J. Phys. A: Math. Theor. **46** (2013) 105302 (15pp) doi:10.1088/1751-8113/46/10/105302

Semiclassical approach to squeezing-like transformations in quantum systems with higher symmetries

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Received 8 October 2012, in final form 27 January 2013
Published 21 February 2013
Online at stacks.iop.org/JPhysA/46/105302

Abstract
We provide a coarse but intuitive classification of squeezing in quantum systems with $SU(n)$ symmetries. This classification is based on the non-equivalent paths (classical trajectories) in the corresponding phase-space. The example of $SU(3)$ is studied in details.

PACS numbers: 42.50.Dv, 03.65.Ta, 03.65.Fd

(Some figures may appear in colour only in the online journal)

1. Introduction

Squeezing—by which we mean the reduction of the fluctuations of some observable below a threshold to be described below—is important on account of its deep connections with entanglement and quantum metrology. It is physically understood as reflecting the presence of 'quantum correlations' between basis elements of the Hilbert space appropriate for the description of a quantum system, and has been much discussed for the quantum harmonic oscillator or for quantum spin (or $su(2)$) systems. The objective of this paper is to generalize a definition of squeezing beyond those two examples to quantum systems described by observables in the Lie algebra $su(n)$, and to propose a rough classification of squeezing in system with higher symmetries. Such systems include, for instance: Bose–Einstein condensates for which quantum tunnelling between several finite wells is important; ensembles of many n-level atoms; n-beam splitters, and others.

To establish a threshold for squeezing, we will use coherent states [1]. For a fixed representation, these states are obtained by the action of a global $SU(n)$ group transformation on the highest weight state. This highest weight can be expressed as a product of an appropriate number of copies of the highest weight state of the fundamental representation, and can thus be considered as separable. Any coherent state can therefore also be considered as separable and so cannot as a matter of definition exhibit the kind of quantum correlations we will associate

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with squeezing. (There can exist in the Hilbert space highly correlated states—examples are the so-called Dicke states in spin-like systems; these kinds of correlations however are not the ones that lead to squeezing).

Squeezing in quantum systems with $SU(2)$ symmetry has been extensively studied in [2–9], and reviewed in [10]. A general squeezing criterion for a collection of particles with higher spins has been proposed in [11]. This definition directly generalizes the definition of [12]. Other definitions are possible [2].

Using coherent states, we observe there exists a family of observables, related by subgroup transformations and depending on $2n-3$ continuous parameters, such that the fluctuations of any operator of the set, when evaluated in a coherent state, are isotropic, i.e. are unchanged by the subgroup transformations and do not depend on those parameters. This invariance property allows us to set the threshold for quantum correlations as the fluctuation of specified observables evaluated in a coherent state, and geometrically understand squeezing as a deformation of the probability distribution in the appropriate phase space so it is no longer invariant under the subgroup transformations.

In systems with higher symmetries, correlations leading to squeezing, i.e. to the reduction below the isotropic limit of the fluctuations of the observables for which fluctuations are uniform in the coherent states, can be produced in several non-equivalent ways. A coarse classification can be done on the basis of the type of correlations generated between states: different types of phase-space deformations of quasi probability distributions are related to a labelling scheme for states based on the (recursive) subgroup chain

$$SU(n) \supset SU(n-1) \otimes U(1) \supset SU(n-2) \otimes U(1) \otimes U(1) \ldots . \tag{1}$$

with $SU(2) \supset U(1)$ as the last link.

The properties of phase space are themselves inherited from the coset structure associated with coherent states. For the symmetric (i.e. one-rowed) representations of $SU(n)$, the highest weight is invariant (up to an overall phase) under $SU(n-1) \otimes U(1) \sim U(n-1)$ and the resulting geometry is that of the $SU(n)/U(n-1)$ sphere.

For instance, in spin-like states, states transforms naturally under the group $SU(2)$ and are labelled through the $SU(2) \supset U(1)$ chain using the index m. There is only one class of correlations, based on $U(1)$ invariance: even if the so-called one- and two-axes squeezing transformations are functionally different, both generate correlations between every $U(1)$-invariant subspace (spanned by a single $|jm\rangle$ state) of the entire $SU(2)$ representation.

An $SU(3)$ representation decomposes into a direct sum of $SU(2) \otimes U(1)$ subspaces, which further decomposes to a sum of $U(1) \otimes U(1)$ weight subspaces. In systems with $SU(3)$ symmetry, we identify two types of essentially inequivalent correlations: the first type is due to correlations between coherent states in these different $SU(2) \otimes U(1)$ subspaces; the second is a result of correlations between individual states in the representation, labelled by the complete $SU(3) \supset SU(2) \otimes U(1) \supset U(1) \otimes U(1)$ chain.

