat{w}_S^{(s)}(\Omega)$. The actual expression for symbols of even simple physical states can be quite involved, especially for the $s = 0$ (Wigner) mapping. In the limit of large spin, $S \gg 1$, an asymptotic expression for $\hat{w}_S^{(0)}(\Omega)$, valid for integer S, can be obtained [94] as

$$\hat{w}_S^{(0)}(\Omega) \simeq (-1)^S \left(1 + \frac{\hat{\mathbf{S}} \cdot \mathbf{n}}{S}\right) \mathrm{e}^{-\mathrm{i}\pi \hat{\mathbf{S}} \cdot \mathbf{n}}, \tag{3.23}$$

$$\hat{w}_S^{(0)}(0) = (-1)^S \sum_m \left(1 + \frac{m}{S}\right) \mathrm{e}^{-\mathrm{i}\pi m} | S, m\rangle\langle S, m|, \tag{3.24}$$

where $\hat{\mathbf{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ and $\mathbf{n}$ is the unit vector of equation (3.15).

For instance, the Wigner function of the state $|S, m\rangle$ acquires the following asymptotic form

$$W_m(\Omega) \simeq (-1)^S \, d^S_{mm}(2\theta) \left(1 + \frac{m}{S} \cos\theta\right) \tag{3.25}$$

$$+ \frac{(-1)^S}{2S} \sin\theta \left(\mathrm{e}^{-\mathrm{i}\phi} d^S_{mm+1}(2\theta) \sqrt{(S-m)(S+m+1)} + \mathrm{c.c.}\right). \tag{3.26}$$

The symbols of general coherent states are very often useful in applications; these can be easily obtained using the covariance property of equation (3.14), i.e. by rotating the symbol of the state $|S, S\rangle$ so that $W^{(0)}_{m=S}(\mathbf{n}) \simeq n_z^{2S}(1 + n_z)$. In this manner, for instance, the Wigner function of an equatorial coherent state $|\theta_\circ = \pi/2, \phi_\circ = 0\rangle$ is just

$$W_{\pi/2}(\Omega) = (\sin\theta \cos\phi)^{2S-1} \left[1 + \sin\theta \cos\phi\right]. \tag{3.27}$$

We also note that the approximate kernel equation (3.23) leads to an accurate form of Wigner function, including situations involving macroscopic quantum superpositions such as Schrödinger cat states $\sim |S, S\rangle + |S, -S\rangle$.

The asymptotic form of the star-product operator $\mathbf{L}^{(s)}_{f,g}(\Omega)$ in the limit $S \gg 1$ is also considerably simplified

$$\mathbf{L}^{(s)}_{f,g}(\Omega) = \mathbb{1} \otimes \mathbb{1} + \frac{\varepsilon}{2} \left[(1-s) \mathbb{S}_f^- \otimes \mathbb{S}_g^+ - (1+s) \mathbb{S}_f^+ \otimes \mathbb{S}_g^-\right] + \mathcal{O}(\varepsilon^2), \tag{3.28}$$

$$\mathbb{S}^\pm := \left(-\partial_\theta \mp \frac{\mathrm{i}}{\sin\theta} \partial_\phi\right). \tag{3.29}$$

where the action of the operation $\otimes$ on the product of symbols is defined as

$$(\mathbb{S}_f^- \otimes \mathbb{S}_g^+) W_f^{(s)}(\Omega) W_g^{(s)}(\Omega) := (\mathbb{S}^- W_f^{(s)}(\Omega))(\mathbb{S}^+ W_g^{(s)}(\Omega)). \tag{3.30}$$

A consequence of equation (3.28) is reflected in the expansion of the Moyal bracket of equation (2.30), which reduces to the Poisson bracket on the $\mathcal{S}^2$ sphere in the large spin limit [45]:

$$\partial_t W_\rho^{(s)}(\Omega) = 2\varepsilon \{ W_\rho^{(s)}(\Omega), W_H^{(s)}(\Omega) \}_P + s\mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^3), \tag{3.31}$$

where $\{\cdot,\cdot\}_P$ denotes the Poisson brackets on the sphere:

$$\{\cdot,\cdot\}_P = \frac{1}{\sin\theta}(\partial_\phi \otimes \partial_\theta - \partial_\theta \otimes \partial_\phi) = \hat{P}. \tag{3.32}$$

The first-order correction terms on the right of equation (3.31) have the form

$$\text{correction terms} = \varepsilon^2 s \Big[\mathcal{L}^2 \{ W_\rho^{(s)}, W_H^{(s)} \}_P - \{ \mathcal{L}^2 W_\rho^{(s)}, W_H^{(s)} \}_P \Big] + \mathcal{O}(\varepsilon^3), \tag{3.33}$$

where

$$\mathcal{L}^2 = -\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right], \tag{3.34}$$

is the $su(2)$ Casimir operator on the sphere, so that $\mathcal{L}^2 Y_{LM}(\theta,\phi) = L(L+1) Y_{LM}(\theta,\phi)$.

As an example of the type of second-order correction terms that occur, we may observe that the Hamiltonian

$$\hat{H} = \chi \hat{S}_z^2, \tag{3.35}$$

leads to the following exact evolution equation for the Wigner function

$$\partial_t W_\rho(\Omega) = -\chi \left[\frac{1}{2\varepsilon} \cos\theta \Phi(\mathcal{L}^2) - \varepsilon \left(\frac{1}{2}\cos\theta + \sin\theta\partial_\theta \right) \Phi^{-1}(\mathcal{L}^2) \right] \partial_\phi W_\rho(\Omega), \tag{3.36}$$

where the function $\Phi(\mathcal{L}^2)$ is defined as:

$$\Phi(\mathcal{L}^2) = \left[2 - \varepsilon^2(2\mathcal{L}^2 + 1) + 2\sqrt{(1-\varepsilon^2\mathcal{L}^2)^2 - \varepsilon^2} \right]^{1/2}. \tag{3.37}$$

If the limit of $\varepsilon \ll 1$, the leading term of equation (3.36) is $\sim \varepsilon^{-1}$; the terms of $\mathcal{O}(1)$ vanish and the first correction terms are $\sim \varepsilon$:

$$\partial_t W_\rho(\Omega) = -\chi \left[\varepsilon^{-1} \cos\theta \partial_\phi + \varepsilon \, \hat{\Xi} \right] W_\rho(\Omega), \tag{3.38}$$

where $\hat{\Xi}$ is a diffusion-like operator containing higher order derivatives:

$$\hat{\Xi} = -\frac{1}{2} \left[\cos\theta(\mathcal{L}^2 + 1) + \sin\theta\partial_\theta \right] \partial_\phi. \tag{3.39}$$

3.2.2. Correspondence rules. In applications, the *correspondence rules* (also called Bopp operators or D-algebra elements) are very useful [42, 88, 90]:

$$\left. \begin{array}{l} \hat{\rho}\hat{S}_z \\ \hat{S}_z\hat{\rho} \end{array} \right\} \leftrightarrow \left\{ \left(\mp \frac{1}{2}\mathbb{L}_z + \Lambda_0^{(s)}(\Omega) \right) W_\rho^{(s)}(\Omega), \right. \tag{3.40}$$

$$\left.\begin{array}{l}\hat{\rho}\hat{S}_{\pm}\\ \hat{S}_{\pm}\hat{\rho}\end{array}\right\}\leftrightarrow\left\{\left(\mp\frac{1}{2}\mathbb{L}_{\pm}+\Lambda_{\pm}^{(s)}(\Omega)\right)W_{\rho}^{(s)}(\Omega),\right. \tag{3.41}$$

where $\mathbb{L}_{\pm,z}$ are the first order differential operators,

$$\mathbb{L}_{\pm}=\mathrm{e}^{\pm\mathrm{i}\phi}\big(\pm\partial_{\theta}+\mathrm{i}\cot\theta\partial_{\phi}\big),\quad \mathbb{L}_{z}=-\mathrm{i}\partial_{\phi}. \tag{3.42}$$

satisfying the $su(2)$ commutation relations.

The operators $\Lambda_{0,\pm}^{(s)}(\Omega)$ have an exact simple form for $s=\pm1$ [89]:

$$\Lambda_{0}^{(\pm1)}(\Omega)=\frac{1}{2}\left(\frac{1}{\varepsilon}\cos\theta+s\cos\theta+s\sin\theta\partial_{\theta}\right), \tag{3.43}$$

$$\Lambda_{\pm}^{(\pm1)}(\Omega)=\mathrm{e}^{\pm\mathrm{i}\phi}\frac{\sin\theta}{2\varepsilon}\mp\frac{s}{2}\left[\cos\theta\,\mathbb{L}_{\pm}-\mathrm{e}^{\pm\mathrm{i}\phi}\sin\theta(\mathbb{L}_{z}\pm1)\right]. \tag{3.44}$$

Unfortunately the corresponding exact expressions for the Wigner function ($s=0$) are rather intricate (but provided in equations (B.10) and (B.11)), although the operators $\hat{\Lambda}_{0}^{(0)}(\Omega)$ and $\hat{\Lambda}_{\pm}^{(0)}(\Omega)$ do have good asymptotic properties:

$$\Lambda_{0}^{(0)}(\Omega)=\frac{1}{2\varepsilon}\cos\theta+\mathcal{O}(\varepsilon),\quad \Lambda_{\pm}^{(0)}(\Omega)=\mathrm{e}^{\pm\mathrm{i}\phi}\frac{\sin\theta}{2\varepsilon}+\mathcal{O}(\varepsilon), \tag{3.45}$$

where no first-order terms appear in the expansion of $\Lambda_{0,\pm}^{(0)}(\Omega)$ on ε.

3.3. Contraction limits

Two interesting non-compact groups can be obtained as contractions [84, 85] from the $SO(3)$ group: the Heisenberg–Weyl group and the group $E(2)$ of rigid motions of the 2-dimensional Euclidean plane. It results that the contraction procedure can be performed also on the level of mapping operators so as to obtain from equation (3.8) (for integer values of **S**) kernels for HW and $E(2)$ groups.

3.3.1. Contraction to the Heisenberg–Weyl group. Let us assume the density matrix for the system has a sharp maximum in the vicinity of the lowest state of the $2S+1$ dimensional representation, $\rho_{k-S,j-S}$, with $k,j\ll S$. Geometrically, these states with $m\approx-S$ are concentrated near the south pole of the Bloch sphere, so that in the kernel of equation (3.8) one should consider $\theta\to\pi$ [94, 95].

For $S\gg1$, the matrix elements of $su(2)$ generators between states in this region become indistinguishable (up to scaling) from those of Heisenberg–Weyl algebra [84, 96]:

$$\hat{S}_{+}\mapsto\sqrt{2S}\,\hat{a}^{\dagger},\qquad \hat{S}_{-}\mapsto\sqrt{2S}\,\hat{a},\qquad \hat{S}_{z}\mapsto\hat{N}-S, \tag{3.46}$$

where $\hat{N}$ satisfies

$$[\hat{N},\hat{a}^{\dagger}]=\hat{a}^{\dagger},\quad [\hat{N},\hat{a}]=-\hat{a},\quad [\hat{a},\hat{a}^{\dagger}]=1-\hat{N}/S,\quad \hat{N}\to\hat{a}^{\dagger}\hat{a}. \tag{3.47}$$

Using the asymptotic form of equation (3.23) with $\theta\to\pi$ and $S\to\infty$ so the product $r=\sqrt{S/2}\,\sin\theta$ remains finite, one obtains

$$\hat{w}_{S}^{(0)}(\Omega)\to2\exp\left[\mathrm{i}\pi(\hat{a}^{\dagger}-\alpha^{*})(\hat{a}-\alpha)\right]=\hat{w}^{(0)}(\alpha), \tag{3.48}$$

$$\alpha = r\,\mathrm{e}^{-\mathrm{i}\phi}, \tag{3.49}$$

i.e. exactly the Wigner kernel for Heisenberg–Weyl group [97].

The contraction of $\hat{w}_S^{(\pm 1)}(\Omega)$ to the corresponding HW kernels is done by observing that the spin coherent states of equation (3.4) are generated by applying the displacement operator $T(\omega)$ of equation (3.3) for $(\theta = \pi - \vartheta, \phi)$ to the lowest state of the representation, $|S, -S\rangle$, so that in the vicinity of the south pole, where $\vartheta \ll 1$, the displacement on the sphere $T(\Omega)$ contracts to a displacement in the plane:

$$T(\Omega) \to T(\alpha) = \exp(\alpha a^\dagger - \alpha^* a), \tag{3.50}$$

with α given in equation (3.49). Thus, identifying $|S, -S\rangle$ with $|0\rangle$ in the Fock basis we recover the projector on the harmonic oscillator coherent state. A similar contraction to HW is obtained in the vicinity of the north pole of the Bloch sphere.

3.3.2. Contraction to the E(2) group. If the elements of the density matrix $(\hat{\rho})_{kk'}$ that differ significantly from 0 are such that $|k|, |k'| \ll S$, the states are now concentrated in a band not too far from the equator of the Bloch sphere. For these states the matrix elements of $su(2)$ generators become [84], in the limit $S \to \infty$, indistinguishable from those of the $e(2)$ algebra, i.e. with the scaling

$$\frac{\hat{S}_x}{S} \mapsto \hat{e}_x, \quad \frac{\hat{S}_y}{S} \mapsto \hat{e}_y, \quad \hat{S}_z \mapsto \hat{e}_z \tag{3.51}$$

the commutation relations are now those of $e(2)$, the Euclidean algebra in two dimensions:

$$[\hat{e}_z, \hat{e}_x] = \mathrm{i}\hat{e}_y, \quad [\hat{e}_y, \hat{e}_z] = \mathrm{i}\hat{e}_x, \quad [\hat{e}_x, \hat{e}_y] = 0. \tag{3.52}$$

In $e(2)$, the operators $(\hat{e}_x, \hat{e}_y)$ generate translations, while $\hat{e}_z$ still generates rotations in the xy plane. The representation space is spanned by orthonormalized basis $|m\rangle$ of eigenstates of the operator $\hat{e}_z$:

$$\hat{e}_z|m\rangle = m|m\rangle, \quad m = -\ell, ..., \ell \qquad \text{with } \ell \to \infty. \tag{3.53}$$

From the asymptotic form of equation (3.24), and using in the covariant representation of equation (3.10), one obtains

$$\hat{w}_S^{(0)}(\Omega) \to \hat{w}(z) = (-1)^S T(z) \mathrm{e}^{-\mathrm{i}\pi\hat{e}_z} T^\dagger(z), \tag{3.54}$$

$$T(z) = \exp\left[-\mathrm{i}(z^* \hat{e}_+ + z\hat{e}_-)\right], \tag{3.55}$$

where $z = \tan\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi}$ and $\hat{e}_\pm = \hat{e}_x \pm \mathrm{i}\hat{e}_y$.

4. Generalized *SU*(2) Wigner-like mapping

The standard $SU(2)$ mapping cannot be directly applied to systems with variable spin, for which the representation space is not restricted to a single $SU(2)$ invariant subspace. Although any state can still be expanded on the angular momentum basis, the density matrix and some observables cannot be represented in terms of projectors on distinct $SU(2)$ irreducible subspaces. Simple realizations of this type of systems include two coupled angular momenta, the linear rigid rotor, two classically-pumped interacting field modes are of considerable interest in physical applications.

In order to map states and observables of this type of quantum systems into a set of c-number functions with good properties under global $SU(2)$ transformations, it is mandatory to use an operational basis that spans the whole angular momentum space. This basis is formed by the generalized $SU(2)$ tensors [98].

Moreover, a generalized kernel $\hat{\mathsf{w}}_j^{(s)}(\phi,\theta,\psi)$ [49] which enables this mapping depends not only on the spherical angles θ,ϕ but also on a third 'angle' ψ and a new discrete index j; the latter two are initially formal parameters but they will be seen to take on physical meaning when we later deal with applications [73].

4.1. Generalized tensors

Suppose the Hilbert space $\mathbb{H}$ contains multiple $SU(2)$ irreps and is thus spanned by

$$\mathbb{H} = \mathrm{Span}\{|J,m\rangle; m=-J,\dots,J; J=0,\tfrac{1}{2},1,\tfrac{3}{2},\dots\}. \tag{4.1}$$

We first construct a set of tensor operators which connect different $SU(2)$ subspaces [98]:

$$\hat{T}_{Kq}^{J'J} = \sum_{mm'} \sqrt{\frac{2K+1}{2J'+1}} \left\langle \begin{matrix} J \\ m \end{matrix}; \begin{matrix} K \\ q \end{matrix} \middle| \begin{matrix} J' \\ m' \end{matrix} \right\rangle |J',m'\rangle\langle J,m|. \tag{4.2}$$

The tensors transform like the $su(2)$ basis states $|K,q\rangle$ under commutation with the $su(2)$ generators. They form a complete set and any operator in $\mathbb{H}$ can be expanded as

$$\hat{f} = \sum_{J,J'=0,\frac{1}{2},1\dots}^{\infty} \sum_{K=|J'-J|}^{J'+J} \sum_{q=-K}^{K} f_{Kq}^{J'J}\hat{T}_{Kq}^{J'J}, \qquad f_{Kq}^{J'J} = \mathrm{Tr}((\hat{T}_{Kq}^{J'J})^{\dagger}\hat{f}). \tag{4.3}$$

The expansion of equation (4.3) can be re-arranged in the form of direct sum on the sectors with fixed values of $j=J'+J$:

$$\hat{f} = \sum_{j=0,\frac{1}{2},1\dots}^{\infty} \hat{f}_j, \quad \hat{f}_j = \sum_{K=\{0,\frac{1}{2}\}}^{j} \sum_{q,q'=-K}^{K} f_{Kq}^{\frac{1}{2}(j+q')\,\frac{1}{2}(j-q')}\,\hat{T}_{Kq}^{\frac{1}{2}(j+q')\,\frac{1}{2}(j-q')}, \tag{4.4}$$

where the sum over K starts at $K=0$ when j is integer, and starts at $K=\frac{1}{2}$ when j is half-integer.

These tensors will in general be represented by rectangular matrices with $(2J'+1)$ columns and $(2J+1)$ rows. Operators $\hat{f}_j$ in equation (4.4) for which j is half-integer are necessarily rectangular and constructed from rectangular tensors of the form given in equation (4.2). On the other hand, operators $\hat{f}_j$ for which j is an integer may or may not be square tensors. The *square* tensors of equation (3.5) are, in this notation, matrices of dimension $(j+1)\times(j+1)$ since $j=2S\in\mathbb{Z}^+$ in this case; for the $\hat{T}_{LM}^{S}$ tensors of equation (3.5) we have $q'=0$ (so that j is necessarily integer) in the expansion of equation (4.4); in addition, these tensors act entirely within a $(j+1)$-dimensional $SU(2)$ subspace.

Figure 1 is a pictorial description of how various elements of these non-square tensors appear in the decomposition of $\hat{f}$ on j-sectors.

4.2. Generalized kernel

For the operators of equation (4.3), a convenient $SU(2)$ covariant Wigner-like mapping to c-number functions, depending on *three* Euler angles $\omega=(\phi,\theta,\psi)$

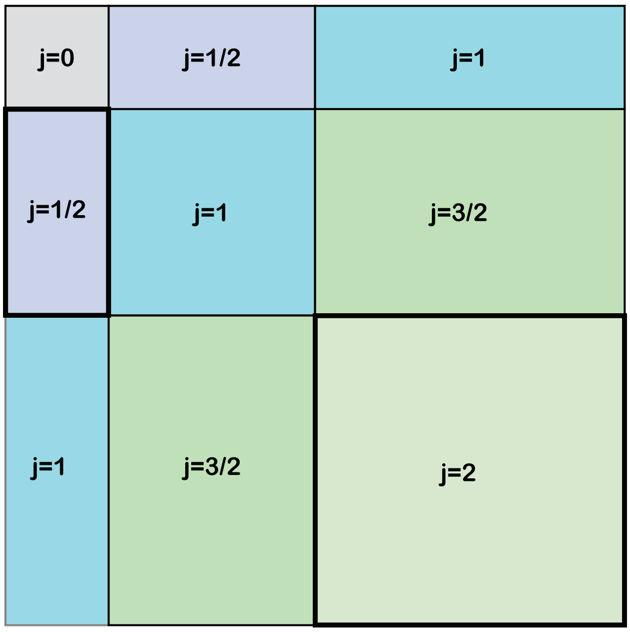

Figure 1. A pictorial representation of the various tensorial j-subspaces arising in the decomposition of operators into square and rectangular tensors.

$$\hat{f} \Leftrightarrow \{W_f^{j\,(s)}(\omega),\ j=0,\tfrac{1}{2},1,...\},\tag{4.5}$$

was proposed in [49]. The j-symbols

$$W_f^{j\,(s)}(\omega) = \mathrm{Tr}(\hat{f}\ \hat{\mathrm{w}}_j^{(s)}(\omega)),\tag{4.6}$$

are obtained by using a generalized kernel

$$\hat{\mathrm{w}}_j^{(s)}(\omega) = \sum_{K=\left\{0,\frac{1}{2}\right\}}^{j} \sqrt{\frac{2K+1}{j+1}}\tag{4.7}$$

$$\times \sum_{q,q'=-K}^{K} \left[\sqrt{\frac{j-q'+1}{j+1}} \left\langle \begin{matrix} \frac{1}{2}(j-q') \\ \frac{1}{2}(j-q') \end{matrix}; \begin{matrix} K \\ q' \end{matrix} \middle| \begin{matrix} \frac{1}{2}(j+q') \\ \frac{1}{2}(j+q') \end{matrix} \right\rangle \right]^{-s} D_{qq'}^{K}(\omega)\hat{T}_{Kq}^{\frac{1}{2}(j+q')\,\frac{1}{2}(j-q')},\tag{4.8}$$

where the $SU(2)$ functions $D_{qq'}^{K}(\omega)$ are defined in equation (3.7). The kernel of equation (4.8) can be represented in an explicitly covariant form

$$\hat{\mathrm{w}}_j^{(s)}(\omega) = T(\omega)\hat{\mathrm{w}}_j^{(s)}(0)T^{\dagger}(\omega),\tag{4.9}$$

with

$$\hat{\mathrm{w}}_j^{(s)}(0) = \sum_{K=\{0,1/2\}}^{j} \sum_{q=-K}^{K} \sqrt{\frac{2K+1}{j+1}}\tag{4.10}$$

$$\times\left(\sqrt{\frac{j-q+1}{j+1}}\left\langle\begin{array}{c}\frac{1}{2}(j-q)\\ \frac{1}{2}(j-q)\end{array};\begin{array}{c}K\\ q\end{array}\middle|\begin{array}{c}\frac{1}{2}(j+q)\\ \frac{1}{2}(j+q)\end{array}\right\rangle\right)^{-s}\hat{T}_{K\,q}^{\frac{1}{2}(j+q)\,\frac{1}{2}(j-q)}. \tag{4.11}$$

We point out that $\hat{\mathrm{w}}_j^{(-1)}(0)$ is not a diagonal rank-one tensor as in the $SU(2)$ case of equation (3.8); nevertheless we have the relation

$$\sum_{j=0,1/2,1,..}(j+1)\hat{\mathrm{w}}_j^{(-1)}(0)=|\Phi\rangle\langle\Phi|, \tag{4.12}$$

$$|\Phi\rangle=\sum_{S=0,1/2,\ldots}^{\infty}\sqrt{2S+1}\,|S,S\rangle. \tag{4.13}$$

The kernels of equation (4.8) are Hermitian and normalized

$$\frac{j+1}{16\pi^2}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\,\sin\theta\int_0^{4\pi}\mathrm{d}\psi\,\hat{\mathrm{w}}_j^{(s)}(\phi,\theta,\psi) \tag{4.14}$$

$$=\begin{cases}\mathbb{1}_{j+1}, & j\text{ integer},\\ 0, & j\text{ half-integer}.\end{cases} \tag{4.15}$$

In addition, they satisfy the following trace and orthogonality conditions:

$$\mathrm{Tr}\Big(\hat{\mathrm{w}}_j^{(s)}(\omega)\Big)=\begin{cases}1, & j\text{ integer},\\ 0, & j\text{ half-integer},\end{cases} \tag{4.16}$$

$$\mathrm{Tr}(\hat{\mathrm{w}}_j^{(s)}(\omega)\hat{\mathrm{w}}_{j'}^{(-s)}(\omega'))=\delta_{j,j'}\delta_j(\omega,\omega'), \tag{4.17}$$

where $\delta_{j,j'}$ is the usual Kronecker symbol while $\delta_j(\omega,\omega')$ is the reproductive kernel

$$\int\mathrm{d}\omega W_f^{j'}(\omega)\delta_j(\omega,\omega')=\delta_{j,j'}W_f^{j'}(\omega'). \tag{4.18}$$

that functions as an analog of the δ-function on the group. Here, $\mathrm{d}\omega=\mathrm{d}\phi\,\sin\theta\mathrm{d}\theta\,\mathrm{d}\psi$ is the $SU(2)$ volume element.

It follows from equation (4.17) that the map (4.6) is explicitly invertible:

$$\hat{f}_j=\frac{j+1}{16\pi^2}\int\mathrm{d}\omega\,W_f^{j\,(s)}(\omega)\hat{\mathrm{w}}_j^{(-s)}(\omega), \tag{4.19}$$

and thus establishes a one-to-one correspondence between the components $\hat{f}_j$ appearing in equation (4.19) and the symbols $W_f^{j\,(s)}(\omega)$ on the subspaces labeled by j.

For every fixed value of the discrete parameter j, the map of equation (4.6) is a function of three Euler angles and thus cannot be a representation of operators in a classical phase space, which by definition is even-dimensional. Nevertheless, in contrast with the standard $SU(2)$ covariant Stratonovich–Weyl approach, the generalized map offers the possibility of reconstructing, through equations (4.4) and (4.19), the whole operator rather than only its projection on irreducible subspaces.

We note that, when $\hat{f}$ acts in a *single* $SU(2)$ subspace, $q'=0$ in equation (4.4), j is an integer and only square tensors $\hat{T}_{Kq}^{j/2j/2}$ enter in the map of equations (4.5) and (4.6). In this case we recover the standard Stratonovich kernel of equation (3.8) with $S=j/2$, where $\hat{w}_{S=j/2}^{(s)}(\theta,\phi)$, is independent of the third angle ψ.

The kernel $\hat{w}_S^{(s)}(\Omega)$ of equation (3.8) is simply related to the generalized kernel $\hat{\mathsf{w}}_j^{(s)}(\omega)$ of equation (4.8) by averaging over the angle ψ:

$$\hat{w}_{S=j/2}^{(s)}(\Omega) = \int_0^{4\pi} \frac{d\psi}{4\pi}\, \hat{\mathsf{w}}_j^{(s)}(\omega). \tag{4.20}$$

The physical significance of the phase ψ will be discussed in greater details later.

The overlap relation can be seen to take the form

$$\mathrm{Tr}(\hat{f}\,\hat{g}) = \sum_{j=0,\frac{1}{2},1\ldots}^{\infty} \frac{j+1}{16\pi^2} \int d\omega\, W_f^{j\,(s)}(\omega)\, W_g^{j\,(-s)}(\omega). \tag{4.21}$$

The symbol of the identity operator $\mathbb{1}$ on the entire space is obviously

$$W_{\mathbb{1}}^{j}(\omega) = \sum_{n=0,1,2\ldots}^{\infty} \delta_{jn} = \begin{cases} 1 & j \in \mathbb{Z}^*, \\ 0 & \text{otherwise} \end{cases} \tag{4.22}$$

This implies the normalization condition

$$\mathrm{Tr}(\hat{f}\,) = \sum_{j=0,1,2,\ldots}^{\infty} \frac{j+1}{16\pi^2} \int d\Theta\, W_f^{j}(\omega). \tag{4.23}$$

An explicit differential form for the star-product acting directly on j-symbols,

$$W_{fg}^{j\,(s)}(\omega) = \sum_{j_1,j_2} \mathbf{L}_{f,g}^{j,j_1j_2\,(s)}(W_f^{j_1\,(s)}(\omega) W_g^{j_2\,(s)}(\omega)), \tag{4.24}$$

can be found in [73] and is reproduced in appendix C. It has a local form $\mathbf{L}_{f,g}^{j,j_1j_2\,(s)} \sim \delta_{j,j_1}\delta_{j,j_2}$ and reduces to the standard product of equation (B.1) when the operators $\hat{f}$ and $\hat{g}$ are elements of the enveloping algebra of $su(2)$.

The structure of the correspondence rules (Bopp operators) for the generalized kernels is not unexpectedly more involved than for the simple $SU(2)$ mappings of equations (3.40)–(3.41): derivatives with respect to the angle ψ appear even for the description of the action of the angular momentum operators [100]. The asymptotic form of correspondence rules are most useful in practice.

4.3. Some examples

Broadly speaking the generalized symbols $W_f^{j\,(s)}(\omega)$ have more complicated form than their standard Stratonovich–Weyl expressions. In addition, not all of them have an intuitive interpretation, as can be seen from equation (4.29) below. Nevertheless, within the framework of this generalized formalism, one can find using equations (4.5)–(4.9) images of operator that do not belong to the $su(2)$ enveloping algebra. This becomes especially attractive in the semi-classical limit, when these symbols 'become functions defined' in a symplectic phase space. We emphasize that the index j can only take integer values for symbols independent of ψ. When there is no ψ dependence, the operators act exclusively within an $SU(2)$ irreducible subspace, and are functions of the angular momentum operators.

For instance, the image of the total angular momentum operator $\hat{J}^2$ is

$$W_{J^2}^{j(s)}(\omega) = \frac{j}{2}\left(\frac{j}{2}+1\right) \sum_{n\in\mathbb{Z}^+} \delta_{j,n} \tag{4.25}$$

and independent of the parameter s.

On the other hand, non-diagonal tensors constructed as per equation (4.2) which mix $SU(2)$ irreps always depend on the angle ψ, both for integer and half-integer values of of the index j. For instance the operator $\hat{n}_z = \widehat{\cos\theta}$, corresponding to the z-component of the unit vector of equation (3.15), has an expansion in terms in the non-squared tensors of (4.2) as

$$\hat{n}_z = \sum_{j=1,3,5...} \sqrt{\frac{j+1}{2}} \left(T_{10}^{\frac{j+1}{2}\frac{j-1}{2}} - T_{10}^{\frac{j-1}{2}\frac{j+1}{2}} \right), \tag{4.26}$$

and can also be expanded in a coordinate basis as

$$\hat{n}_z = \int \mathrm{d}\varphi \, \sin\vartheta \mathrm{d}\vartheta \, \cos\vartheta |\varphi, \vartheta\rangle\langle\varphi, \vartheta|, \tag{4.27}$$

$$|\varphi, \vartheta\rangle = \sum_{j=0,1,...} \sum_{m=-j}^{j} Y_{jm}^{*}(\varphi, \vartheta)|j, m\rangle, \tag{4.28}$$

where φ and ϑ are angles in the configuration space. The j-symbol for $\hat{n}_z$ is then obtained as

$$W_{n_z}^{j(s)}(\omega) = \left(\frac{j}{j+1} \right)^{-s/2} \sin\theta \cos\psi \sum_{n=0,1,..} \delta_{j,2n+1}. \tag{4.29}$$

The counter-intuitive form of this symbol will be discussed later.

Introducing the bi-polar spherical harmonics [93] $\{Y_\ell(\varphi_1, \vartheta_1) \otimes Y_{\ell'}(\varphi_2, \vartheta_2)\}_{Kq}$, one may observe that for $j \in \mathbb{Z}^+$, the kernel of (4.11) can be expressed for $s = 0$ using the $|\varphi, \vartheta\rangle$ basis as

$$\hat{\mathrm{w}}_j^{(0)}(0) = \int \mathrm{d}\,\Omega_1 \, \mathrm{d}\,\Omega_2 |\varphi_2, \vartheta_2\rangle\langle\varphi_1, \vartheta_1| \tag{4.30}$$

$$\times \sum_{K=0}^{j} \sum_{q=-K}^{K} \sqrt{\frac{2K+1}{j+1}} (-1)^{\frac{j-q}{2}} \left\{ Y_{\frac{j+q}{2}}(\varphi_1, \vartheta_1) \otimes Y_{\frac{j-q}{2}}(\varphi_2, \vartheta_2) \right\}_{Kq}. \tag{4.31}$$

This representation clearly reveals its the angular momentum coupling nature of the mapping kernel.

As another example consider the realization of angular momentum states $|JM\rangle$ in terms of boson operators (the Schwinger representation):

$$|J, M\rangle \mapsto \frac{(\hat{a}^\dagger)^{J+M}(\hat{b}^\dagger)^{J-M}}{\sqrt{(J+M)!\,(J-M)!}} |0\rangle|0\rangle \tag{4.32}$$

The boson annihilation operator $\hat{a}$ clearly connects $|JM\rangle$ and $|J'M'\rangle$, where $J' = J - \frac{1}{2}$ and $M' = M - 1/2$. Using equation (4.32) the Heisenberg–Weyl annihilation operator $\hat{a}$ can be expanded as

$$\hat{a} = \sum_{j=\frac{1}{2},\frac{3}{2},...} \sqrt{\frac{(j+1/2)(j+3/2)}{2}} \, \hat{T}_{1/2\,-1/2}^{\frac{1}{2}(j-1/2),\frac{1}{2}(j+1/2)}, \tag{4.33}$$

and its symbol is given by

$$W_a^{j\,(s)} = \sqrt{\frac{(j+1/2)^{1-s}(j+3/2)}{(j+1)^{1-s}}} \cos\frac{1}{2}\theta \, \mathrm{e}^{-\mathrm{i}(\phi+\psi)/2} \sum_{k=\frac{1}{2},\frac{3}{2},...}^{\infty} \delta_{j,k}, \tag{4.34}$$

showing that only half-integer values of j survive for this essentially 'quantum' operator. This should be contrasted with $\hat{n}_z$, an operator that admits a classical interpretation.

An important application is to the product of two Heisenberg–Weyl coherent states

$$|\alpha\beta\rangle := |\alpha\rangle_a|\beta\rangle_b, \tag{4.35}$$

$$|\gamma\rangle = \mathrm{e}^{-|\gamma|^2/2}\sum_{n=0,1,..}^{\infty}\frac{\gamma^n}{\sqrt{n!}}|n\rangle, \tag{4.36}$$

This state is decomposable into a direct sum of the $SU(2)$ coherent states $|\theta_0,\phi_0;S\rangle$ equation (3.4) as

$$|\alpha\beta\rangle = \sum_{S=0,\frac{1}{2},1,\dots}^{\infty}\mathrm{e}^{-r^2/2}\frac{r^{2S}}{\sqrt{(2S)!}}\mathrm{e}^{\mathrm{i}j\psi_0}|\theta_0,\phi_0;S\rangle, \tag{4.37}$$

with the coherent state parameters α and β are given by

$$\alpha = r\cos\left(\frac{1}{2}\theta_0\right)\mathrm{e}^{-\mathrm{i}(\phi_0+\psi_0)/2}, \qquad \beta = r\sin\left(\frac{1}{2}\theta_0\right)\mathrm{e}^{\mathrm{i}(\phi_0-\psi_0)/2}. \tag{4.38}$$

The sum over S clearly points to the existence of non-diagonal elements in the $SU(2)$ decomposition of the density matrix $|\alpha\beta\rangle\langle\alpha\beta|$, so the standard map of equation (3.8) cannot be used. Within the generalized approach one calculates that the j-Wigner symbols of the sate equation (4.37) depend only on a single continuous parameter:

$$W_{\alpha\beta}^{j(0)}(\omega) = \frac{r^{2j}\mathrm{e}^{-r^2}}{\sqrt{j+1}}\sum_{K=\{0,\frac{1}{2}\}}^{j}\frac{2K+1}{\sqrt{(j+K+1)!\,(j-K)!}}\chi^K(\nu), \tag{4.39}$$

where $\chi^K(\nu)$ is the $SU(2)$ group character

$$\chi^K(\nu) = \frac{\sin[(2K+1)\frac{\nu}{2}]}{\sin\frac{\nu}{2}}, \tag{4.40}$$

with the angle ν implicitly given by

$$\begin{aligned}\cos\tfrac{1}{2}\nu = {}&\cos\tfrac{1}{2}(\theta-\theta_0)\cos\tfrac{1}{2}(\phi-\phi_0)\cos\tfrac{1}{2}(\psi-\psi_0)\\ &-\cos\tfrac{1}{2}(\theta+\theta_0)\sin\tfrac{1}{2}(\phi-\phi_0)\sin\tfrac{1}{2}(\psi-\psi_0).\end{aligned} \tag{4.41}$$

The corresponding Q-function also has a simple form:

$$W_{\alpha\beta}^{j(-1)}(\omega) = \mathrm{e}^{-r^2}\left(r\cos\tfrac{1}{2}\theta'\right)^{2j}\sum_{q'=-j}^{j}\frac{1}{\sqrt{(j+q')!\,(j-q')!}}\mathrm{e}^{2\mathrm{i}q'(\phi'+\psi')}, \tag{4.42}$$

where the angles $\omega' = (\phi',\theta',\psi')$ are obtained as a composition $\omega\circ\omega_0$ of the Euler angles ω_0 (4.38) and ω in the standard way [93].