Because squeezing is well described in the semiclassical limit as a deformation of an initial phase space distribution, it is appropriately convenient to describe squeezing in terms of the geometrical picture provided by phase space dynamics. In this paper, we will study the evolution of an initially coherent $SU(3)$ state generated by a Hamiltonian nonlinear in $su(3)$ generators. We will illustrate how inequivalent Hamiltonians produce different types of correlations. Using semiclassical methods for $SU(n)$ systems [13–15], we will relate the deformation of the Wigner function associated with a suitable initial state to specific types of correlations.

In systems with $SU(n)$ symmetries, the ideas sketched above can be obviously generalized: there will be a hierarchy of squeezings, due to correlations between $SU(n-1)$,

$SU(n-2),\dots,SU(2)$ coherent states, all the way down to individual states in the representation. These squeezings can all be described geometrically through a deformation of a phase-space quasi-distribution.

2. Permutation-symmetric systems with *SU*(3) symmetry

We will discuss exclusively irreducible representations of $SU(3)$ of the type $(\lambda,0)$. These are also sometimes known as symmetric or 'one-rowed' representation as the Young diagram associated with this representation contains a single row of λ boxes.

2.1. *SU*(3) *coherent states*

A convenient realization of $su(3)$ is obtained by using creation and destruction operators for harmonic oscillator states. Starting with nine $\hat{C}_{ij}=a_i^\dagger a_j$, $i,j=1,2,3$, we have

$$[\hat{C}_{ij},\hat{C}_{k\ell}]=\hat{C}_{i\ell}\delta_{jk}-\hat{C}_{kj}\delta_{i\ell}. \tag{2}$$

The algebra $su(3)$ is spanned by the six ladder operators $\hat{C}_{ij}$, $i\neq j$ and two diagonal Cartan elements, which we take as

$$\hat{h}_1=2\hat{C}_{11}-\hat{C}_{22}-\hat{C}_{33},\qquad \hat{h}_2=\tfrac{1}{2}(\hat{C}_{22}-\hat{C}_{33}). \tag{3}$$

The operators $\hat{C}_{ij}$ act in the standard way on the three-dimensional oscillator kets $|\nu_1\nu_2\nu_3\rangle$.

The set $\{|\nu_1\nu_2\nu_3\rangle,\nu_1+\nu_2+\nu_3=\lambda\}$ is a basis for the irrep $(\lambda,0)$ of dimension $\frac{1}{2}(\lambda+1)(\lambda+2)$. The highest weight state of the irrep is $|\lambda00\rangle$.

Elements in the $SU(3)$ groups are parametrized, following [16], as

$$\begin{aligned}R(\tilde{\omega})&=R_{23}(\alpha_1,\beta_1,-\alpha_1)R_{12}(\alpha_2,\beta_2,-\alpha_2)T(\alpha_3,\beta_3,\gamma_1,\gamma_2)\\ T(\alpha_3,\beta_3,\gamma_1,\gamma_2)&\equiv R_{23}(\alpha_3,\beta_3,-\alpha_3)\,\mathrm{e}^{-\mathrm{i}\gamma_1(2\nu_1-\nu_2-\nu_3)}\,\mathrm{e}^{-\mathrm{i}\gamma_2(\nu_2-\nu_3)/2},\end{aligned} \tag{4}$$

where $\tilde{\omega}\equiv(\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\gamma_1,\gamma_2)$ and $R_{ij}(\eta,\theta,\varphi)$ is a transformation in the $SU_{ij}(2)$ subgroup $(i\neq j)$. This subgroup is obtained by exponentiation of elements in $su_{ij}(2)$ subalgebra, spanned by

$$su_{ij}(2)=\langle\hat{C}_{ij},\hat{C}_{ji},\tfrac{1}{2}[\hat{C}_{ij},\hat{C}_{ji}]\rangle. \tag{5}$$

In the fundamental 3×3 representations of $SU(3)$, R_{ij} is a block matrix transforming only lines i and j of basis vectors.