4.4. Semiclassical limit

In the semi-classical limit, when typical values of the index j are sufficiently large, the exact expression of equation (4.24) for the star-product reduces to the elegant form

$$W^{j(s)}_{fg}(\omega) \approx \mathrm{e}^{\varepsilon(s-1)\mathbb{J}^0\otimes\mathbb{J}^0 - \frac{\varepsilon}{2}(\mathbb{J}^+\otimes\mathbb{J}^- - \mathbb{J}^-\otimes\mathbb{J}^+) - \frac{\varepsilon s}{2}(\mathbb{J}^+\otimes\mathbb{J}^- + \mathbb{J}^-\otimes\mathbb{J}^+)}$$
$$\times \int_0^{4\pi} \frac{\mathrm{d}\varphi\mathrm{d}\varphi'}{(4\pi)^2} \sum_{j_1,j_2} (\mathrm{e}^{\mathrm{i}(j_2-j+J^0)\varphi'} W^{j_1(s)}_f(\omega))(\mathrm{e}^{\mathrm{i}(j_1-j-J^0)\varphi} W^{j_2(s)}_g(\omega)), \tag{4.43}$$

with

$$\mathbb{J}^\pm = \mathrm{i}\mathrm{e}^{\mp\mathrm{i}\psi}\left[\mathrm{i}\frac{\partial}{\partial\theta} \pm \cot\theta\frac{\partial}{\partial\psi} \mp \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right], \quad \mathbb{J}^0 = -\mathrm{i}\frac{\partial}{\partial\psi}. \tag{4.44}$$

More precisely, this semi-classical limit corresponds to situations where components with large values of j_1 and j_2 are the most important in the decomposition of the operators $\hat{f}$ and $\hat{g}$ given by equation (4.4). Since $j_1 = J_1 + J_1'$ and $j_2 = J_2 + J_2'$, this occurs when J_1' and J_1, and when J_2' and J_2, are large in equation (4.3).

Operationally, the tensors appearing in the expansion of equation (4.4) should be predominantly of low rank. When the density matrices satisfies this criteria, it describes so-called semi-classical states. The semi-classical parameter $\varepsilon = (j+1)^{-1}$ in this case is different in every sector labelled by the index j.

The approximate expression for the star-product given in equation (4.43) is considerably simplified in the limit where we consider j as a formal continuous parameter. Since symbols with integer and half-integer values of the index j behave in fundamentally different ways, the star-product takes a slightly different form in the continuous limit.

For integer j one can show that, to leading order in the semiclassical parameter, one obtains

$$\begin{aligned} W^{j(s)}_{fg} &\approx \left(\mathbb{1}\otimes\mathbb{1} + \varepsilon\partial_\psi\otimes\partial_\psi\right)\left(W^j_f W^j_g + W^{j+1/2}_f W^{j+1/2}_g\right) \\ &+\mathrm{i}\left(\partial_\psi\otimes(\varepsilon\cot\theta\partial_\theta+\partial_j) - (\varepsilon\cot\theta\partial_\theta+\partial_j)\otimes\partial_\psi\right)\left(W^j_f W^j_g + W^{j+1/2}_f W^{j+1/2}_g\right) \\ &+\mathrm{i}\varepsilon\hat{P}\left(W^j_f W^j_g + W^{j+1/2}_f W^{j+1/2}_g\right) - \frac{\partial_j\otimes\mathbb{1} + \mathbb{1}\otimes\partial_j}{2} W^{j+1/2}_f W^{j+1/2}_g, \end{aligned} \tag{4.45}$$

whereas, for the half-integer j case we have instead

$$\begin{aligned} W^{j(s)}_{fg} &\approx \left(\mathbb{1}\otimes\mathbb{1} + \varepsilon\partial_\psi\otimes\partial_\psi\right)\left(W^{j+1/2}_f W^j_g + W^j_f W^{j+1/2}_g\right) \\ &+\mathrm{i}\left(\partial_\psi\otimes(\varepsilon\cot\theta\partial_\theta+\partial_j) - (\varepsilon\cot\theta\partial_\theta+\partial_j)\otimes\partial_\psi\right)\left(W^{j+1/2}_f W^j_g + W^j_f W^{j+1/2}_g\right) \\ &+\mathrm{i}\varepsilon\hat{P}\left(W^{j+1/2}_f W^j_g + W^j_f W^{j+1/2}_g\right) + \frac{\mathbb{1}\otimes\partial_j}{2} W^{j+1/2}_f W^j_g + \frac{\partial_j\otimes\mathbb{1}}{2} W^j_f W^{j+1/2}_g, \end{aligned} \tag{4.46}$$

where $\hat{P}$ is the Poisson bracket operator on $\mathcal{S}^2$ given in equation (3.32). Both of these expression contain derivative *w/r* to the parameter j, which is now understood in this limit to be continuous.

The form of the star-product given above can be used to obtain the semi-classical evolution equation for the j-symbols of the density matrix. Summing and extracting (4.45) and (4.46) one obtains the approximate Moyal equation (2.29) (to leading order in ε) for the linear combinations

$$W^{(s)}_{\rho_\pm}(\omega, j) := W^{j\,(s)}_\rho(\omega) \pm W^{j+1/2\,(s)}_\rho(\omega) \tag{4.47}$$

in a form of [73] analogous to the standard TWA for the pure SU(2) case, given in equation (3.31)

$$\partial_t W^{(s)}_{\rho\pm}(\omega,j) \approx \left[-\frac{2\cot\theta}{j+1}(\partial_\theta\otimes\partial_\psi - \partial_\psi\otimes\partial_\theta) + \frac{2}{(j+1)\sin\theta}(\partial_\theta\otimes\partial_\phi - \partial_\phi\otimes\partial_\theta) + 2(\partial_\psi\otimes\partial_j - \partial_j\otimes\partial_\psi)\right] W^{(s)}_{H\pm}(\omega,j) W^{(s)}_{\rho\pm}(\omega,j), \tag{4.48}$$

which we can represent as a Poisson bracket on a 4-dimensional manifold:

$$\partial_t W^{(s)}_{\rho_\pm}(\omega,j) \approx 2\{W^{(s)}_{H_\pm}(\omega,j), W^{(s)}_{\rho_\pm}(\omega,j)\} \tag{4.49}$$

$$= \sum_{a,b=1}^{4} \omega^{ab} \frac{\partial W^{(s)}_{H_\pm}(\omega,j)}{\partial \xi^a} \frac{\partial W^{(s)}_{\rho_\pm}(\omega,j)}{\partial \xi^b}, \tag{4.50}$$

$$W^{(s)}_{H_\pm}(\omega,j) := W^{j\,(s)}_{H}(\omega,j) \pm W^{j+1/2\,(s)}_{H}(\omega,j), \tag{4.51}$$

with $\xi_1 = \phi, \xi_2 = \theta, \xi_3 = \psi, \xi_4 = j$; here, ω_{ab} are the components of a non-degenerate closed 2-form $\hat{\omega}$:

$$\alpha = (j+1)\mathrm{d}\psi - (j+1)\cos\theta\mathrm{d}\phi, \tag{4.52}$$

$$\hat{\omega} = -\mathrm{d}\alpha \tag{4.53}$$

$$= \mathrm{d}\phi \wedge d((j+1)\cos\theta) + \mathrm{d}\psi \wedge d(j+1). \tag{4.54}$$

The (continuous) index j thus emerges as a new dynamical variable conjugate to the angle ψ.

The canonical (Darboux) pairs $(\phi, (j+1)\cos\theta)$ and $(\psi, j+1)$ can be conveniently interpreted in terms of a rigid rotor motion [99]: the projection $(j+1)\cos\theta$ of the angular momentum $j+1$ on the fixed axis z produces a phase shift ϕ, while the precession of the phase ψ is generated by total angular moment $j+1$ in the body-fixed frame.

The 2-form $\hat{\omega}$ of equation (4.54) defines a metric on the cotangent bundle $T^*\mathcal{S}^2$ corresponding to the co-adjoint orbit of the $E(3)$ group fixed by the values of the Casimir operators $\mathbf{R}^2 = 1$ and $\mathbf{S}\cdot\mathbf{R} = 0$, where the (commuting) generators of translations $\mathbf{R} = (\hat{X}, \hat{Y}, \hat{Z})$ together with the components of the angular momentum operators $\mathbf{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ close on $e(3)$, the Euclidean algebra in three dimensions:

$$[\hat{S}_i, \hat{S}_j] = \mathrm{i}\varepsilon_{ijk}\hat{S}_k, \qquad [\hat{X}_i, \hat{X}_j] = 0, \qquad [\hat{S}_i, \hat{X}_j] = \mathrm{i}\varepsilon_{ijk}\hat{X}_k. \tag{4.55}$$

The translations operators $\hat{X}_\pm = \hat{X} \pm \mathrm{i}\hat{Y}$ and $\hat{Z}$ act on the angular momentum basis functions by multiplying the spherical harmonics $Y_{jm}(\varphi,\vartheta)$ by factors of $\sin\vartheta\,\mathrm{e}^{\pm\mathrm{i}\varphi}$ and $\cos\vartheta$, respectively.

We note that $W^{j(s)}_{\mathbf{S}\cdot\mathbf{R}}(\omega) = 0$ while

$$W^{j(s)}_{\mathbf{R}^2}(\omega) = \begin{cases} 1 & j \in \mathbb{Z}^+, \\ 0 & j = \frac{1}{2}, \frac{3}{2}\dots. \end{cases} \tag{4.56}$$

As a result of the explicitly covariant form of the kernel given in equation (4.31), the semiclassical approach on $T^*\mathcal{S}^2$ is very convenient to study the dynamics of spin–spin interactions invariant under global rotations. For instance, given the Wigner j-symbol of $\hat{L}_z = \hat{N}_z + \hat{S}_z$ and $\hat{\mathbf{N}}\cdot\hat{\mathbf{S}}$ as

$$W^{j}_{L_z}(\omega) = \frac{1}{2}\sqrt{\frac{j}{2}\left(\frac{j}{2}+1\right)}\cos\theta, \tag{4.57}$$

$$W^{j}_{\hat{\mathbf{N}}\cdot\hat{\mathbf{S}}}(\omega) = \frac{1}{2}\left[\frac{j}{2}\left(\frac{j}{2}+1\right) - N(N+1) - S(S+1)\right], \tag{4.58}$$

the dynamics resulting from the Hamiltonian

$$\hat{H} = \kappa(\hat{N}_z + \hat{S}_z) + \chi\hat{\mathbf{N}}\cdot\hat{\mathbf{S}}, \tag{4.59}$$

is described by the following simple evolution equation in the $T^*\mathcal{S}^2$ phase space:

$$\partial_t W_\rho(\omega, j) = -\kappa\partial_\phi W_\rho(\omega, j) - 2\chi(j+1)\partial_\psi W_\rho(\omega, j), \tag{4.60}$$

where no higher order derivatives appear. It follows from the solution of Equation (4.60)

$$W_\rho(\omega, j|-t) = W(\phi - \kappa t, \theta, \psi - 2\chi(j+1)t, j|t=0), \tag{4.61}$$

that the evolution of the angle ψ is different in each j-subspace and is an indicator of appearance of spin–spin correlations; this therefore represents an alternative to other correlation measures such as purity or negativity.

The correspondence rules (Bopp operators) are also nicely simplified in the semi-classical limit $j \gg 1$: to $\mathcal{O}(1)$ one obtains [100]

$$W^{j(s)}_{S_+\rho}(\omega) \approx \frac{1}{2}\left[\frac{1}{\varepsilon}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} - \bar{\mathbb{J}}_+\right] W^{j(s)}_{\rho}(\omega), \tag{4.62}$$

$$W^{j(s)}_{S_-\rho}(\omega) \approx \frac{1}{2}\left[\frac{1}{\varepsilon}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} + \bar{\mathbb{J}}_-\right] W^{j(s)}_{\rho}(\omega), \tag{4.63}$$

$$W^{j(s)}_{S_z\rho}(\omega) \approx \frac{1}{2}\left(\frac{1}{\varepsilon}\cos\theta - \mathrm{i}\frac{\partial}{\partial\phi}\right) W^{j(s)}_{\rho}(\omega), \tag{4.64}$$

where the operators $\bar{\mathbb{J}}_\pm$ are defined slightly differently from the $\mathbb{J}^\pm$ equation (C.4): here we have

$$\bar{\mathbb{J}}_\pm = \mathrm{i}\mathrm{e}^{\pm\mathrm{i}\phi}\left[\mp\cot\theta\frac{\partial}{\partial\phi} + \mathrm{i}\frac{\partial}{\partial\theta} \pm \frac{1}{\sin\theta}\frac{\partial}{\partial\psi}\right]. \tag{4.65}$$

On the other hand, for HW creation-annihilation operators we obtain,

$$W^{j(s)}_{a\rho}(\omega) \approx \frac{2}{\sqrt{\varepsilon}}\mathrm{e}^{-\mathrm{i}\frac{\phi+\psi}{2}}\cos\theta/2\, W^{j+\frac{1}{2}(s)}_{\rho}(\omega), \tag{4.66}$$

$$W^{j(s)}_{a^\dagger\rho}(\omega) \approx \frac{2}{\sqrt{\varepsilon}}\mathrm{e}^{\mathrm{i}\frac{\phi+\psi}{2}}\cos\theta/2\, W^{j-\frac{1}{2}(s)}_{\rho}(\omega), \tag{4.67}$$

indicating that, to leading order, the action of generators of the Heisenberg–Weyl algebra is reduced to a change the value of the index j by 1/2.

5. Applications to kinematical problems

5.1. Quantum tomography

One of the simplest applications of the theory of quasi-distributions for spin-like systems is an explicit reconstruction scheme of the density matrix from measured probabilities (see e.g. [15, 86] and references therein). For fixed spin systems one has

$$\hat{\rho} = \frac{2S+1}{4\pi} \int_{S^2} \mathrm{d}\,\Omega \; \hat{w}_S^{(1)}(\theta,\phi) Q_\rho(\theta,\phi), \tag{5.1}$$

where

$$Q_\rho(\theta,\phi) = \langle S, S | \hat{\rho}(\theta,\phi) | S, S \rangle, \tag{5.2}$$

$$\hat{\rho}(\theta,\phi) = \hat{D}^\dagger(\theta,\phi)\,\hat{\rho}\,\hat{D}(\theta,\phi). \tag{5.3}$$

The quantities $Q_\rho(\theta,\phi)$ are frequently called *tomograms*.

As mentioned in connection with equation (2.23), a direct tomographic reconstruction of the $SU(2)$ Wigner function

$$W_\rho(\Omega') = \frac{4\pi}{2S+1} \int_{S_2} \mathrm{d}\,\Omega\, Q_\rho(\Omega) K_S(\Omega,\Omega'), \tag{5.4}$$

$$K_S(\Omega,\Omega') = \sum_{L=0}^{2S} \left\langle \begin{matrix} S \\ S \end{matrix}; \begin{matrix} L \\ 0 \end{matrix} \middle| \begin{matrix} S \\ S \end{matrix} \right\rangle P_L(\cos\zeta), \tag{5.5}$$

with $P_L(z)$ the Legendre polynomial and ζ defined in equation (3.19), can be more appropriate especially for large values of the effective spin S (in which case the kernel $K_S(\Omega,\Omega')$ is a smooth function) and/or when the precise value of S is unknown (due—say—to a weak dependence of the kernel equation (5.5) on the spin size when $S \gg 1$) [101].

Tomographic schemes like the ones mentioned above are highly redundant and in practice are discretized. For instance, the density matrix of spin S system can be recovered by measuring probabilities to obtain the maximum spin projection for $(2S+1)^2$ appropriately chosen direction [102]. On the other hand, the coefficients ρ_{Kq} of the multipole expansion (3.16), can be found from the moments $\langle (\mathbf{n}\cdot\hat{\mathbf{S}})^\ell \rangle$, $1 \leqslant \ell \leqslant K$ measured in suitable directions.

The $SU(2)$ maps are widely used for the tomography of two polarization modes (H and V) within polarization sectors [76] constituted by states with fixed photon numbers and thus corresponding to $SU(2)$ invariant subspaces. This kind of reconstruction of the polarization density matrix (in each subspace) can be efficiently performed in experiment [103] and the results can be conveniently represented as a distribution on the S^2 sphere.

In the case of systems with variable spin or having a variable number of excitations, the reconstruction relation of equation (4.19) can be rewritten as [104]

$$\hat{\rho} = \sum_{j=0,1/2,1\ldots}^{\infty} \frac{j+1}{16\pi^2} \int \mathrm{d}\omega Q_\rho^j(\omega) \hat{\mathrm{w}}_j^{(1)}(\omega), \tag{5.6}$$

where $Q_\rho^j(\omega) = \hat{W}_j^{s=-1}(\omega) = \mathrm{Tr}(\hat{\rho}(\omega)\hat{Q}^j(0))$, where $\hat{\rho}(\omega) = \hat{T}^\dagger(\omega)\,\hat{\rho}\,\hat{T}(\omega)$ is the $SU(2)$-transformed density matrix. In other words, the full density matrix can be reconstructed by measuring the expectation values of the operators (written in the angular momentum basis):

$$\hat{Q}^j(0) = \sum_{q=-j}^{j} \left| \tfrac{1}{2}(j+q), \tfrac{1}{2}(j+q) \right\rangle \left\langle \tfrac{1}{2}(j-q), \tfrac{1}{2}(j-q) \right|. \tag{5.7}$$

on $\hat{\rho}(\omega)$.

One example of a variable spin system important in applications is the two polarization mode states (H and V), considered in the whole space, and not only inside individual polarization sectors. In this case the operator $\hat{Q}^j(0)$ takes the specific form (in the photon polarization basis)

$$\hat{Q}^{j}(0) = \sum_{q=-j}^{j} |j+q\rangle_{HH}\langle j-q| \otimes |0\rangle_{VV}\langle 0|, \tag{5.8}$$

with factorized states always containing the vacuum in one mode. Then, the data required for reconstruction of the complete state of an arbitrary two-mode polarized light field can be obtained from a conditional balanced homodyne tomographic setup [104].

5.2. Phase distribution problem

5.2.1. SU(2) relative phase POVM. The map (3.8) can be used to introduce a distribution function for the phase ϕ associated with the projection of the angular momentum on the z axis. In quantum optics, the phase ϕ is frequently related to the relative phase between polarization modes [105] defined in each polarization sector. Such an operator valued measure (POVM) can be defined [105, 106] by integrating over θ the kernel of the equation (3.8) for $s=-1$, corresponding to the Q-function

$$\hat{w}_S^{(s=-1)}(\Omega) = |\Omega; S\rangle\langle\Omega; S|, \tag{5.9}$$

where $|\Omega; S\rangle$ is the $SU(2)$ coherent state (3.4). The phase POVM

$$\hat{\Delta}_S(\phi) = \frac{2S+1}{4\pi}\int_0^{\pi} \mathrm{d}\theta\, \sin\theta\, \hat{w}_S^{(-1)}(\Omega), \tag{5.10}$$

is positively defined, Hermitian, normalized

$$\hat{\Delta}_S(\phi) = \hat{\Delta}_S^{\dagger}(\phi), \qquad \int_0^{2\pi} \mathrm{d}\phi \hat{\Delta}_S(\phi) = \mathbb{1}, \tag{5.11}$$

and satisfies the covariance condition,

$$\mathrm{e}^{-\mathrm{i}\phi_0\hat{S}_z}\hat{\Delta}_S(\phi)\mathrm{e}^{\mathrm{i}\phi_0\hat{S}_z} = \hat{\Delta}_S(\phi+\phi_0). \tag{5.12}$$

The resulting probability distribution

$$P(\phi) = \mathrm{Tr}\,[\hat{\Delta}_S(\phi)\hat{\rho}] \tag{5.13}$$

can be used to evaluate averages of phase observables according to

$$\langle f(\hat{\phi})\rangle = \int_0^{2\pi} \mathrm{d}\phi P(\phi) f(\phi). \tag{5.14}$$

Although in many situations the explicit form of (5.13) is quite complicated, the asymptotic for $S \gg 1$ frequently is notoriously simplified. For instance, for a coherent state (3.4) $|\Omega_0; S\rangle$ one has

$$P(\phi) \approx \exp\left(-\frac{\hat{L}^2}{2S+1}\right)\delta(\phi-\phi_0), \quad 0<\theta_0<\pi, \tag{5.15}$$

where $\hat{L}^2$ is the differential realization of the Casimir operator on $\mathcal{S}^2$. For $\theta_0 \neq 0$ or π the operator $\hat{L}^2$, when acting on $\delta(\phi-\phi_0)$, is reduced to

$$\hat{L}^2 = -\frac{1}{\sin^2\theta_0}\partial^2_{\phi_0}, \tag{5.16}$$

while at $\theta_0 = 0$ or π (i.e. for highest and lowest states $|S, \pm S\rangle$ of the representation), it leads to a flat distribution: $P(\phi) = 1/2\pi$. Then, for instance, for typical angle observables we obtain

$$\langle \cos^k \phi \rangle = \exp\left(-\frac{\hat{L}^2}{2S+1} \right) \cos^k \phi_0. \tag{5.17}$$

In the contraction limit described in equations (3.46)–(3.50), the POVM equation (5.10) is reduced to the phase distribution operator for the Heisenberg–Weyl group, which also allows to consistently define the phase operator itself [107].

5.2.2. Joint $\psi - \phi$ POVM. The map of equations (4.6)–(4.8) allows a generalization of the concept of phase distribution to systems with variable spin. The phase ψ in this case is formally one of the angle variables defining a point in (ϕ, θ, ψ, j) space. As discussed in section 4.4 this phase also appears in the semi-classical limit as a variable conjugate to j, and thus can be interpreted as a phase shift generated by the total angular momentum rather than to its projection on a given fixed axis. Algebraically, the phase ψ carries information about coherences between sectors (labelled by j) of a density matrix. Thus, in the framework of the generalized $SU(2)$ approach one can introduce a joint ϕ-ψ POVM in a way that generalizes equation (5.10):

$$\hat{\Delta}(\phi, \psi) = \sum_{j=0,\frac{1}{2},1,..} \frac{j+1}{16\pi^2} \int_0^\pi \sin\theta \mathrm{d}\theta \ \hat{\mathrm{w}}_j^{(-1)}(\omega), \tag{5.18}$$

where $\hat{\mathrm{w}}_j^{(-1)}(\omega)$ is defined in equation (4.8), and the positivity of (5.18) follows from (4.13). The operator $\hat{\Delta}(\phi, \psi)$ is normalized

$$\int_0^{4\pi} \mathrm{d}\psi \int_0^{2\pi} \mathrm{d}\phi \hat{\Delta}(\phi, \psi) = \mathbb{1}, \tag{5.19}$$

and covariant under $\hat{S}_z$ rotations,

$$\mathrm{e}^{-\mathrm{i}\phi' \hat{S}_z} \hat{\Delta}(\phi, \psi) \mathrm{e}^{\mathrm{i}\phi' \hat{S}_z} = \hat{\Delta}(\phi + \phi', \psi), \tag{5.20}$$

and $\hat{S}_0$-displacements,

$$\mathrm{e}^{-\mathrm{i}\psi' \hat{S}_0} \hat{\Delta}(\phi, \psi) \mathrm{e}^{\mathrm{i}\psi' \hat{S}_0} = \hat{\Delta}(\phi, \psi + \psi'), \tag{5.21}$$

where the operator

$$\hat{S}_0 = \frac{1}{2} \sum_{N=0,1,...}^{\infty} N \sum_{k=-N/2}^{N/2} |N/2, k\rangle \langle N/2, k|, \tag{5.22}$$

plays the role of the total photon number in the case of two-polarized field modes; $\hat{S}_0$ also commutes with the angular momentum operators. Integration of $\hat{\Delta}(\phi, \psi)$ over ψ gives a direct sum of $SU(2)$ phase POVM given in equation (5.10):

$$\int_0^{4\pi} \mathrm{d}\psi \hat{\Delta}(\phi, \psi) = \sum_{j=0,1,...}^{\infty} \hat{\Delta}_{j/2}(\phi). \tag{5.23}$$

The average values are computed by using the joint probability distribution function $P(\phi, \psi) = \mathrm{Tr}\,[\hat{\Delta}(\phi, \psi)\hat{\rho}]$ in the standard manner:

$$\langle f(\hat{\phi}) g(\hat{\psi}) \rangle = \int_0^{2\pi} \mathrm{d}\phi \int_0^{4\pi} \mathrm{d}\psi P(\phi, \psi) f(\phi) g(\psi), \tag{5.24}$$

and in general $\langle f(\hat{\phi}) g(\hat{\psi}) \rangle \neq \langle f(\hat{\phi}) \rangle \langle g(\hat{\psi}) \rangle$.

For the two-mode coherent state of equation (4.35), where in the parametrization of equation (4.38) the angle ψ_0 plays the role of 'phase' between $SU(2)$ irreducible subspaces (4.37), the joint distribution function in the limit of large average photon number, $|\alpha|^2 + |\beta|^2 = \bar{n} \gg 1$, has the asymptotic form,

$$P(\phi, \psi) \approx \mathrm{e}^{-\mathbf{J}^2/2\bar{n}} \delta(\phi - \phi_0)\delta(\psi - \psi_0), \tag{5.25}$$

where

$$\mathbf{J}^2 = -\frac{1}{\sin^2\theta_0}(\partial^2_{\phi_0} - 2\cos\theta_0 \partial_{\phi_0}\partial_{\psi_0} + \partial^2_{\psi_0}), \tag{5.26}$$

so that in particular,

$$\langle \cos^k \hat{\phi} \rangle \approx \mathrm{e}^{-\mathbf{J}^2/2\bar{n}} \cos^k \phi_0, \quad \langle \cos^k \hat{\psi} \rangle \approx \mathrm{e}^{-\mathbf{J}^2/2\bar{n}} \cos^k \psi_0, \tag{5.27}$$

have sharp maxima at ψ_0 and ϕ_0, while the correlation disappears for high intensity fields

$$\langle \cos\hat{\phi} \cos\hat{\psi} \rangle \approx \mathrm{e}^{-\mathbf{J}^2/2\bar{n}}(\cos\phi_0 \cos\psi_0) = \langle \cos\hat{\phi} \rangle \langle \cos\hat{\psi} \rangle + O(1/\bar{n}). \tag{5.28}$$

Observables depending only on the phase ψ can be computed using a POVM obtained by integrating equation (5.18) over the phase ϕ,

$$\hat{\Delta}(\psi) = \int_0^{2\pi} \mathrm{d}\phi \hat{\Delta}(\phi, \psi), \tag{5.29}$$

which satisfies the covariance condition (5.21), but it is *not* invariant under the action of $\hat{S}_z$. In the particular case of the state equation (4.37) one has

$$P(\psi) \approx \mathrm{e}^{-\mathbf{J}^2/2\bar{n}} \delta(\psi - \psi_0), \quad \mathbf{J}^2 = -\frac{1}{\sin^2\theta_0}\partial^2_{\psi_0}. \tag{5.30}$$

6. Applications to semi-classical dynamics

In this section we present some examples of semi-classical dynamics using the Truncated Wigner Approximation for spin-like systems with fixed and variable values of spin. In general, the TWA is appropriate for description of dynamics of semi-classical states represented in phase space as smooth localized functions, the typical size of which is much less than characteristic dimensions of the classical potential, and located in stable regions of the symbol of Hamiltonian. In addition, the TWA can also describe evolution of states located in unstable classical regions, although the time scale for the validity of this description is necessarily shorter than for the stable motion. This more limited validity is related to a comparatively larger sensitivity to small fluctuations leading to a faster deviation from classical trajectories.

The main effect captured by TWA is a deformation of the initial distribution, since different points in the distribution move with different velocities (with the exception of evolutions generated by Hamiltonians linear in the generators, for which the evolution of the state amounts to a rigid translation in phase space of the original function). Physical effects such as e.g. spin squeezing [108] which arise from this type of Hamiltonian are well described in the TWA and thus are purely classical in nature in the sense that all 'quantumness' of this effect is in the initial distribution rather than in the evolution.

The situation becomes more interesting for multipartite systems. Here, the semi-classical evolution occurs in a phase space which is a direct product of phase spaces of individual subsystems. The semi-classical evolution may then 'entangle' the interacting subsystems in the sense that the initially independent distributions associated with different physical subsystems become strongly interrelated without affecting the purity of the initial state. In this case there may exists more than one semi-classical parameter.

The semi-classical time τ_{sem} for which the TWA describes relatively well the quantum evolution of higher order correlation functions of spin observables, strongly depends both on the Hamiltonian and the initial state. In many physical applications the (dimensionless) semi-classical time is $\tau_{\text{sem}} \sim \varepsilon^{-\lambda}$, $0 < \lambda \leqslant 1$, where ε is a semi-classical parameter. Interestingly, for some selected observables the time validity of the TWA can be significantly longer than the semi-classical time τ_{sem}. In the case of spin-variable systems the semi-classical parameter is usually taken as averaged over the sectors that have the largest contribution in the decomposition of the initial state.

The possibility of simulating the full quantum dynamics by semi-classical dynamics can be fruitfully applied for optimization purposes; for instance, to the determination of optimum values of the interaction constants needed to achieve the best squeezing or entanglement in spin-like systems.

Finally, we note that one important ingredient of the semiclassical approach is that the Wigner function of the physically relevant initial states (such as coherent states, squeezed states, etc) can be nicely approximated in the limit $S \gg 1$ using the asymptotic form of the mapping kernel given in equation (3.23). The asymptotic form of the kernel eliminates numerical errors appearing in the computation of Clebsch–Gordan coefficients for representations of large dimensions.

6.1. Fixed S dynamics

Suppose the density matrix decomposes into $SU(2)$-invariant subspaces as in equation (2.20). Then, in each invariant subspace the semi-classical dynamics is governed by classical trajectories on $\mathcal{S}^2$:

$$W_\rho(\theta, \phi|t) = W_\rho(\theta(-t), \phi(-t)|t = 0), \tag{6.1}$$

where $\theta(t)$, $\phi(t)$ are solutions of the classical Hamiltonian equations and $W_\rho(\theta, \phi) := W_\rho^{(0)}(\theta, \phi)$.

The time scale τ_{sem} for which TWA remains valid is heavily dependent on the highest degree of the polynomial in the $su(2)$ generators entering in the Hamiltonian. For typical physical applications where interaction Hamiltonians are no more than quadratic in the generators and assuming the interaction constant χ of the highest degree operator is independent of the $SU(2)$ irrep label S , the TWA usually fails before $\tau_{\text{sem}} \sim S^{-1/2}$ and $\tau_{\text{sem}} \sim S^{-1}$ for stable and unstable regimes, respectively. Nevertheless, typical observables such as some average values and fluctuations of the spin operators, may evolve according to the semi-classical picture even up to times $\tau \sim 1$, where the quantum interference effects become predominant and in general cannot be neglected.

6.1.1. The finite Kerr-like interaction.

The simplest non-trivial example is the Kerr-like Hamiltonian

$$\hat{H} = \chi \hat{S}_z^2. \tag{6.2}$$

The symbol of the Hamiltonian is given in table 1 as

$$W_H(\theta, \phi) \propto \cos^2 \theta \tag{6.3}$$

and leads to the following evolution of the Wigner function [57, 109, 110]:

$$W_\rho(\Omega|t) = W_\rho\left(\theta, \phi - \frac{\chi\, t}{\varepsilon}\cos\theta | t = 0\right), \tag{6.4}$$

i.e. parts of the initial distribution located at different values of θ precess with different angular velocities about the z axis.

If the initial state is the equatorial coherent state of (3.27), with $\theta_0 = \pi/2, \phi_0 = 0$, so the phase space distribution is localized at the stable fixed point of $W_H(\Omega) \sim \cos^2\theta$, the solution to equation (6.4) leads to a very good description of squeezing effect: the first and the second moments of the spin operators are described up to times $\chi t \sim 1$. By contrast, the evolution of arbitrary observables is usually well described by TWA, in this particular case, only up to times $\chi\tau_{\text{sem}} \lesssim S^{-1/2}$. For longer times, telltale signs of self-interference leading to Schrödinger cats on the sphere start to appear, as can be noticed for instance by analyzing the behaviour of the fourth moment $m_4(t)$ of the Wigner function using Equation (2.39).

6.1.2. The Meshkov–Lipkin interaction. The TWA is also applicable to the less trivial case of the Meshkov–Lipkin interaction [111]:

$$\hat{H} = \chi\hat{S}_z^2 + g\hat{S}_x. \tag{6.5}$$

The semi-classical (i.e. large angular momentum) trajectories of this Hamiltonian have been by investigated by Bohr and Mottelson in the context of the nuclear cranking model [113].

The resulting symbol for the Hamiltonian is given by

$$W_H(\Omega) = g\frac{\sin\theta\cos\phi}{2\varepsilon} + \frac{\chi\cos^2\theta}{4\varepsilon^2}. \tag{6.6}$$

This Hamiltonian can be seen to exhibit a second order phase transition for $g\varepsilon = \chi$.

When the initial state is a spin coherent state $|\theta_0 = \pi/2, \phi_0 = 0\rangle$ located at the minimum of $W_H(\Omega)$ given in equation (6.6), the corresponding truncated evolution equation leads to the standard equations for (θ, ϕ)

$$\partial_t\theta = -g\sin\phi, \tag{6.7}$$

$$\partial_t\phi = -g\cot\theta\cos\phi + \frac{\chi}{\varepsilon}\cos\theta, \tag{6.8}$$

providing a good description of the time-dependence of the two first moments of the angular momentum operators below the phase-transition point $g\varepsilon \leqslant \chi$ up to times $\chi t \lesssim S^{-\lambda}$, $\lambda < 1/2$. Above the phase-transition point, $g\varepsilon > \chi$, the same state $|\pi/2, 0\rangle$ is on the classical separatrix but the evolution of the first two moments is still sufficiently well approximated for $\chi t \lesssim S^{-\lambda}, \lambda \sim 1/2$ even in this unstable regime [112].

The deviation of the semi-classical evolution from the exact evolution of the fourth order moment m_4 of equation (2.39) is quite different above and below the phase transition: below the transition, when $g\varepsilon \leqslant \chi$, the semi-classical approximation hold to $\chi\tau_{\text{sem}} \lesssim S^{-1/2}$ whereas above, when $g\varepsilon > \chi$ it holds to $\chi\tau_{\text{sem}} \lesssim S^{-1}$.

6.1.3. Bi-partite systems. For bi-partite systems, when the Wigner distribution is defined on $\mathcal{S}^2 \times \mathcal{S}^2$, the evolution along classical trajectories leads to inter-dependence of phase variables in the individual sub-manifolds, which results in a strong correlation between interacting subsystems. This effect can be observed even on the simplest example of bi-linear interaction

$$\hat{H} = \chi \hat{S}_z^{(1)} \otimes \hat{S}_z^{(2)}, \tag{6.9}$$

which leads to the following evolution of the bi-partite Wigner function in the TWA [57]:

$$W_\rho(\Omega_1, \Omega_2|t) = W_\rho(\theta_1, \phi_1 - \frac{\chi\, t}{2\varepsilon_2}\cos\theta_2; \theta_2, \phi_2 - \frac{\chi\, t}{2\varepsilon_1}\cos\theta_1 | t = 0), \tag{6.10}$$

$$\varepsilon_{1,2} = (2S_{1,2} + 1)^{-1} \tag{6.11}$$

Equation (6.10) provides an excellent description of the purity dynamics $P(t)$ of one of interacting spins

$$P(t) = \mathrm{Tr}(\rho_1^2) \sim \int \mathrm{d}\,\Omega_1\, \mathrm{d}\,\Omega_2\, \mathrm{d}\,\Omega_2'\, W_\rho(\Omega_1, \Omega_2|t) W_\rho(\Omega_1, \Omega_2'|t), \tag{6.12}$$

frequently used for the characterization of the spin–spin entanglement for pure states.