The highest weight state for the irrep $(\lambda,0)$ is $|\lambda00\rangle$. It is stable under $T(\alpha_3,\beta_3,\gamma_1,\gamma_2)$. The set of such transformations generates a $U_{23}(2)$ subgroup which we write as $\mathcal{H}$. Coherent states are labelled by points on $SU(3)/U_{23}(2)\sim S^4$. We use $\omega=(\alpha_1,\beta_1,\alpha_2,\beta_2)$ as coordinates on S^4. Thus, the $SU(3)$ coherent state

$$\begin{aligned}|\omega\rangle=D(\omega)|\lambda00\rangle&\equiv R_{23}(\alpha_1,\beta_1,-\alpha_1)R_{12}(\alpha_2,\beta_2,-\alpha_2)|\lambda00\rangle\\ &=R_{23}(\omega_1)R_{12}(\omega_2)|\lambda00\rangle.\end{aligned} \tag{6}$$

The highest weight state $|\lambda00\rangle$ can be expressed as the tensor product of λ copies of the highest weight state $|100\rangle$ for a 'single qutrit':

$$|\lambda00\rangle=|100\rangle_1\otimes|100\rangle_2\otimes\cdots\otimes|100\rangle_\lambda. \tag{7}$$

As a result, $|\omega\rangle$ can also be expressed as a product of λ one-qutrit states

$$|\omega\rangle\propto|\omega\rangle_1\otimes|\omega\rangle_2\otimes\cdots\otimes|\omega\rangle_\lambda, \tag{8}$$

$$|\omega\rangle_a=\cos\tfrac{1}{2}\beta_2|100\rangle_a+\mathrm{e}^{\mathrm{i}\alpha_2}\cos\tfrac{1}{2}\beta_1\sin\tfrac{1}{2}\beta_2|010\rangle_a+\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\sin\tfrac{1}{2}\beta_1\sin\tfrac{1}{2}\beta_2|001\rangle_a. \tag{9}$$

One verifies without difficulty that the variance of the observable

$$\mathcal{K}(\alpha_3,\beta_3,\gamma_1,\gamma_2)\equiv T(\alpha_3,\beta_3,\gamma_1,\gamma_2)(\hat{C}_{13}+\hat{C}_{31})T^{-1}(\alpha_3,\beta_3,\gamma_1,\gamma_2), \tag{10}$$

when evaluated using the highest weight state $|\lambda 00\rangle$, is independent of the angles $(\alpha_3, \beta_3, \gamma_1, \gamma_2)$ and equal to λ. Moreover, $\mathcal{K}(\alpha_3, \beta_3, \gamma_1, \gamma_2)$ actually depends only on the combination $\chi \equiv 6\gamma_1 + \gamma_2$ so we will henceforth write $\mathcal{K}(\alpha_3, \beta_3, \chi)$.

It follows from this definition that the variance of the shifted observable

$$\mathcal{K}(\omega; \alpha_3, \beta_3, \chi) \equiv D(\omega)\mathcal{K}(\alpha_3, \beta_3, \chi)D^{-1}(\omega), \tag{11}$$

when evaluated in the coherent state $D(\omega)|\lambda 00\rangle$, is also uniform and equal to λ. We use $(\Delta\mathcal{K}(\omega; \alpha_3, \beta_3, \chi))^2 = \lambda$ as our squeezing threshold and define an $su(3)$ state $|\psi\rangle$ as squeezed if there is an observable of the form $\mathcal{K}(\omega; \tilde{\alpha}_3, \tilde{\beta}_3, \tilde{\chi})$ for which

$$(\Delta\mathcal{K}(\omega; \tilde{\alpha}_3, \tilde{\beta}_3, \tilde{\chi}))^2 < \lambda \tag{12}$$

when evaluated in $|\psi\rangle$.

2.2. *Two types of nonlinear squeezing transformations*

A simple way to produce correlations is to consider evolutions generated by Hamiltonians that are nonlinear functions in the Cartan elements.

2.2.1. *Pure SU(3) correlations.*

Start with

$$\begin{aligned} |\omega\rangle &= R_{23}(\omega_1)[R_{12}(\omega_2)|\lambda 00\rangle] \\ &= R_{23}(\omega_1)\left[\sum_p |\lambda - p, p, 0\rangle D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda - p, \frac{1}{2}\lambda}(\omega_2)\right] \end{aligned} \tag{13}$$

$$= \sum_p [R_{23}(\omega_1)|\lambda - p, p, 0\rangle] D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda - p, \frac{1}{2}\lambda}(\omega_2) \tag{14}$$

with $D^{\ell}_{mm'}(\omega_2)$ the standard $SU(2)$ Wigner D-function. Note that states of the form $|\lambda - p, p, 0\rangle$ are $SU_{23}(2)$ highest weight for the irrep of dimension $p + 1$.