Since purity is not a physical observable, the limitations for its applicability are not the same as for moments of the angular momentum operators. In particular, the results of TWA basically coincide with exact quantum computations for large spins [57] for arbitrary times.

In situations where the spin–spin interaction is modified by the presence of a magnetic field, the interaction Hamiltonian has more complicated structure:

$$\hat{H} = \chi \hat{S}_z^{(1)} \otimes \hat{S}_z^{(2)} + g(\hat{S}_y^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes \hat{S}_y^{(2)}), \tag{6.13}$$

and in particular reveals a phase transition at $2g\varepsilon = \chi$. The TWA describes well a short time evolution of bi-partite correlations even in the unstable regime and can be used for optimization of spin–spin entanglement as a function of the parameter g/χ in the limit of large spins.

Finally, we add the TWA also describes the long-time dynamics of some other properties of multi-partite systems, such as the negativity [71], which characterizes spin–spin correlations [112].

6.2. Variable spin dynamics

Following equations (4.50)–(4.54), the semi-classical dynamics of a quantum system with variable spin—or alternatively systems for which the initial state contains non-negligible contributions from several large $SU(2)$ subspaces—is defined by trajectories in the four-dimensional symplectic manifold $T^*\mathcal{S}^2$. These trajectories can be different for $W_{\rho_\pm}^{(s)}(\omega, j) = W_\rho^{j\,(s)}(\omega) \pm W_\rho^{j+1/2\,(s)}(\omega)$. The computation of average values of operators containing only $\hat{T}_{KQ}^{\ell\ell'}$ tensors for $\ell + \ell' =$ even in its decomposition (and in particular the square tensors describing operators from the $su(2)$ enveloping algebra) is completed using $\frac{1}{2}(W_{\rho_+}^j(\omega) + W_{\rho_-}^j(\omega))$:

$$\left\langle \hat{T}_{KQ}^{\ell\ell'} \right\rangle = \int_0^\infty \mathrm{d}j\, \frac{j+1}{16\pi^2} \tag{6.14}$$

$$\times \int \mathrm{d}\omega\, W_{T_{KQ}^{\ell\ell'}}(\omega, j) \frac{W_{\rho+}(\omega(-t), j(-t)) + W_{\rho-}(\omega(-t), j(-t))}{2}, \tag{6.15}$$

where $\omega(t)$ and $j(t)$ are classical trajectories on $T^*\mathcal{S}^2$.

For the expectation values of non-squared tensors, for which $\ell + \ell' =$ odd, we need to use $W_\rho^{j+1/2}(\omega) = \frac{1}{2}(W_{\rho+}^j(\omega) - W_{\rho-}^j(\omega))$:

$$\left\langle \hat{T}^{\ell\ell'}_{KQ} \right\rangle = \int_0^\infty \mathrm{d}j \, \frac{j+1}{16\pi^2} \tag{6.16}$$

$$\times \int \mathrm{d}\omega \; W_{T^{\ell\ell'}_{KQ}}(\omega, j) \frac{W_{\rho+}(\omega(-t), j(-t)) - W_{\rho-}(\omega(-t), j(-t))}{2}, \tag{6.17}$$

where $\ell - \ell' = n + 1/2, n \in \mathbb{Z}$.

There are two basic physical situations for which the semi-classical picture on $T^*\mathcal{S}^2$ is appropriate, depending if the expansion of the Hamiltonian contains only square tensors, or rectangular and square tensors.

6.2.1. Hamiltonian with square tensors only. If the decomposition of the Hamiltonian contains only square tensors so $\hat{H}$ is block-diagonal, and if an exact or perturbative solution in each $SU(2)$-subspace is available, the whole evolution operator is recovered by summing over irreducible subspaces with coefficients from the initial state expansion.

On the other hand, the density matrix will typically contain contributions from multiple subspaces, its phase space image, in general, will contain non-square tensors coupled by the Hamiltonian: the evolution of the density matrix is not a simple sum of maps in each $SU(2)$-irreducible subspace.

This is where decomposing in terms of j-sectors becomes particularly convenient: whereas various $SU(2)$-subspaces of W_ρ may be mixed, different j-sector are NOT mixed under evolution: when the Hamiltonian is block-diagonal and preserves irreducible $SU(2)$-subspaces the Wigner function evolves in each j-sector as

$$W^j_\rho(\omega|t) = W_\rho(\phi(-t), \theta(-t), \psi + \Psi(\theta, \phi|t), j), \tag{6.18}$$

where $\theta(t), \phi(t)$ are classical trajectories on $\mathcal{S}^2$ and the form of $\Psi(\theta, \phi, j|t)$ depends on the Hamiltonian function.

For instance, it follows readily from equation (4.25) that the total angular momentum, $\hat{H} = \omega \hat{J}^2$, generates only rotations of the ψ-angle in the (integer) j-symbols,

$$W^j_\rho(\omega|t) = W_\rho(\phi, \theta, \psi - (j+1)\omega t, j|t=0). \tag{6.19}$$

As another example, consider the non-linear two-mode Hamiltonian

$$\hat{H} = \frac{\chi}{2}((\hat{a}^\dagger \hat{a})^2 + (\hat{b}^\dagger \hat{b})^2) + \frac{g}{2}(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger), \tag{6.20}$$

that preserves the integral of motion $\hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}$ and can be recast, up to the constant operator $\frac{1}{2}\hat{N}(\frac{1}{2}\hat{N}+1) \equiv \hat{J}^2$ representing the total angular momentum, in terms of the $su(2)$ algebra generators in form of the Lipkin-Meshkov model given in equation (6.5), with

$$\hat{S}_x = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger), \quad \hat{S}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}). \tag{6.21}$$

If the initial state belongs to a single $SU(2)$ irrep, the approximation given in equation (6.1) can be applied in the limit of large average excitation numbers $\langle \hat{N} \rangle$.

For an arbitrary initial state, for instance the product of coherent states given in equation (4.35) which does not belong to a single $SU(2)$ subspace, where at least one of the fields is in a strong coherent state, the evolution equation on $T^*\mathcal{S}^2$ should be used. The angle ψ in the general solution (6.18) then evolves according to

$$\psi(t)=\psi+\frac{2}{3}\chi(j+1)t+g\int_0^t\frac{\cos\phi(\tau)}{\sin\theta(\tau)}\mathrm{d}\tau. \tag{6.22}$$

We should point out that it is not always convenient to use a map from a two-mode Hamiltonian into an $su(2)$ form because of the complexity of computing the symbol of the Hamiltonian. For instance, a direct application of the generalized map of equations (4.6) and (4.8) and in particular of equation (4.34) is more appropriate in the case of the down-conversion Hamiltonian,

$$\hat{H}=g(\hat{a}^{\dagger 2}\hat{b}+\hat{b}^{\dagger}\hat{a}^2). \tag{6.23}$$

The j symbol of this Hamiltonian contains only half-integer j and for $j\gg 1$ simplifies to

$$W_H^j(\omega)\approx 2(j+1)^{3/2}\cos^2\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(3\phi+\psi), \tag{6.24}$$

while the representation of (6.23) in terms of $su(2)$ generators contains square roots and their Wigner symbols do not have simple expressions.

Solutions in the form equation (6.18) should be used for computation of averages of non-squared operators, as for instance, $\hat{a}^{\dagger},\hat{a}$ in case of equation (6.5) or (6.23).

6.2.2. Hamiltonians containing non-square tensors. Suppose the expansion of the Hamiltonian of the system contains non-square operators $\hat{T}_{KQ}^{\ell\ell'}$, $\ell\neq\ell'$. A simple physical situation where this would occur consists e.g. in adding to an $SU(2)$ invariant two-mode Hamiltonian a (non-invariant) pumping term $\sim(\hat{a}^{\dagger}+\hat{a})$.

For this type of Hamiltonian the two-mode Heisenberg–Weyl TWA does not always lead to satisfactory results, especially when one of the non-linearly coupled field is initially in a vacuum state. This kind of system cannot be treated in terms of the standard Stratonovich–Weyl approach either, but is tractable in the formalism of generalized j-mappings [100].

As an example of this situation, we consider the Lipkin–Meshkov Hamiltonian of equation (6.20) in the presence of external pumping $\mu(\hat{a}^{\dagger}+\hat{a})$.

In this case, the evolution of both $W_{\rho_+}(\omega,j)$ and $W_{\rho_-}(\omega,j)$ are required. The evolution of $W_{\rho_+}(\omega,j)$ yields Hamilton equations defining classical trajectories on $T^*\mathcal{S}^2$:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-g\sin\phi, \tag{6.25}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}=-g\cot\theta\cos\phi+\chi(j+1)\cos\theta, \tag{6.26}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}=g\frac{\cos\phi}{\sin\theta}+\frac{2}{3}\chi(j+1), \tag{6.27}$$

$$\frac{\mathrm{d}j}{\mathrm{d}t}=-2\mu\sqrt{j+1}\,\cos(\frac{1}{2}\theta)\sin(\frac{1}{2}(\phi+\psi)). \tag{6.28}$$

The first two equations are similar to (6.7) with $\varepsilon^{-1} \to j+1$ but now contain an evolution of the index j.

The evolution of $W_{\rho_-}(\omega, j)$ is obtained with minimal effort by changing $\mu \to -\mu$ in equation (6.28), with the other Hamilton equations remaining the same.

As in the single-S $SU(2)$ case, the initial Wigner function can be well approximated in the limit $j \gg 1$. For instance, the Wigner function corresponding to the state $|\alpha = 0\rangle|\beta = r\rangle$ in the limit of a highly excited b-mode, $r^2 \gg 1$ has the form

$$\left.\begin{aligned} W^j_\rho(\omega) &= 2\sqrt{\frac{j}{r^2}}\,\mathrm{e}^{-(j-r^2)^2/2r^2}\mathrm{e}^{-2j\sin(\frac{1}{2}\nu)^2}, \\ W^{j+1/2}_\rho(\omega) &= 2\mathrm{e}^{-(j-r^2)^2/2r^2}\mathrm{e}^{-2j\sin(\frac{1}{2}\nu)^2}\cos\frac{1}{2}\nu, \end{aligned}\right\} \qquad j \in \mathbb{Z}^+, \tag{6.29}$$

where

$$\cos\tfrac{1}{2}\nu = \sin\tfrac{1}{2}\theta\cos\left(\tfrac{1}{2}(\phi - \psi)\right). \tag{6.30}$$

6.3. The linear rigid rotor

The linear rigid rotor is an example of a system for which typical Hamiltonians contain non-square tensors. This example deserves a separate treatment due to its importance in many applications, such as linear molecules in external fields.

To connect with Classical Mechanics, we need a change of coordinate so that the symmetry axis of the rotor (rather than the total angular momentum axis) is now the quantization axis. This is achieved by first applying a $\pi/2$-rotation about the y-axis. Applying this rotation to the kernel of equation (4.11) yields

$$\hat{\mathrm{w}}^y_j(0) = \mathrm{e}^{-\mathrm{i}\pi/2\hat{S}_y}\hat{\mathrm{w}}(0)\mathrm{e}^{\mathrm{i}\pi/2\hat{S}_y}. \tag{6.31}$$

The full kernel

$$\hat{\mathrm{w}}^y_j(\omega) = T(\omega)\hat{\mathrm{w}}^y_j(0)T^\dagger(\omega) \tag{6.32}$$

leads to the standard classical picture [114]. This can be verified by noting for instance that, with this change of axis, the symbol of $\hat{n}_z$ operator (4.27) has now an intuitive form

$$W^j_{n_z}(\omega) \approx \cos\theta \sum_{n=0,1,..} \delta_{j,n} + \mathcal{O}(j^{-1}). \tag{6.33}$$

This should be compared with the 'unrotated' expression of equation (4.29).

The Darboux coordinates become (p_θ, θ) and (p_ϕ, ϕ), where the conjugate momenta to coordinates θ and ϕ are

$$p_\theta = (j+1)\sin\psi, \tag{6.34}$$

$$p_\phi = (j+1)\sin\theta\cos\psi, \tag{6.35}$$

and $p_\psi = 0$ always.

Using this scheme a simple semi-classical approach to linear rigid rotor evolution can be developed, as for instance, the alignment dynamics in an external field described by

$$\hat{H} = \hat{J}^2 - \kappa\hat{n}_z^2. \tag{6.36}$$

To leading order in $1/j$, the j-symbol is just a classical Hamiltonian

$$W_H^j(\omega, j) = \frac{j}{2}\left(\frac{j}{2}+1\right) - \kappa\cos^2\theta, \tag{6.37}$$

where j takes only integer values. The truncated Wigner evolution for $W_\rho(\omega, j)$ leads to the following evolution:

$$\partial_t\theta = -(j+1)\sin\psi, \tag{6.38}$$

$$\partial_t\phi = -(j+1)\frac{\cos\psi}{\sin\theta}, \tag{6.39}$$

$$\partial_t\psi = -(j+1)\cot\theta\cos\psi + \frac{4\kappa\cos\theta\sin\theta\cos\psi}{j+1} \tag{6.40}$$

$$\partial_t j = 4\kappa\cos\theta\sin\theta\sin\psi, \tag{6.41}$$

which are equivalent to the standard Hamilton equations for the canonical variables (p_θ, θ) and (p_ϕ, ϕ) and describe well the alignment dynamics for times $\kappa t \sim 1$. Alternative phase space-like approaches were developed in [115].

6.4. Dissipation

As a final application we consider the semi-classical dynamics of dissipative systems, governed by the Lindblad equation:

$$\partial_t\hat{\rho} = -\mathrm{i}\,[\hat{H}, \hat{\rho}] + \sum_j \gamma_j\hat{L}_j(\hat{\rho}), \tag{6.42}$$

where $\hat{L}_j$ is a Lindblad superoperator.

Phase space methods allow the rewriting (sometimes exactly as for HW symmetry, or to some approximation in the $SU(2)$ case) of the Lindblad equation in the form of a diffusion equation, thus helping the analysis of physical effects using the intuitive language of classical mechanics [116, 117].

In the semi-classical limit of the TWA, the Schrödinger equation is mapped to the classical Hamiltonian equation (in the sense that the image of the commutator is the appropriate Poisson brackets), so that zeroth order terms in a semi-classical parameter appearing in the expansion of the star-product of equations (3.28), (4.45) and (4.46) (or of the correspondence rules of equations (3.40), (3.41), (4.62)–(4.64)) cancel out.

As a result of this cancellation, the star-product is not formally required in order to obtain the evolution equation in the semi-classical limit within the framework of the standard classical phase space approach. To represent the action of the superoperators $\hat{L}_j$ in phase space, both zeroth and first order terms in the semi-classical expansions (3.40), (3.41), (4.62)–(4.64) are needed.

The dissipative phase-space dynamics of spins were studied quite intensively using mainly the P- and Q-symbols. The Wigner map is sometimes more convenient for physical analysis [118]; the evolution equation takes a slightly different form than for the P- and Q-representations. Here, we provide a few equations appearing in typical spin relaxation problems.

6.4.1. Single spin systems. When the system is in equilibrium with a thermal bath at temperature T and limited to a single value of S, the dissipative terms are given by

$$\hat{L}_1(\hat{\rho}) = \gamma_1(2\hat{S}_-\hat{\rho}\hat{S}_+ - \hat{S}_+\hat{S}_-\hat{\rho} - \hat{\rho}\hat{S}_+\hat{S}_-), \qquad \gamma_1 = \frac{g(\nu+1)}{2}, \tag{6.43}$$

where $\nu = [\exp(\hbar\omega_0/kT) - 1]^{-1}$ and

$$\hat{L}_2(\hat{\rho}) = \gamma_2(2\hat{S}_+\hat{\rho}\hat{S}_- - \hat{S}_-\hat{S}_+\rho - \hat{\rho}\hat{S}_-\hat{S}_+), \qquad \gamma_2 = \frac{g\nu}{2}, \tag{6.44}$$

with g a system-bath effective coupling. The Lindblad equation (6.42) was discussed in the framework of the $SU(2)$ phase-space approach in numerous papers (see for instance [41] [117] [118] and references therein). The evolution equation for the Wigner function is obtained by using the semi-classical expansion of the correspondence rules for the $su(2)$ generators:

$$\hat{L}_{1,2}(\hat{\rho}) \;\to \Bigg[-\mathcal{L}^2 - \frac{\partial^2}{\partial\phi^2} \mp \varepsilon^{-1}(2\cos\theta + \sin\theta\partial_\theta) \tag{6.45}$$

$$\pm\frac{\varepsilon}{2}\left(2(\mathcal{L}^2+1)\cos\theta + \left(\frac{3}{2} + \frac{2\mathcal{L}^2+1}{4}\right)\sin\theta\partial_\theta\right)\Bigg]W_\rho(\Omega). \tag{6.46}$$

with $\mathcal{L}$ is given in equation (3.34).

A contribution from a dispersion-like decoherence

$$\hat{L}_d(\hat{\rho}) = \frac{\gamma_d}{2}(2\hat{S}_z\hat{\rho}\hat{S}_z - \hat{S}_z^2\hat{\rho} - \hat{\rho}\hat{S}_z^2) \tag{6.47}$$

takes the form of a phase-diffusion differential operator:

$$\hat{L}_d(\hat{\rho}) \to \frac{\partial^2 W_\rho(\Omega)}{\partial\phi^2}. \tag{6.48}$$

In the high temperature limit where $\nu \gg 1$, we find the simplified expression for the super-operator image in the phase space:

$$\hat{L}(\hat{\rho}) \simeq \frac{\nu}{2}(\hat{L}_1 + \hat{L}_2)(\hat{\rho}) \to -\frac{\nu}{2}\left(\mathcal{L}^2 + \frac{\partial^2}{\partial\phi^2}\right)W_\rho(\Omega). \tag{6.49}$$

6.4.2. Multiple S values. In this case a contribution from the $\hat{S}_0$ operator, defined in equation (5.22) and such that $\mathcal{L}^2 = \hat{S}_0(\hat{S}_0+1)$ may arise in order to describe a decoherence between different invariant subspaces. This effect appears in the description of depolarizing channel in two-mode system:

$$\hat{L}_s(\hat{\rho}) = \frac{\Gamma}{2}(2\hat{S}_0\hat{\rho}\hat{S}_0 - \hat{S}_0^2\hat{\rho} - \hat{\rho}\hat{S}_0^2). \tag{6.50}$$

In this case the phase space dynamics is described by

$$\hat{L}_s(\hat{\rho}) \to \frac{\partial^2 W_\rho(\omega, j)}{\partial\psi^2}. \tag{6.51}$$

7. Generalizations: *SU*(*n*) mappings

7.1. The general case

The formalism of Stratonovich–Weyl maps can be fairly easily generalized to some systems with $SU(n)$ dynamical symmetry. A straightforward construction of an s-parametrized $\hat{w}^{(s)}(\Omega)$ mapping kernel for systems with $SU(n)$ symmetry group was proposed in [32] and contains certain ambiguities. An accurate deduction of such kernel for symmetric representations of $SU(n)$ has been done in [23] (see also [81, 122]).