Thus, the states $R_{23}(\omega_1)|\lambda - p, p, 0\rangle$ are $SU_{23}(2)$ coherent states for each different p

$$|\omega_1; p\rangle \equiv R_{23}(\omega_1)|\lambda - p, p, 0\rangle = \sum_q D^{\frac{1}{2}p}_{\frac{1}{2}(p-2q), \frac{1}{2}p}(\omega_1)|\lambda - p, p - q, q\rangle. \tag{15}$$

With this we can expand the $SU(3)$ coherent state (6) on the basis of the $SU_{23}(2)$ coherent states:

$$|\omega\rangle = \sum_{p=0}^{\lambda} |\omega_1; p\rangle\, D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda - p, \frac{1}{2}\lambda}(\omega_2). \tag{16}$$

The $su(3)$ Cartan element $\hat{h}_1$ commutes with the $SU(2)_{23}$ transformations generating the coherent state $|\omega\rangle$. Thus, a unitary operator of the general form

$$U_1(\xi) = \exp(-\mathrm{i}\xi H_1(\hat{h}_1)), \tag{17}$$

where $H_1(\hat{h}_1)$ is an arbitrary function of $\hat{h}_1$, will produce

$$U_1(\xi)|\omega\rangle = \sum_p [R_{23}(\omega_1)U_1(\xi)|\lambda - p, p, 0\rangle] D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda - p, \frac{1}{2}\lambda}(\omega_2) \tag{18}$$

$$= \sum_p [\mathrm{e}^{-\mathrm{i}\xi H_1(2\lambda - 3p)}|\omega_1; p\rangle] D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda - p, \frac{1}{2}\lambda}(\omega_2) \tag{19}$$

where $H_1(2\lambda - 3p)$ depends only on the $SU(2)_{23}$ index p (and the $SU(3)$ index λ of course).

Equation (19) has a transparent meaning: it is a superposition of $SU(2)$ Dicke-like coherent states with a nonlinear phase dependent on each $SU_{23}(2)$ coherent state. This phase is at the origin of the 'squeezing', i.e. appearance of specific quantum correlation between the $|\omega_1; p\rangle$ coherent states of equation (15). We will call such correlations 'pure' $SU(3)$ squeezing. As it will be shown later, the corresponding Wigner function is deformed in a specific way, and appropriately chosen observables become 'squeezed', i.e. their initially isotropic fluctuations of the initial coherent state decrease below the standard limit λ as a result of the correlations introduced by the nonlinear phase.

2.2.2. $SU(2)$ correlations in $SU(3)$ states. The $su(3)$ algebra contains a second Cartan element, $\hat{h}_2$, which does not commute with the $SU_{23}(2)$ transformation that generates $|\omega\rangle$.

Thus, with $H_2(\hat{h}_2)$ some nonlinear function in $\hat{h}_2$, we now consider

$$U_2(\xi) = \exp(-\mathrm{i}\xi H_2(\hat{h}_2)). \tag{20}$$

Using again the notation of equation (15), we see that

$$U_2(\xi)|\omega\rangle = \sum_p [\exp(-\mathrm{i}\xi H_2(p-2q))|\omega_1; p\rangle] D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda-p,\frac{1}{2}\lambda}(\omega_2), \tag{21}$$

where $[\exp(-\mathrm{i}\xi H_2(p-2q))|\omega_1; p\rangle]$ is an $SU(2)$ squeezed state [12]:

$$|\omega_1; \xi; p\rangle = \sum_q \mathrm{e}^{-\mathrm{i}\xi H_2(p-2q)}|\lambda - p, p-q, q\rangle D^{\frac{1}{2}p}_{\frac{1}{2}(p-2q),\frac{1}{2}p}(\omega_1). \tag{22}$$

The transformation $U_2(\xi)$ thus generates $SU(2)$-like squeezed states inside each $SU_{23}(2)$ subspace through a nonlinear phase that depends on the basis index q.

In addition, $U_2(\xi)$ will in general produce correlations between $SU_{23}(2)$ subspaces since the index p labelling $SU_{23}(2)$ subspaces varies in a nonlinear way between the $SU_{23}(2)$ subspaces. This last type of correlation strongly depends on amplitudes $D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda-p,\frac{1}{2}\lambda}(\omega_2)$ of the various $SU_{23}(2)$ coherent state in the decomposition of the initial state, i.e. in the geometrical 'position' of the initial $SU(3)$ coherent state on S^4 sphere.

Different patches of the initial quasidistribution will evolve at different rates so that, over times of order $\xi t \approx 1$ ($\hbar = 1$ throughout), we can expect complicated self-interference effects to occur when some patches of the initial quasidistribution catch up with others. On the other hand, for time short enough to neglect this self-interference, the dynamics generated by diagonal Hamiltonians is much simpler and leads only to evolutions of the phase angles $\alpha_{1,2}$, i.e. the 'amplitude' angles $\beta_{1,2}$ are not affected. In particular, the Hamiltonian $H_1(\hat{h}_1)$, which commutes with R_{23}-type transformations, cannot change the parameters of $SU_{23}(2)$ coherent states and produces evolution only of the angle α_2. On the other hand, both α_1 and α_2 evolve with time when the dynamics of the system is governed by $H_2(\hat{h}_2)$.

2.3. $SU(3)$ phase space representation

Following the prescription of [15], we associate to an operator $\hat{X}$ a phase-space symbol

$$\hat{X} \mapsto W_X(\Omega) = \mathrm{tr}(\hat{w}(\Omega)\hat{X}) \tag{23}$$

using the quantization kernel $\hat{w}(\Omega)$:

$$\hat{w}(\Omega) \equiv \Lambda(\Omega)\hat{P}\Lambda^\dagger(\Omega), \qquad \Omega \in SU(3)/U(2). \tag{24}$$

Here, $\Lambda(\Omega)$ is the matrix representation for the element Ω in the irrep $(\lambda, 0)$. The essential information about the mapping is contained in the $SU_{23}(2)$-invariant operator $\hat{P}$ given explicitly by

$$\hat{P} = \sum_{\sigma=0}^{\lambda} \hat{T}^{\lambda}_{(\sigma,\sigma)(\sigma\sigma\sigma)0} \left(\frac{\dim(\sigma)}{\dim(\lambda)}\right)^{1/2} \tag{25}$$

with $\hat{T}^{\lambda}_{(\sigma\sigma)(\sigma\sigma\sigma)0}$ the zero-weight $(\sigma\sigma\sigma)$, $I = 0$ component of the tensor operator transforming by the $su(3)$ irrep (σ, σ). Notational details are found in [15] or [16]. (Note that equation (25) corrects a misprint in [15]).

The Poisson bracket on S^4 obtained using the parametrization (6) is found to be

$$\{f, g\} = \frac{4}{\sin\beta_1 \sin^2\frac{1}{2}\beta_2}\left(\frac{\partial f}{\partial\alpha_1}\frac{\partial g}{\partial\beta_1} - \frac{\partial g}{\partial\alpha_1}\frac{\partial f}{\partial\beta_1}\right) - \frac{2\tan\frac{1}{2}\beta_1}{\sin^2\frac{1}{2}\beta_2}\left(\frac{\partial f}{\partial\alpha_2}\frac{\partial g}{\partial\beta_1} - \frac{\partial g}{\partial\alpha_2}\frac{\partial f}{\partial\beta_1}\right) + \frac{4}{\sin\beta_2}\left(\frac{\partial f}{\partial\alpha_2}\frac{\partial g}{\partial\beta_2} - \frac{\partial g}{\partial\alpha_2}\frac{\partial f}{\partial\beta_2}\right), \tag{26}$$

where f and g are any two functions on $SU(3)/U(2)$.

The density operator $\hat{\rho}_\omega = |\omega\rangle\langle\omega| = D(\omega)|\lambda 00\rangle\langle\lambda 00|D^{-1}(\omega)$ for the coherent state $|\omega\rangle$ is mapped to the Wigner function $W_{\rho_\omega}(\Omega) = W_\lambda(\omega^{-1}\Omega)$. Here, the argument $\omega^{-1}\Omega$ of the Wigner function is understood as the coset representative of the product of the corresponding group elements. Moreover, $W_\lambda(\Omega)$ is the symbol corresponding to the highest weight $|\lambda 00\rangle$ state

$$W_\lambda(\Omega) = \sum_{\sigma=0}^{\lambda} \tilde{C}^{(\sigma\sigma)(\sigma\sigma\sigma)0}_{\lambda 00;\lambda 00} \sqrt{\frac{2(\sigma+1)^3}{(\lambda+1)(\lambda+2)}}\, D^{(\sigma,\sigma)}_{(\sigma\sigma\sigma)0;(\sigma\sigma\sigma)0}(\Omega), \tag{27}$$

with $\tilde{C}^{(\sigma\sigma)(\sigma\sigma\sigma)0}_{\lambda 00;\lambda 00}$ the matrix element

$$\tilde{C}^{(\sigma\sigma)(\sigma\sigma\sigma)0}_{\lambda 00;\lambda 00} = \langle(\lambda,0)\lambda 00; 0|\, T^{\lambda}_{(\sigma\sigma)(\sigma\sigma\sigma)0}|(\lambda,0)\lambda 00; 0\rangle. \tag{28}$$

An explicit expression for $\tilde{C}^{(\sigma\sigma)(\sigma\sigma\sigma)0}_{\nu_1\nu_2\nu_3;\nu_1\nu_2\nu_3}$ is given in the appendix.