Denote by $\mathbb{H}$ the Hilbert space of a quantum system. Suppose a compact Lie group $\mathfrak{G}$, which we take to be $SU(n)$ acts irreducibly on $\mathbb{H}$ so that $\mathbb{H}$ carries the irrep λ of $\mathfrak{G}$. Elements $\omega \in \mathfrak{G}$ act linearly in $\mathbb{H}$ via the matrix representation $T(\omega)$.

Let $\mathfrak{H}$ be the largest subgroup of $SU(n)$ that leaves $|\lambda; \text{h.w.}\rangle$ invariant (to within a phase), and the phase space for the corresponding classical system is isomorphic to the coset $\mathfrak{M} = SU(n)/\mathfrak{H}$ [83], so that $SU(n)$ acts on $\mathfrak{M}$ by canonical transformations.

Operators acting in $\mathbb{H}$ will transform according to the irrep $\lambda \otimes \lambda^*$, where λ^* the irrep conjugate to λ.

The product $\lambda \otimes \lambda^*$ is reducible so, in addition to the labeling of states in irrep λ, we must consider the labeling of tensors from a general irrep $\sigma = (\sigma_1, \sigma_2, ..., \sigma_{n-1})$. In general, a weight $\boldsymbol{\alpha}$ in irrep σ may occur multiple times; the label I_α distinguishes between multiple occurrences of this weight.

Ambiguities occur when the irrep σ occurs more than once in the decomposition $\lambda \otimes \lambda^*$, and how to proceed in such cases remains an open question [23].

Assuming σ is not repeated in the decomposition $\lambda \otimes \lambda^*$, a tensor $\hat{T}^{\lambda}_{\sigma;\gamma I_\gamma}$ is given explicitly by

$$\hat{T}^{\lambda}_{\sigma;\gamma I_\gamma} = \sum_{\alpha I_\alpha \beta I_\beta} |\lambda; \boldsymbol{\alpha} I_\alpha\rangle\langle\lambda; \boldsymbol{\beta} I_\beta| \tilde{C}^{\sigma\gamma I_\gamma}_{\lambda \boldsymbol{\alpha} I_\alpha; \lambda^* \boldsymbol{\beta}^* I_\beta}, \tag{7.1}$$

where λ^* the irrep conjugate to λ, β^* the weight conjugate to β, and σ an irrep in the decomposition of $\lambda \otimes \lambda^*$.

The coefficients $\tilde{C}^{\sigma\gamma I_\gamma}_{\lambda\boldsymbol{\alpha} I_\alpha;\lambda^* \boldsymbol{\beta}^* I_\beta}$ satisfy the orthogonality relation

$$\sum_{\alpha I_\alpha \beta I_\beta} \left(\tilde{C}^{\sigma' \boldsymbol{\nu}' I_{\nu'}}_{\lambda\, \boldsymbol{\alpha} I_\alpha; \lambda^* \boldsymbol{\beta}^* I_\beta}\right)^* \tilde{C}^{\sigma\, \boldsymbol{\nu} I_\nu}_{\lambda\, \boldsymbol{\alpha} I_\alpha; \lambda^* \boldsymbol{\beta}^* I_\beta} = \delta_{\sigma\sigma'} \delta_{\boldsymbol{\nu}\boldsymbol{\nu}'} \delta_{I_\nu I_{\nu'}} \tag{7.2}$$

and are elements of a unitary matrix. The irreducible tensors of equation (7.1) satisfy

$$[\hat{h}_i, \hat{T}^{\lambda}_{\sigma;\boldsymbol{\alpha} I_\alpha}] = \alpha_i \hat{T}^{\lambda}_{\sigma;\boldsymbol{\alpha} I_\alpha}. \tag{7.3}$$

which implies the tensors are trace orthogonal over σ, $\boldsymbol{\alpha}$ and I_α:

$$\mathrm{Tr}\,[(\hat{T}^{\lambda}_{\sigma;\boldsymbol{\alpha} I_\alpha})^* \hat{T}^{\lambda}_{\sigma';\boldsymbol{\alpha}' I'_{\alpha'}}] = \delta_{\sigma\sigma'} \delta_{\alpha\alpha'} \delta_{I_\alpha I'_{\alpha'}}. \tag{7.4}$$

For $SU(n)$ irreps of the type $(\lambda, 0, \ldots, 0)$, basis states will be written $|\lambda; \boldsymbol{\nu}\rangle$, where the ith component of the weight $\boldsymbol{\nu} = [\nu_1, \ldots, \nu_{n-1}]$ is the eigenvalue of the ith Cartan element on the state:

$$\hat{h}_i |\lambda; \boldsymbol{\nu}\rangle = \nu_i\, |\lambda; \boldsymbol{\nu}\rangle. \tag{7.5}$$

For these irreps there is no weight multiplicity and the additional label I is not necessary. Moreover, an irrep σ occurs at most once in $\lambda \otimes \lambda^*$, where $\lambda^* = (0, \ldots, \lambda)$. Thus, for instance:

$$(\lambda, 0) \otimes (0, \lambda) = \bigoplus_{\sigma=0}^{\lambda} (\sigma, \sigma) \qquad \text{for } SU(3), \tag{7.6}$$

$$(\lambda, 0, 0) \otimes (0, 0, \lambda) = \bigoplus_{\sigma=0}^{\lambda} (\sigma, 0, \sigma) \quad \text{for } SU(4), \text{etc.} \tag{7.7}$$

The highest weight vector of $(\lambda, 0, \ldots, 0)$ is $U(n-1)$ invariant; this subgroup acts only on the last $n-2$ components of a weight. The kernel is then of the form

$$\hat{w}_{\lambda}^{(s)}(\Omega) = \sum_{\sigma} F_{\sigma}^{(s)} \sum_{\beta I_{\beta}} D^{\sigma}_{\beta I_{\beta};\mathbf{0}\mathbf{0}}(\Omega) \hat{T}^{\lambda}_{\sigma;\beta I_{\beta}}, \tag{7.8}$$

with

$$D^{\sigma}_{\beta I_{\beta};\mathbf{0}\mathbf{0}}(\Omega) \equiv \langle \sigma; \beta I_{\beta} | T(\Omega) | \sigma; \mathbf{0}\mathbf{0} \rangle, \;\; \Omega \in \mathfrak{M}, \tag{7.9}$$

an $SU(n)$ group function for the irrep $\sigma \equiv (\sigma, 0, \ldots, 0, \sigma)$ [119], and

$$F_{\sigma}^{(s)} = \left[\frac{\dim(\sigma)}{\dim(\lambda)} \right]^{(s+1)/2} \left(\tilde{C}^{\sigma \mathbf{0}\mathbf{0}}_{\lambda\,\text{h.w.};\lambda^*\,\text{h.w.}^*} \right)^{-s}. \tag{7.10}$$

Here, the notation $\mathbf{0}0$ is meant to imply that the state $|\sigma; \mathbf{0}0\rangle$ has zero-weight and is a scalar (invariant) under $U(n-1)$ transformations.

7.2. The case of SU(3) irreps of the type $(\lambda, 0)$

For the simplest non-trivial example of the $SU(3)$ group, the general equations (7.8)–(7.10) can be further developed. We follow [119] for the labeling and construction of basis states for an irrep (λ, μ). Defining

$$\hat{C}_{ij} = a^{\dagger}_{i1} a^{\dagger}_{j1} + a^{\dagger}_{i2} a^{\dagger}_{j2}, \tag{7.11}$$

the Lie algebra $su(3)$ is spanned by the six ladder operators $\{\hat{C}_{ij}, i \neq j = 1, 2, 3\}$ together with the two diagonal Cartan elements

$$\hat{h}_1 = 2\hat{C}_{11} - \hat{C}_{22} - \hat{C}_{33}, \qquad \hat{h}_2 = \hat{C}_{22} - \hat{C}_{33}. \tag{7.12}$$

The root diagram is given in figure 2. Basis states are then given by

$$|(\lambda, \mu)(\nu_1\nu_2\nu_3)I\rangle = \sum_{m_1m_2m_3(N)} \left\langle \begin{matrix} \tfrac12\nu_3 \\ m_3 \end{matrix} ; \begin{matrix} \tfrac12\nu_2 \\ m_2 \end{matrix} \middle| \begin{matrix} I \\ N \end{matrix} \right\rangle \left\langle \begin{matrix} I \\ N \end{matrix} ; \begin{matrix} \tfrac12\nu_1 \\ m_1 \end{matrix} \middle| \begin{matrix} \tfrac12\lambda \\ \tfrac12\lambda \end{matrix} \right\rangle |\tfrac12\nu_1 m_1\rangle |\tfrac12\nu_2 m_2\rangle |\tfrac12\nu_3 m_3\rangle, \tag{7.13}$$

$$|s_i m_i\rangle = \frac{(\hat{a}^{\dagger}_{i1})^{s_i+m_i} (\hat{a}^{\dagger}_{i2})^{s_i-m_i}}{\sqrt{(s_i+m_i)!\,(s_i-m_i)!}} |0\rangle. \tag{7.14}$$

with $\left\langle \begin{matrix} j_1 \\ m_1 \end{matrix} ; \begin{matrix} j_2 \\ m_2 \end{matrix} \middle| \begin{matrix} J \\ M \end{matrix} \right\rangle$ an $SU(2)$ Clebsch–Gordan coefficient.

The weight $\boldsymbol{\nu} \equiv [\nu_1 - \nu_2, \nu_2 - \nu_3]$ of the state is extracted from a triple $(\nu_1\nu_2\nu_3)$ of non-negative integers constrained by $\nu_1 + \nu_2 + \nu_3 = \lambda + 2\mu$. The label ν_k is seen, from equations (7.13) and (7.14), to be the total number of quanta in mode k of a 3-dimensional oscillator with two

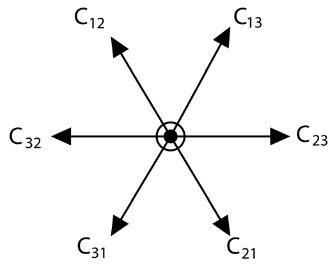

Figure 2. The root diagram for the complexification of $su(3)$.

internal degrees of freedom. The states of equation (7.13) transform as angular momentum I states under the $su(2)$ subalgebra spanned by $\hat{C}_{23}$, $\hat{C}_{32}$ and $\hat{h}_2$.

In general, states satisfy

$$\begin{aligned}\hat{h}_1|(\lambda,\mu)(\nu_1\nu_2\nu_3)I\rangle &= (2\hat{C}_{11}-\hat{C}_{22}-\hat{C}_{33})|(\lambda,\mu)(\nu_1\nu_2\nu_3)I\rangle,\\ &= (2\nu_1-\nu_2-\nu_3)|(\lambda,\mu)(\nu_1\nu_2\nu_3)I\rangle,\end{aligned} \tag{7.15}$$

$$\begin{aligned}\hat{h}_2|(\lambda,\mu)(\nu_1\nu_2\nu_3)I\rangle &= (\hat{C}_{22}-\hat{C}_{33})|(\lambda,\mu)(\nu_1\nu_2\nu_3)I\rangle,\\ &= (\nu_2-\nu_3)|(\lambda,\mu)(\nu_1\nu_2\nu_3)I\rangle.\end{aligned} \tag{7.16}$$

Since the irrep $(\lambda,0)$ does not have weight multiplicities, i.e. each weight $\boldsymbol{\nu}$ occurs at most once, the multiplicity label I for states of this irrep is redundant and often not indicated. In this special case the basis states are simply

$$|(\lambda,0)\nu_1\nu_2\nu_3\rangle = \frac{(\hat{a}^{\dagger}_{11})^{\nu_1}(\hat{a}^{\dagger}_{21})^{\nu_2}(\hat{a}^{\dagger}_{31})^{\nu_3}}{\sqrt{\nu_1!\,\nu_2!\,\nu_3!}}|0\rangle, \qquad I=\frac{1}{2}(\nu_2+\nu_3). \tag{7.17}$$

The highest weight is $|(\lambda,0)\lambda 00\rangle$ and invariant under U_{23} transformations of the form $R_{23}(\omega)\,\mathrm{e}^{-\mathrm{i}\gamma_1\hat{h}_1}$. Hence the coherent states have the form

$$|\Omega;\lambda\rangle = T(\Omega)|(\lambda,0)\lambda 00\rangle, \tag{7.18}$$

where $\Omega := (\alpha_1,\beta_1,\alpha_2,\beta_2)$ is in the coset $SU(3)/U_{23}(2)\simeq\mathbb{CP}^2$ and $T(\Omega)$ is the coset transformation

$$T(\Omega) := R_{23}(\alpha_1,\beta_1,-\alpha_1)R_{12}(\alpha_2,\beta_2,-\alpha_2). \tag{7.19}$$

Here, $R_{ij}(\eta,\theta,\varphi)$ are transformations of the $SU(2)$ subgroup with subalgebra spanned by $\hat{C}_{ij}$, $\hat{C}_{ji}$, $\frac{1}{2}[\hat{C}_{ij},\hat{C}_{ji}]$, and with parameter range $0\leqslant\alpha_1,\alpha_2\leqslant 2\pi$, $0\leqslant\beta_1,\beta_2\leqslant\pi$.

As the transformations of equation (7.19) are products of $SU(2)$ transformations (albeit from different subgroups), the coset functions - they are group function in irreps of the type (σ,σ) as per equation (7.6)— are a sum of products of $SU(2)$ $D^J_{mm'}$ functions [119]:

$$\begin{aligned}D^{(\sigma,\sigma)}_{\boldsymbol{\nu}I;(\sigma\sigma\sigma)0}(\Omega) &= \langle(\sigma,\sigma)\boldsymbol{\nu}I|R_{23}(\alpha_1,\beta_1,-\alpha_1)R_{12}(\alpha_2,\beta_2,-\alpha_2)|(\sigma,\sigma)(\sigma\sigma\sigma)0\rangle\\ &= (-1)^{\sigma}\sqrt{2I+1}\,D^I_{\frac{1}{2}(\nu_2-\nu_3),\frac{1}{2}(\sigma-\nu_1)}(\alpha_1,\beta_1,-\alpha_1)\\ &\times\sum_{\ell=|M|}^{\sigma}(-1)^{\ell}\frac{(2\ell+1)}{\sigma+1}\begin{Bmatrix}\frac{1}{2}\sigma & \frac{1}{2}(2\sigma-\nu_1) & I\\ \frac{1}{2}\nu_1 & \frac{1}{2}\sigma & \ell\end{Bmatrix}D^{\ell}_{M,0}(\alpha_2,\beta_2,-\alpha_2),\end{aligned} \tag{7.20}$$

with $M = \nu_1 - \sigma$ and $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ an $su(2)$ 6j-symbol [119]. The measure $\mathrm{d}\,\Omega$ on the coset and the range of integration of the parameters are given by

$$\int \mathrm{d}\,\Omega = \int_0^{2\pi} \mathrm{d}\alpha_2 \int_0^{2\pi} \mathrm{d}\alpha_1 \int_0^{\pi} \sin\beta_1 \mathrm{d}\beta_1 \int_0^{\pi} \frac{\mathrm{d}\beta_2 1 - \cos\beta_2}{4} \sin\beta_2. \tag{7.21}$$

The $SU(3)$ tensor operators are constructed using $SU(3)$ Clebsch–Gordan coefficients for the coupling $(\lambda, 0) \otimes (0, \lambda) \to (\sigma, \sigma)$. Indeed we find

$$\begin{aligned} \hat{T}^{(\sigma,\sigma)}_{\nu_1\nu_2\nu_3;J} = & \sum_{\boldsymbol{\alpha\beta}} |(\lambda, 0)\alpha_1\alpha_2\alpha_3\rangle\langle(\lambda, 0)\beta_1\beta_2\beta_3| \\ & \times \left\langle \begin{matrix} (\lambda, 0) \\ \alpha_1\alpha_2\alpha_3 \end{matrix} ; \begin{matrix} (0, \lambda) \\ \lambda - \beta_1, \lambda - \beta_2, \lambda - \beta_3 \end{matrix} \middle| \begin{matrix} (\sigma, \sigma) \\ \nu_1\nu_2\nu_3; J \end{matrix} \right\rangle (-1)^{\lambda-\beta_2} \end{aligned} \tag{7.22}$$

where the $SU(3)$ Clebsch–Gordan coefficient can be factored in the usual way as

$$\left\langle \begin{matrix} (\lambda, 0) \\ \alpha_1\alpha_2\alpha_3 \end{matrix} ; \begin{matrix} (0, \lambda) \\ \lambda - \beta_1, \lambda - \beta_2, \lambda - \beta_3 \end{matrix} \middle| \begin{matrix} (\sigma, \sigma) \\ \nu_1\nu_2\nu_3; J \end{matrix} \right\rangle \tag{7.23}$$

$$= \left\langle \begin{matrix} (\lambda, 0) \\ \alpha_1 \end{matrix} ; \begin{matrix} (0, \lambda) \\ \lambda - \beta_1 \end{matrix} \middle\| \begin{matrix} (\sigma, \sigma) \\ \nu_1; J \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{1}{2}(\alpha_2 + \alpha_3) \\ \frac{1}{2}(\alpha_2 - \alpha_3) \end{matrix} ; \begin{matrix} \frac{1}{2}\beta_1 \\ \frac{1}{2}(\beta_3 - \beta_2) \end{matrix} \middle| \begin{matrix} J \\ \frac{1}{2}(\nu_2 - \nu_3) \end{matrix} \right\rangle. \tag{7.24}$$

The coefficient $\left\langle \begin{matrix} (\lambda, 0) \\ \alpha_1 \end{matrix} ; \begin{matrix} (0, \lambda) \\ \lambda - \beta_1 \end{matrix} \middle\| \begin{matrix} (\sigma, \sigma) \\ \nu_1; J \end{matrix} \right\rangle$ is the reduced $SU(3)$ CG, an expression for which is given in equation (D.1) of the appendix.