Finally, one can verify that the $SU(3)$ D-function $D^{(\sigma,\sigma)}_{(\sigma\sigma\sigma)0;(\sigma\sigma\sigma)0}(\Omega)$ collapses to

$$D^{(\sigma,\sigma)}_{(\sigma\sigma\sigma)0;(\sigma\sigma\sigma)0}(\Omega) = \left(\frac{P_{\sigma+1}(\cos\beta_2) - P_\sigma(\cos\beta_2)}{(\cos\beta_2 - 1)(\sigma+1)}\right) \tag{29}$$

with P_ℓ a Legendre polynomial of order ℓ so $W_\lambda(\Omega)$ actually depends on $\cos\beta_2$ only:

$$W_\lambda(\Omega) = W_\lambda(\beta_2). \tag{30}$$

2.4. *Semiclassical evolution leading to squeezing*

2.4.1. General remarks and some Hamiltonians. For irreps of the type $(\lambda, 0)$, in the semiclassical limit where $\lambda \gg 1$, the short-time dynamics of an initially localized state is well described by the Liouville-type equation for the Wigner function [17–19]:

$$\partial_t W_{\rho_\omega}(\Omega) = \varepsilon\{W_{\rho_\omega}(\Omega), W_H(\Omega)\}_P + O(\varepsilon^3), \tag{31}$$

where $W_H(\Omega)$ is the symbol of the Hamiltonian and $\varepsilon = \frac{1}{2\sqrt{\lambda(\lambda+3)}}$ is the semiclassical parameter. The solution to (31) can be written in general form as

$$W(\Omega|t) = W(\Omega(t)), \tag{32}$$

where $\Omega(t)$ denotes classical trajectories on the classical manifold. Each point of the initial distribution thus evolves along a classical trajectory. In the special case of Hamiltonians linear in the generators of $su(3)$, the motion in time of the initial distribution is simply a rigid translation of the distribution in phase space. When the Hamiltonian is nonlinear in the generators, the distribution is distorted in time in phase space, reflecting the presence of correlations that can be associated with squeezing.

For simplicity we will consider Hamiltonians quadratic in the Cartan elements:

$$H_1(\hat{h}_1) = \hat{h}_1^2 - \frac{2\lambda + 3}{5}\hat{h}_1, \qquad H_2(\hat{h}_2) = \hat{h}_2^2 + \frac{2\lambda + 3}{60}\hat{h}_1. \tag{33}$$

To simplify the analysis and focus on squeezing, terms proportional to $\hat{h}_1$ have been inserted to remove any rigid motion of the distribution.

We will select as initial state a coherent state located over the minimum of the Hamiltonian W_H on the S^4 sphere so as to maximize the applicability of the semiclassical description of the evolution.

2.4.2. Evolution generated by H_1. The symbol for H_1 in equation (33) is given (up to constant factors) by

$$W_{H_1} = \tfrac{9}{40}\sqrt{(\lambda-1)\lambda(\lambda+3)(\lambda+4)}(4\cos\beta_2 + 5\cos(2\beta_2)). \tag{34}$$

The resulting evolution is easily obtained as

$$\alpha_2(t) = \alpha_2(0) - \tfrac{9}{5}\sqrt{(\lambda-1)(\lambda+4)}(1 + 5\cos\beta_2)t. \tag{35}$$

We specify the initial state by choosing a coherent state with coordinates (A_1, B_1, A_2, B_2) so it 'sits' above the minimum in the Hamiltonian, i.e. is located at $A_1 = B_1 = A_2 = 0$ and $B_2 = \arccos(-1/5)$.

This Hamiltonian produces an evolution only in the angle α_2. This angle enters in amplitude of each $SU_{23}(2)$ coherent state in the expansion of the full $SU(3)$ coherent state, as per equation (16).

If we parametrize the coset representative of $\omega^{-1}\Omega$ by $(\bar{\alpha}_1, \bar{\beta}_1, \bar{\alpha}_2, \bar{\beta}_2)$, we find

$$|\omega\rangle = R_{12}(0, B_2, 0)|\lambda 00\rangle, \tag{36}$$

$$\begin{aligned}\cos\bar{\beta}_2 = {} & 2\cos^2(\tfrac{1}{2}B_2)\cos^2(\tfrac{1}{2}\beta_2) + 2\cos^2(\tfrac{1}{2}\beta_1)\sin^2(\tfrac{1}{2}B_2)\sin^2(\tfrac{1}{2}\beta_2) \\ & + \cos(\alpha_2)\cos(\tfrac{1}{2}\beta_1)\sin(\beta_2)\sin(B_2) - 1\end{aligned} \tag{37}$$

$$W_{\rho_\omega}(\Omega) = W_\lambda(\bar{\beta}_2). \tag{38}$$

Thus, the time evolution of the system is obtained by the replacement $\alpha_2 \to \alpha_2(t)$ in the argument $\cos\bar{\beta}_2$ of the Wigner function:

$$W_\lambda(\bar{\beta}_2|t) = W_\lambda(\bar{\beta}_2(t)), \tag{39}$$

with $\alpha_2(t)$ given in equation (35). All the other variables are constant in time.