Since $\dim(\lambda, 0) = \frac{1}{2}(\lambda + 1)(\lambda + 2)$ and $\dim(\sigma, \sigma) = (\sigma + 1)^3$ the factor $F^{(s)}_\sigma$ of equation (7.10) specializes to:

$$F^{(s)}_\sigma = \left((-1)^\lambda \left\langle \begin{matrix} (\lambda, 0) \\ \lambda 00 \end{matrix} ; \begin{matrix} (0, \lambda) \\ 0\lambda\lambda \end{matrix} \middle| \begin{matrix} (\sigma, \sigma) \\ \sigma\sigma\sigma; 0 \end{matrix} \right\rangle \right)^{-s} \left(\frac{2(\sigma + 1)^3}{(\lambda + 1)(\lambda + 2)} \right)^{(s+1)/2}, \tag{7.25}$$

$$= \left((-1)^\lambda \lambda! \sqrt{\frac{2(\sigma + 1)^3}{(\lambda + \sigma + 2)!\,(\lambda - \sigma)!}} \right)^{-s} \left(\frac{2(\sigma + 1)^3}{(\lambda + 1)(\lambda + 2)} \right)^{(s+1)/2}, \tag{7.26}$$

where we have used the expression for $\left\langle \begin{matrix} (\lambda, 0) \\ \lambda \end{matrix} ; \begin{matrix} (0, \lambda) \\ 0 \end{matrix} \middle\| \begin{matrix} (\sigma, \sigma) \\ \sigma; 0 \end{matrix} \right\rangle$ given in equation (D.9) of the appendix.

These expressions allow a complete construction of the kernel $\hat{w}^{(s)}_\lambda(\Omega)$ of equation (7.8). The Wigner-type symbols, $s = 0$, of some of the $su(3)$ generators are provided in table 2.

Average values are computed in the usual manner

$$\langle \hat{A} \rangle = \frac{(\lambda + 1)(\lambda + 2)}{8\pi^2} \int \mathrm{d}\,\Omega\; W_\rho(\Omega) W_A(\Omega). \tag{7.27}$$

Combining the expressions of equations (7.26) and (7.22) for $F^{(0)}_\sigma$ and $\hat{T}^\lambda_{\sigma;\mathbf{00}}$, the Wigner kernel at $\Omega = 0$ takes the form

$$\hat{w}_{\lambda}^{(0)}(0) = \sum_{\sigma=0}^{\lambda} F_{\sigma}^{(0)}\,\hat{T}_{\sigma;\mathbf{00}}^{\lambda}, \tag{7.28}$$

$$= \sum_{n_1+n_2+n_3=\lambda} |(\lambda,0)n_1n_2n_3\rangle\langle(\lambda,0)n_1n_2n_3|\,\mathcal{C}_{n_1n_2n_3}^{\lambda}(-1)^{n_1}, \tag{7.29}$$

where $n_1 + n_2 + n_3 = \lambda$. On the other hand, for $s = -1$, the kernel at the origin simplifies to the expected form

$$\hat{w}_{\lambda}^{(-1)}(0) = \sum_{\sigma} F_{\sigma}^{(-1)}\hat{T}_{\sigma\mathbf{00}}^{\lambda} = |(\lambda,0)\lambda00\rangle\langle(\lambda,0)\lambda00|. \tag{7.30}$$

In the limit $\lambda \gg 1$, the coefficient $\mathcal{C}_{n_1n_2n_3}^{\lambda}$ can be approximated [23] as:

$$\mathcal{C}_{n_1n_2n_3}^{\lambda} \simeq \begin{cases} 0 & \text{if } n_1 < \lambda/3, \\ -2 + \dfrac{6}{\lambda}n_1 & \text{if } n_1 > \lambda/3, \end{cases} \tag{7.31}$$

in a manner reminiscent of the asymptotic form of the $SU(2)$ kernel at the origin given in equation (3.23).

Again in the $\lambda \gg 1$ regime, the Wigner symbol for the highest weight state $|(\lambda,0)\lambda00\rangle$ is well approximated by

$$W_{\rho}(\Omega) \approx A\mathrm{e}^{\lambda(\cos\beta_2 - 1)}, \tag{7.32}$$

where A is a normalization constant chosen so that

$$\frac{(\lambda+1)(\lambda+2)}{8\pi^2}\int \mathrm{d}\,\Omega\, W_{\rho}(\Omega) = 1 \tag{7.33}$$

7.3. Some applications

An interesting application the general formalism presented in equations (7.8)–(7.10) is to the quantum tomography of systems with $SU(n)$ symmetry. This can be done by a formal application of the reconstruction formula of equation (2.21). Several proposal for the implementation of such protocols were discussed in [123–125], where additional delicate situations were analyzed. The reduction to the form equation (2.23) is also possible, although the general expression becomes quite cumbersome.

Another direct application is the problem of $SU(3)$ relative phase POVM, similar to the corresponding $SU(2)$ case considered in section 5.2.1. The POVM for two relative phases α_1 and α_2 can be obtained by integrating the kernel $\hat{w}_{\lambda}^{(-1)}(\Omega)$ over β_1, β_2:

$$\hat{\Delta}(\alpha_1,\alpha_2) = \frac{\dim((\lambda,0))}{4\pi^2}\int_0^{\pi}\sin\beta_1\mathrm{d}\beta_1\int_0^{\pi}\frac{\mathrm{d}\beta_2 1-\cos\beta_2}{4}\sin\beta_2\hat{w}_{\lambda}^{(-1)}(\Omega), \tag{7.34}$$

where the normalization factor is chosen so that

$$\int_0^{2\pi}\mathrm{d}\alpha_1\int_0^{2\pi}\mathrm{d}\alpha_2\,\hat{\Delta}(\alpha_1,\alpha_2) = \mathbb{1}. \tag{7.35}$$

Under shifts generated by the action of the Cartan operators the POVM equation (7.34) is transformed to

Table 2. Elements of the $su(3)$ and their symbols.

Operator $\hat{A}$	$W_A^{(0)}(\Omega)$
$\hat{h}_1$	$=\sqrt{2\lambda(\lambda+3)}\,D^{(1,1)}_{(111)0;(111)0}(\Omega)$
	$=\sqrt{\frac{\lambda(\lambda+3)}{8}}\,(1+3\cos(\beta_2))$
$\hat{h}_2$	$=-\sqrt{\frac{2\lambda(\lambda+3)}{3}}\,D^{(1,1)}_{(111)1;(111)0}(\Omega)$
	$=\sqrt{\frac{\lambda(\lambda+3)}{2}}\,\cos\beta_1\sin^2(\frac{1}{2}\beta_2)$
$\hat{C}_{12}+\hat{C}_{21}$	$=\sqrt{\frac{2\lambda(\lambda+3)}{3}}\,(-D^{(1,1)}_{(201)1/2;(111)0}(\Omega)+D^{(1,1)}_{(021)1/2;(111)0}(\Omega))$
	$=\sqrt{\lambda(\lambda+3)}\,\cos(\alpha_2)\cos(\frac{1}{2}\beta_1)\sin(\beta_2)$
$-\mathrm{i}(\hat{C}_{12}-\hat{C}_{21})$	$=\mathrm{i}\sqrt{\frac{2\lambda(\lambda+3)}{3}}\,(D^{(1,1)}_{(201)1/2;(111)0}(\Omega)+D^{(1,1)}_{(021)1/2;(111)0}(\Omega))$
	$=\sqrt{\lambda(\lambda+3)}\,\sin(\alpha_2)\cos\left(\frac{1}{2}\beta_1\right)\sin(\beta_2)$
$\hat{C}_{13}+\hat{C}_{31}$	$=\sqrt{\lambda(\lambda+3)}\,\cos(\alpha_1+\alpha_2)\sin(\frac{1}{2}\beta_1)\sin(\beta_2)\,,$
$-\mathrm{i}(\hat{C}_{13}-\hat{C}_{31})$	$=-\sqrt{\lambda(\lambda+3)}\,\sin(\alpha_1+\alpha_2)\sin(\frac{1}{2}\beta_1)\sin(\beta_2)\,,$

$$\mathrm{e}^{-\mathrm{i}\gamma_1\hat{h}_1}\mathrm{e}^{-\mathrm{i}\gamma_2\hat{h}_2}\,\hat{\Delta}(\alpha_1,\alpha_2)\mathrm{e}^{\mathrm{i}\gamma_1\hat{h}_1}\mathrm{e}^{\mathrm{i}\gamma_2\hat{h}_2}=\hat{\Delta}(\alpha_1+2\gamma_2,\alpha_2+3\gamma_1-\gamma_2), \tag{7.36}$$

as expected [120].

In the particular case of a single SU(3) system, corresponding to irrep (1,0) the kernel has the form

$$\hat{w}^{(-1)}_{(1,0)}(\Omega)=\begin{pmatrix} \cos^2\left(\frac{\beta_2}{2}\right) & \frac{1}{2}\mathrm{e}^{-\mathrm{i}\alpha_2}\sin(\beta_2)\cos\left(\frac{\beta_1}{2}\right) & \frac{1}{2}\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\sin(\beta_2) \\ \frac{1}{2}\mathrm{e}^{\mathrm{i}\alpha_2}\sin(\beta_2)\cos\left(\frac{\beta_1}{2}\right) & \sin^2\left(\frac{\beta_2}{2}\right)\cos^2\left(\frac{\beta_1}{2}\right) & \frac{1}{2}\mathrm{e}^{-\mathrm{i}\alpha_1}\sin(\beta_1)\sin^2\left(\frac{\beta_2}{2}\right) \\ \frac{1}{2}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\sin(\beta_2) & \frac{1}{2}\mathrm{e}^{\mathrm{i}\alpha_1}\sin(\beta_1)\sin^2\left(\frac{\beta_2}{2}\right) & \sin^2\left(\frac{\beta_1}{2}\right)\sin^2\left(\frac{\beta_2}{2}\right) \end{pmatrix}, \tag{7.37}$$

so that, following equation (7.34), a single-particle 'phase distribution operator' becomes

$$\hat{\Delta}(\alpha_1,\alpha_2)=\frac{1}{16\pi}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\alpha_2} & \mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)} \\ \mathrm{e}^{\mathrm{i}\alpha_2} & 0 & \mathrm{e}^{-\mathrm{i}\alpha_1} \\ \mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)} & \mathrm{e}^{\mathrm{i}\alpha_1} & 0 \end{pmatrix}. \tag{7.38}$$

With the identifications $\alpha_2\leftrightarrow(2\varphi_1-\varphi_2)$, $-\alpha_1\leftrightarrow\varphi_1+\varphi_2$, the off-diagonal relative phases coincide with the polar part of a coherent state realization of $su(3)$ on the torus constructed in [121]. For an arbitrary (coherent) state of the form

$$|\Omega_0\rangle=\begin{pmatrix} \cos\left(\frac{B_2}{2}\right) \\ \mathrm{e}^{\mathrm{i}A_2}\sin\left(\frac{B_2}{2}\right)\cos\left(\frac{B_1}{2}\right) \\ \mathrm{e}^{\mathrm{i}(A_1+A_2)}\sin\left(\frac{B_1}{2}\right)\sin\left(\frac{B_2}{2}\right) \end{pmatrix} \tag{7.39}$$

this leads to the following phase distribution function $P(\alpha_1,\alpha_2)=\mathrm{Tr}(\hat{\Delta}(\alpha_1,\alpha_2)\hat{\rho})$

$$P(\alpha_1,\alpha_2) = \frac{1}{12}\left(\pi\sin(B_1)\sin^2\!\left(\frac{B_2}{2}\right)\cos(A_1-\alpha_1)+\pi\sin(B_2)\cos\!\left(\frac{B_1}{2}\right)\cos(A_2-\alpha_2)\right.$$
$$+\pi\sin\!\left(\frac{B_1}{2}\right)\sin(B_2)\cos(-\alpha_1-\alpha_2+A_1+A_2)+4\sin^2\!\left(\frac{B_1}{2}\right)\sin^2\!\left(\frac{B_2}{2}\right)$$
$$\left.+4\cos^2\!\left(\frac{B_2}{2}\right)+4\sin^2\!\left(\frac{B_2}{2}\right)\cos^2\!\left(\frac{B_1}{2}\right)\right). \tag{7.40}$$

7.4. SU(3) dynamics

7.4.1. Poisson bracket on $SU(3)/U(2)\simeq\mathbb{CP}^2$. Following the general scheme presented in equations (2.28)–(2.32) the semi-classical dynamics of quantum systems with $SU(3)$ symmetry group is developed on $SU(3)/U(2)\simeq\mathbb{CP}^2$, where the local coordinates are determined by the coset parametrization of equation (7.19) as $\Omega=(\alpha_1,\beta_1,\alpha_2,\beta_2)$, and the semi-classical parameter is

$$\varepsilon=\frac{1}{2\sqrt{\lambda(\lambda+3)}}. \tag{7.41}$$

One then obtains the Poisson bracket in the form [82]

$$\{\cdot,\cdot\}=\frac{4}{\sin\beta_1\sin^2\frac{1}{2}\beta_2}(\partial_{\alpha_1}\otimes\partial_{\beta_1}-\partial_{\beta_1}\otimes\partial_{\alpha_1}) \tag{7.42}$$

$$-\frac{2\tan\frac{1}{2}\beta_1}{\sin^2\frac{1}{2}\beta_2}(\partial_{\alpha_2}\otimes\partial_{\beta_1}-\partial_{\beta_1}\otimes\partial_{\alpha_2}) \tag{7.43}$$

$$+\frac{4}{\sin\beta_2}(\partial_{\alpha_2}\otimes\partial_{\beta_2}-\partial_{\beta_2}\otimes\partial_{\alpha_2}). \tag{7.44}$$

Although the $SU(3)$ Wigner functions are not so conveniently illustrated on $\mathbb{CP}^2$, the most important features of the evolution can be captured by analysis of physically meaningful limits in which some semiclassical trajectories can be obtained either analytically or numerically in the TWA scheme. In addition, additional insights can be gained by plotting projections of the Wigner function on some selected submanifolds [80, 82].

7.4.2. Application: two types of SU(3) squeezing. The simplest dynamical application of the $SU(n)$ phase space approach is the analysis of *squeezing*, i.e. the possibility of reducing quantum fluctuations of given observables in multi-level systems below some commonly accepted threshold. It can be shown [82] that an hierarchy of different types of squeezings can be established, caused by dynamically induced correlations between substructures in the spectrum of the system. Analysis of these effects can be performed completely in the framework of the $SU(n)$ TWA.

The simplest $SU(3)$ Hamiltonians that lead to a deformation of an initial distribution resulting in squeezing are of the form $\hat{H}_1\sim\hat{h}_1^2+$ linear terms and $\hat{H}_2\sim\hat{h}_2^2+$ linear terms, where $\hat{h}_1$

and $\hat{h}_2$ are given in equations (7.15) and (7.16). The the linear terms are added for convenience, to remove rigid rotations of the distribution in the phase space.

The effect produced by these Hamiltonians is substantially different:
a) The Hamiltonian

$$\hat{H} = \hat{h}_2^2 + \left(\frac{2\lambda + 3}{60}\right)\hat{h}_1, \tag{7.45}$$

with symbol

$$W_{H_2} = \frac{\sqrt{(\lambda-1)\lambda(\lambda+3)(\lambda+4)}}{480} \sin^4\left(\frac{1}{2}\beta_2\right) \times [3 + 4\cos\beta_2 + 5\cos(2\beta_2) + 20(1 + 3\cos(2\beta_1))], \tag{7.46}$$

generates evolution of the angles α_1 and α_2

$$\alpha_1(t) = \alpha_1(0) - \sqrt{(\lambda-1)(\lambda+4)}\left[\cos\beta_1 \sin^2\left(\frac{1}{2}\beta_2\right)\right]t, \tag{7.47}$$

$$\alpha_2(t) = \alpha_2(0) - \frac{1}{10}\sqrt{(\lambda-1)(\lambda+4)}\left[1 - \cos\beta_1 \sin^2\left(\frac{1}{2}\beta_2\right)\right]t, \tag{7.48}$$

in a very similar way as the Hamiltonian $\hat{H} \sim \hat{S}_z^2$ leads to the phase ϕ dynamics in case of $SU(2)$ symmetry, as shown in equation (6.4). This is especially noticeable from how the symbol $W_{h_2} \sim \cos\beta_1 \sin^2(\frac{1}{2}\beta_2)$ appears in equations (7.47)–(7.48). Since $\hat{h}_2$ is the Cartan element proportional to the operator S_z of the $su_{23}(2)$ subalgebra equation (7.17), $\hat{h}_2^2$ produces $SU(2)$-like correlations in each $SU_{23}(2)$ subrepresentation contained in the original $(\lambda, 0)$ irrep. As a result, the optimal squeezing time and the maximum achievable squeezing scale with λ practically in the same way as they scale with S in spin systems [82].
b) The Hamiltonian

$$\hat{H} = \hat{h}_1^2 - \left(\frac{2\lambda + 3}{5}\right)\hat{h}_1, \qquad \hat{h}_1 = 2\hat{C}_{11} - \hat{C}_{22} - \hat{C}_{33}. \tag{7.49}$$

with symbol

$$W_H = \frac{9}{40}\sqrt{(\lambda-1)\lambda(\lambda+3)(\lambda+4)}\,(3 + 4\cos\beta_2 + 5\cos(2\beta_2)), \tag{7.50}$$

produces, in contradistinction to equation (7.45), correlations between $SU(2)$ subspaces but no correlations inside the subspaces, as $\hat{h}_1$ acts diagonally on $SU(2)$ states. As a result, the type of squeezing generated by $\hat{H} \propto \hat{h}_1^2$ is qualitatively different from $SU(2)$-type squeezing. This difference is reflected in the scaling behaviours with λ of optimal squeezing times and maximum squeezing. This type of squeezing can be considered as a *pure SU(3) squeezing* [82].

From the phase space perspective it is clear that since W_H depends on β_2 alone, the only coordinate to evolve in time, according to equation (7.44), will be α_2. For an initial state of the form

$$|\zeta\rangle = R_{12}(0, B_2, 0)|(\lambda, 0)\lambda 00\rangle \tag{7.51}$$

with $B_2 = \arccos(-1/5)$ (which defines the location of the minimum of the potential function (7.50)), the trajectory can be obtained analytically as

$$\alpha_2(t) = \alpha_2(0) - \frac{9}{5}\sqrt{(\lambda-1)(\lambda+4)}\,(1+5\cos\beta_2)\,t, \tag{7.52}$$

resulting in a squeezing of the initial distribution.

8. Conclusion

This review is a survey of the axiomatic formulation of phase space quantum mechanics, concentrating on spin-like systems and following Stratonovich. Our aim was to highlight some of the uses of phase space methods in a variety of kinematical and dynamical settings, including situations where the states of the system contain more than one value of spin angular momentum and provide some explicit expressions ready for their immediate applications in particular physical situations.

The formalism for higher symmetry group can be generalized without too much difficulty but technology required for practical calculations—various Clebsch–Gordan and recoupling coefficients, the properties of coset functions etc—are still not well developed.

Because of the inherent breadth of applications, we have by design included many technical results, but only references to source papers where fuller derivations and discussions can be found. We hope that in this way the review can serve a springboard for further investigations in a research area where new results are constantly published.

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Appendix A. An integral form of $\hat{w}_S^{(s)}(0)$

One can show that

$$\hat{w}_S^{(s)}(0) = \int_0^{2\pi} \mathrm{d}\omega \mathrm{e}^{\mathrm{i}\omega\,\hat{S}_z}\kappa^{(s)}(\omega), \tag{A.1}$$

$$\kappa^{(s)}(\omega) = \frac{1}{2\pi}\sum_{L=0}^{2S}(-i)^L\frac{2L+1}{2S+1}\chi_L^S(\omega)\left\langle \begin{matrix} S \\ S \end{matrix}; \begin{matrix} L \\ 0 \end{matrix} \middle| \begin{matrix} S \\ S \end{matrix} \right\rangle^{-s} \tag{A.2}$$

where $\chi_L^S(\omega)$ are the generalized characters of the group [93]

$$\chi_L^S(\omega) = i^L\sum_M \mathrm{e}^{-\mathrm{i}M\omega}\left\langle \begin{matrix} S \\ M \end{matrix}; \begin{matrix} L \\ 0 \end{matrix} \middle| \begin{matrix} S \\ M \end{matrix} \right\rangle. \tag{A.3}$$

One can also show that the function $\kappa^{(0)}(\omega)$ in the limit case of large representation dimensions, $S \gg 1$ takes the following asymptotic form [94] (see also [95]),

$$\kappa^{(0)}(\omega) \to (-1)^S\left[\delta(\omega-\pi) - \frac{\mathrm{i}}{S}\frac{\partial}{\partial\omega}\delta(\omega-\pi)\right], \quad S\to\infty, \tag{A.4}$$

where the limit is understood in a weak sense.

Appendix B. Exact expression for the *SU*(2) star-product

The $SU(2)$ star-product operation can be expressed as

$$\mathbf{L}_{f,g}(\Omega) = N_S \sum_n a_n F_S^{s-1}(\mathcal{L}^2)\,[(\mathbb{S}^{+(n)} F_S^{1-s}(\mathcal{L}^2))_f \otimes (\mathbb{S}^{-(n)} F_S^{1-s}(\mathcal{L}^2))_g]\,, \tag{B.1}$$

where we have used the notation $\hat{A}_f \otimes \hat{B}_g(W_f W_g) := (\hat{A} W_f)(\hat{B} W_g)$, and

$$N_S = \sqrt{2S+1}\,[(2S+1)!\,(2S)!]^{s/2}\,, \tag{B.2}$$

$$a_n = \frac{(-1)^n}{n!\,(2S+n+1)!}. \tag{B.3}$$

Here $\mathcal{L}^2$ is Casimir operator on the sphere given in equation (3.34) such that

$$F_S(\mathcal{L}^2) Y_{L,M}(\theta,\phi) = F_S(L) Y_{L,M}(\theta,\phi), \tag{B.4}$$

with $F_S(L)$ succinctly given by

$$F_S(L) = \sqrt{(2S+L+1)!\,(2S-L)!}\,, \tag{B.5}$$

but expressible as a function of $L(L+1)$. In other words

$$F_S(\mathcal{L}^2) = \sum_{L,M} F_S(L) |L,M\rangle\langle L,M|. \tag{B.6}$$

The symbolic powers $\mathbb{S}^{\pm(n)}$ have been introduced in (B.1) according to

$$\mathbb{S}^{\pm(n)} = \Pi_{k=0}^{n-1}\left(k\cot\theta - \partial_\theta \mp \frac{\mathrm{i}}{\sin\theta}\partial_\phi\right). \tag{B.7}$$

The number of terms in the sum (B.1) is finite because the Wigner function $W_f^{(s)}(\Omega)$ of an operator $\hat{f}$ is the polynomial of finite degree in the $SU(2)$ generators: specifically

$$\mathbb{S}^{\pm(n)} W_f^{(s)}(\Omega) = 0, \quad n > \deg \hat{f}\,, \tag{B.8}$$

where the degree of non-linearity, $\deg \hat{f}$, (in the generators of the $su(2)$ algebra) of an operator $\hat{f}$, is defined by the maximum value of ℓ, such that $f_{\ell k} \neq 0$, with $f_{\ell k}$ given in equation (3.16).

As an example, the star-product for $S = 1/2$ is given by

$$\begin{aligned}
&\mathbf{L}_{f,g}(\Omega) \\
&= 2^{(s+1)/2}\left(\frac{1}{2} F_S^{s-1}(\mathcal{L}^2)\,[(F_S^{1-s}(\mathcal{L}^2))_f \otimes (F_S^{1-s}(\mathcal{L}^2))_g]\right. \\
&\quad \left. -\tfrac{1}{3!} F_S^{s-1}(\mathcal{L}^2)\,[(\mathbb{S}^{+(1)} F_S^{1-s}(\mathcal{L}^2))_f \otimes (\mathbb{S}^{-(1)} F_S^{1-s}(\mathcal{L}^2))_g]\right),
\end{aligned} \tag{B.9}$$

where $\tilde{F}^{-1}(\mathcal{L}^2) W_f^{(s)} = 0$ if $\deg \hat{f} \geqslant 2$.

The $\Lambda_{\pm,0}^{(s)}$ operators for $s = \pm 1$ are given in equation (3.44). For the Wigner function ($s = 0$) they are quite involved,

$$\Lambda_0^{(0)} = \tfrac{1}{4\varepsilon}\cos\theta\,\Phi(\mathcal{L}^2) - \tfrac{\varepsilon}{4}(\cos\theta + 2\sin\theta\,\partial_\theta)\Phi^{-1}(\mathcal{L}^2), \tag{B.10}$$

$$\Lambda_{\pm}^{(0)} = \mathrm{e}^{\pm\mathrm{i}\phi}\frac{\sin\theta}{4\varepsilon}\Phi(\mathcal{L}^2) \pm \tfrac{\varepsilon}{4}\left[2\cos\theta l_{\pm} - \mathrm{e}^{\pm\mathrm{i}\phi}\sin\theta(2l_z \pm 1)\right]\Phi^{-1}(\mathcal{L}^2), \tag{B.11}$$

where the function $\Phi(\mathcal{L}^2)$ is given in equation (3.37) and $\mathcal{L}^2$ is the differential realization of the $SU(2)$ Casimir operator (3.34).

Appendix C. Generalized star-product operation

The generalization of the star-product operation for an arbitrary value of of the ordering parameter s can be found following [49], leading to the following explicit expression:

$$\begin{aligned}\mathbf{L}_{f,g}^{jj_1j_2\,(s)} = &\int_0^{4\pi}\frac{\mathrm{d}\varphi'\mathrm{d}\varphi}{(4\pi)^2}\sum_{n=0}^{\infty} a^n_{\frac{j-\hat{\mathbb{J}}^0\otimes\mathbb{1}+\mathbb{1}\otimes\hat{\mathbb{J}}^0}{2}}\left[\frac{(j_1+1)(j_2+1)}{j+1}\right]^{\frac{1-s}{2}} \\ &\times F_j^{s-1}(\hat{J}^2)(\Gamma(j-\hat{\mathbb{J}}^0+2)\Gamma(j+\hat{\mathbb{J}}^0+2))^{-s/2} \\ &\times\left[((\hat{\mathbb{J}}^+)^n F_{j_1}^{1-s}(\hat{J}^2)\mathrm{e}^{\mathrm{i}(j_2-j+\hat{\mathbb{J}}^0)\varphi'}(\Gamma(j_1-\hat{\mathbb{J}}^0+2)\Gamma(j_1+\hat{\mathbb{J}}^0+2))^{s/2})\right. \\ &\left.\bigotimes((\hat{\mathbb{J}}^-)^n F_{j_2}^{1-s}(\hat{J}^2)\mathrm{e}^{\mathrm{i}(j_1-j-\hat{\mathbb{J}}^0)\varphi}(\Gamma(j_2-\hat{\mathbb{J}}^0+2)\Gamma(j_2+\hat{\mathbb{J}}^0+2))^{s/2})\right],\end{aligned} \tag{C.1}$$

where

$$a_K^n = \frac{(-1)^n}{n!\,\Gamma(2K+n+2)}, \tag{C.2}$$

$$F_j(\hat{J}^2)D^k_{nm}(\Theta) = \sqrt{(j+k+1)!\,(j-k)!}\,D^k_{nm}(\Theta), \tag{C.3}$$

$\Gamma(x)$ is the standard Gamma-function, $\Gamma(x+1) = x\Gamma(x)$,

$$\hat{\mathbb{J}}^{\pm} = \mathrm{i}\mathrm{e}^{\mp\mathrm{i}\psi}\left[\mathrm{i}\frac{\partial}{\partial\theta} \pm \cot\theta\frac{\partial}{\partial\psi} \mp \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right], \quad \hat{\mathbb{J}}^0 = -\mathrm{i}\frac{\partial}{\partial\psi}, \tag{C.4}$$

are the contravariant components of the $su(2)$ algebra generators, and

$$\hat{J}^2 = -\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2} - 2\cos\theta\frac{\partial^2}{\partial\phi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right)\right], \tag{C.5}$$

is the Casimir operator.

Appendix D. Reduced *SU*(3) CG coefficients $\left\langle \begin{matrix}(\lambda,0) \\ \lambda-q\end{matrix};\begin{matrix}(0,\lambda) \\ \sigma-p+q\end{matrix} \middle\| \begin{matrix}(\sigma,\sigma) \\ 2\sigma-p;J\end{matrix}\right\rangle$

We have found that the reduced $SU(3)$ CG needed to construct the tensor operators of equation (7.22) are given by the summations

$$\left\langle \begin{matrix} (\lambda,0) \\ \lambda-q \end{matrix}; \begin{matrix} (0,\lambda) \\ \sigma-p+q \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ 2\sigma-p;J \end{matrix} \right\rangle \times \left\langle \begin{matrix} \frac{1}{2}q \\ \frac{1}{2}q \end{matrix}; \begin{matrix} \frac{1}{2}(\sigma-p+q) \\ J-\frac{1}{2}q \end{matrix} \middle| \begin{matrix} J \\ J \end{matrix} \right\rangle$$

$$\times \frac{\langle(\sigma,\sigma)2\sigma-p;J \parallel T^{p/2} \parallel (\sigma,\sigma)2\sigma;\frac{1}{2}\sigma\rangle}{\sqrt{2J+1}}$$

$$= \sum_{s=s_{\min}}^{s_{\max}} (-1)^{\frac{1}{2}(p+\sigma)-J-s} \left\langle \begin{matrix} \frac{1}{2}\sigma \\ -s+\frac{1}{2}\sigma \end{matrix}; \begin{matrix} p/2 \\ J-\frac{1}{2}\sigma+s \end{matrix} \middle| \begin{matrix} J \\ J \end{matrix} \right\rangle$$

$$\times \sqrt{\frac{p!}{(\frac{1}{2}(p-\sigma)+J+s)!\,(\frac{1}{2}(p+\sigma)-J-s)!}}$$

$$\times \sum_{\nu=\nu_{\min}}^{\nu_{\max}} \frac{(-1)^{\nu}}{\sigma!} \sqrt{\frac{2(\lambda-\sigma)!\,(\sigma+\nu+1)!\,(\lambda-\nu)\,!\,(2\sigma+1)!}{(\lambda-\sigma-\nu)\,!\,\nu!\,(\lambda+\sigma+2)!}}$$

$$\times \left\langle \begin{matrix} \frac{1}{2}\nu \\ \frac{1}{2}\nu \end{matrix}; \begin{matrix} \frac{1}{2}(\sigma+\nu) \\ \frac{1}{2}(\sigma-\nu)-s \end{matrix} \middle| \begin{matrix} \frac{1}{2}\sigma \\ \frac{1}{2}\sigma-s \end{matrix} \right\rangle \begin{pmatrix} \frac{1}{2}(p-\sigma)+J+s \\ q-\nu \end{pmatrix} \sqrt{\frac{(\lambda-\nu)!\,q!}{(\lambda-q)!\,\nu!}}$$

$$\times \sqrt{\frac{(\nu+s)!\,(\lambda+p-q-\sigma)!\,(\sigma-s)!}{(\lambda-\sigma-\nu)!\,(J+\frac{1}{2}(\sigma-p))!\,(q+\frac{1}{2}(\sigma-p)-J)}}, \tag{D.1}$$

where

$$\left\langle \begin{matrix} \frac{1}{2}q \\ \frac{1}{2}q \end{matrix}; \begin{matrix} \frac{1}{2}(\sigma-p+q) \\ J-\frac{1}{2}q \end{matrix} \middle| \begin{matrix} J \\ J \end{matrix} \right\rangle = (-1)^{2J+p-\sigma} \sqrt{\frac{(2J+1)!\,q!}{\left(J-\frac{1}{2}(\sigma-p)\right)!\left(J+q+\frac{1}{2}(\sigma-p)+1\right)!}}, \tag{D.2}$$

$$\langle(\sigma,\sigma)2\sigma-p;J \parallel T^{p/2} \parallel (\sigma,\sigma)2\sigma;\frac{1}{2}\sigma\rangle = (-1)^{\frac{1}{2}(p-2J+\sigma)} \sqrt{\frac{(2J+1)\,p\,!\,\sigma\,!\,(2\sigma+1)!}{\left(-J-\frac{p}{2}+\frac{3\sigma}{2}\right)!\left(J-\frac{p}{2}+\frac{3\sigma}{2}+1\right)!}}, \tag{D.3}$$

$$\left\langle \begin{matrix} \frac{1}{2}\nu \\ \frac{1}{2}\nu \end{matrix}; \begin{matrix} \frac{1}{2}(\sigma+\nu) \\ \frac{1}{2}(\sigma-\nu)-s \end{matrix} \middle| \begin{matrix} \frac{1}{2}\sigma \\ \frac{1}{2}\sigma-s \end{matrix} \right\rangle = \sqrt{\frac{(\sigma+1)\sigma!\,(\nu+s)!}{s!\,(\nu+\sigma+1)!}}, \tag{D.4}$$

and the following bounds hold:

$$\mathrm{Min}[J+\frac{1}{2}(p-\sigma),0] \leqslant q \leqslant \lambda+p-\sigma \tag{D.5}$$

$$s_{\min} = \mathrm{Max}[0, \tfrac{1}{2}(\sigma - p) - J] \quad s_{\max} = \mathrm{Min}[\sigma, \tfrac{1}{2}(p+\sigma) - J] \tag{D.6}$$

$$\nu_{\min} = \mathrm{Max}[0, q + \tfrac{1}{2}(\sigma - p) - J - s] \quad \nu_{\max} = \mathrm{Min}[q, \lambda - \sigma] \tag{D.7}$$

Since the $SU(2)$ CGs $\left\langle \begin{smallmatrix} \frac{1}{2}\sigma \\ -s+\frac{1}{2}\sigma \end{smallmatrix} ; \begin{smallmatrix} p/2 \\ J-\frac{1}{2}\sigma+s \end{smallmatrix} \middle| \begin{smallmatrix} J \\ J \end{smallmatrix} \right\rangle$ are known, and indeed have simple expressions given they contain at least one state of maximal weight, the only unknown left in equation (D.1) is the reduced CG we are looking for.

Further simplification occurs when $p = \sigma$ and $J = 0$. In this case we have

$$\left\langle \begin{matrix} (\lambda,0) \\ \alpha_1 \end{matrix} ; \begin{matrix} (0,\lambda) \\ \lambda-\alpha_1 \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \right\rangle = (-1)^{\lambda-\alpha_1} \sqrt{\frac{2(\sigma+1)\,(\lambda-\sigma)!}{(\lambda-\alpha_1+1)\,(\lambda+\sigma+2)!}} \times \sum_{k=\mathrm{Max}[0,\sigma-\alpha_1]}^{\mathrm{Min}[\sigma,\lambda-\alpha_1]} \frac{(-1)^k (k+\alpha_1)!\,(\lambda+\sigma+1-k-\alpha_1)!}{k!\,(\lambda-k-\alpha_1)!\,(k+\alpha_1-\sigma)!\,(\sigma-k)!} \tag{D.8}$$

Finally, the coefficient $\left\langle \begin{smallmatrix} (\lambda,0) \\ \lambda \end{smallmatrix} ; \begin{smallmatrix} (0,\lambda) \\ 0 \end{smallmatrix} \middle\| \begin{smallmatrix} (\sigma,\sigma) \\ \sigma;0 \end{smallmatrix} \right\rangle$ needed to obtain $F_\sigma^{(s)}$ of equation (7.26) is given by

$$\left\langle \begin{matrix} (\lambda,0) \\ \lambda \end{matrix} ; \begin{matrix} (0,\lambda) \\ 0 \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \right\rangle = \frac{\sqrt{2}\,\lambda!\,(\sigma+1)^{3/2}}{\sqrt{(\lambda+\sigma+2)!\,(\lambda-\sigma)!}} \tag{D.9}$$

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