Figure 1 shows the deformation resulting from time evolution under H_1 in equation (33) of the initial state of equation (36) with $\lambda = 10$. The Wigner function is given in the plane $\alpha_1 = \beta_1 = 0$, for $t = 0.026$. This value of t will be seen later to be the one for which the state exhibits maximum squeezing. On the left is the Wigner function evolved using quantum mechanical evolution, while on the right we show the Wigner function evolved using the semi-classical Liouville-like evolution given in equation (31). One notices in the quantum mechanical calculation some ripples and small regions where the distribution is negative; neither of these features can be reproduced semi-classically.

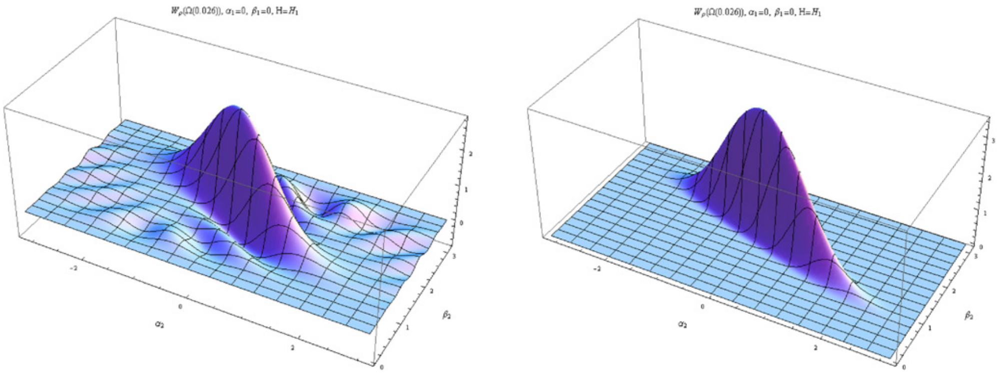

Figure 1. The Wigner function for the initial (coherent) state of equation (36) with $\lambda = 10$, time-evolved under H_1 in equation (33) to the optimal squeezing time $t = 0.026$, using quantum mechanical evolution (left) and the semi-classical evolution (right).

2.4.3. Evolution generated by H_2. The symbol for H_2 is (up to constant factors)

$$W_{H_2} = \frac{1}{480}\sqrt{(\lambda-1)\lambda(\lambda+3)(\lambda+4)} \\ \times\left[3 + 4\cos\beta_2 + 5\cos(2\beta_2) + 20(1+3\cos(2\beta_1))\sin^4\left(\tfrac{1}{2}\beta_2\right)\right] \tag{40}$$

with resulting classical trajectories

$$\alpha_1(t) = \alpha_1(0) - \sqrt{(\lambda-1)(\lambda+4)}\cos\beta_1\sin^2\left(\frac{\beta_2}{2}\right)t, \tag{41}$$

$$\alpha_2(t) = \alpha_2(0) - \tfrac{1}{20}\sqrt{(\lambda-1)(\lambda+4)}(2 + 5\cos\beta_1(\cos\beta_2 - 1))t. \tag{42}$$

This Hamiltonian generates a two-dimensional dynamics. The evolution of α_1 results in the deformation of individual $SU_{23}(2)$ coherent states, i.e. to the usual $SU(2)$ squeezing. On the other hand, the evolution of α_2 produces another type of deformation of the initial $SU(3)$ coherent state that corresponds to correlations between $SU_{23}(2)$ coherent states as discussed in section 2.2.2.

The minimum of W_{H_2} occurs at $B_1 = \frac{1}{2}\pi$, $B_2 = \pi$ so we have

$$|\omega\rangle = R_{23}\left(0, \tfrac{1}{2}\pi, 0\right)R_{12}(0, \pi, 0)|\lambda 00\rangle, \tag{43}$$

$$\cos\bar{\beta}_2 = \sin^2\left(\tfrac{1}{2}\beta_2\right) + \cos\alpha_1\sin\beta_1\sin^2\left(\tfrac{1}{2}\beta_2\right) - 1, \tag{44}$$

$$W_{\rho_\omega}(\Omega) = W_\lambda(\bar{\beta}_2). \tag{45}$$

The value of $B_2 = \pi$ collapses the D-functions in equation (16) to a δ so the initial coherent state reduces for this particular value to a single $SU_{23}(2)$ coherent state, and the motion becomes one-dimensional, i.e. only the angle α_1 evolves in time.

As before, the time evolution of the system is obtained by the replacement $\alpha_1 \to \alpha_1(t)$ in the argument $\cos\bar{\beta}_2$ of the Wigner function:

$$W_\lambda(\bar{\beta}_2|t) = W_\lambda(\bar{\beta}_2(t)), \tag{46}$$

with $\alpha_1(t)$ given in equation (41).

Note that $W_\lambda(\bar{\beta}_2(t))$ does not depend on α_2 so this angle evolves without affecting the shape of the distribution. This can be observed on figure 2, where once again we present on the left the quantum mechanical calculation, with ripples and regions of negativity in the quantum mechanical calculations, and on the right the semi-classical calculation. For H_2 the results must be plotted in the $\alpha_2 = \beta_2 = 0$ plane.

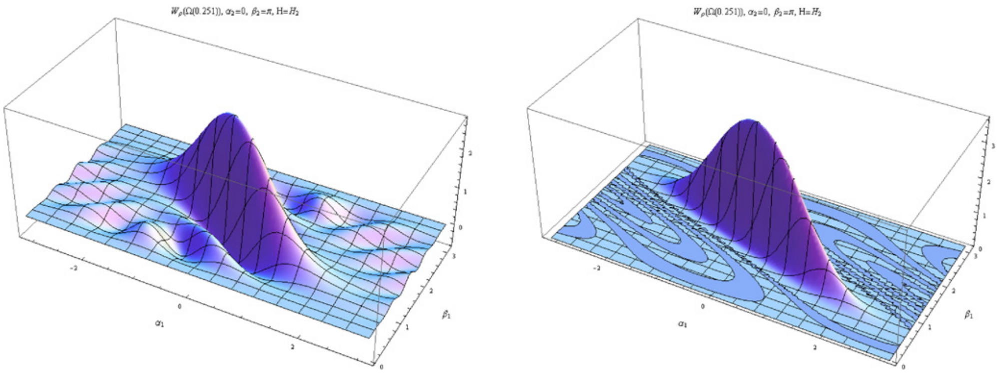

Figure 2. The Wigner function for the initial (coherent) state of equation (43) with $\lambda = 10$, time-evolved under H_2 in equation (33) to the optimal squeezing time $t = 0.251$, using quantum mechanical evolution (left) and the semi-classical evolution (right).

Although qualitatively similar the results obtained using H_1, the evolution under H_2 differ from the evolution under H_1 in some essential manner. The optimal squeezing times are different and will be seen to scale differently with λ. Moreover, because the evolution is in α_1 only, one must examine slides in planes orthogonal to those used for the analysis of the evolution under H_1.

As a final remark, we note that although both examples of evolutions provided in this discussion yield $SU(2)$ correlations inside a single $SU(2)$ subspace, the correlations are necessarily inequivalent because they involve different quantum numbers.

2.5. Squeezing

In section 2.2, we distinguished two types of squeezing transformations in $SU(3)$ system. The first is related to correlations between $SU(2)$ coherent states and the second is related to squeezing of individual $SU(2)$ coherent states.

Even if figures 1 and 2 both display a geometrical squeezing of the initially isotropic $SU(3)$ coherent state, both types of correlations yield different scaling behaviours with λ in time and in the way they distort the initial state.

For pure $SU(3)$ squeezing, the optimal squeezing time is shorter than for $SU(2)$-type squeezing. This is because it is easier to correlate already localized states like $SU(2)$ coherent states than individual basis states. In the case of squeezing generated by H_2, the initial state takes the specialized form of a single $SU(2)$ coherent state. Hence, the squeezing behaviour generated by H_2 for the initial coherent state located at the minimum of $W_{H_2}(\Omega)$ is identical to that generated by S_z^2 in $SU(2)$ systems for an initial state located on the equator of the S^2 sphere. Figure 3 presents a more quantitative comparison of the optimal squeezing as a function of t, generated by H_1 and H_2.

The difference between the two types of squeezing can be assessed more quantitatively by considering the scaling behaviour with λ of location in time of the minimum of the squeezing curve, as well as the location of this minimum on the vertical axis. The scaling behaviour is summarized in table 1, where we have added the values of the parameters for $SU(2)$ so as to make comparison easy.

We see that the scaling behaviours under $H = H_2$ are indeed those of $SU(2)$ while the scaling factors for the evolution under $H = H_1$ are considerably different from the $SU(2)$ values.

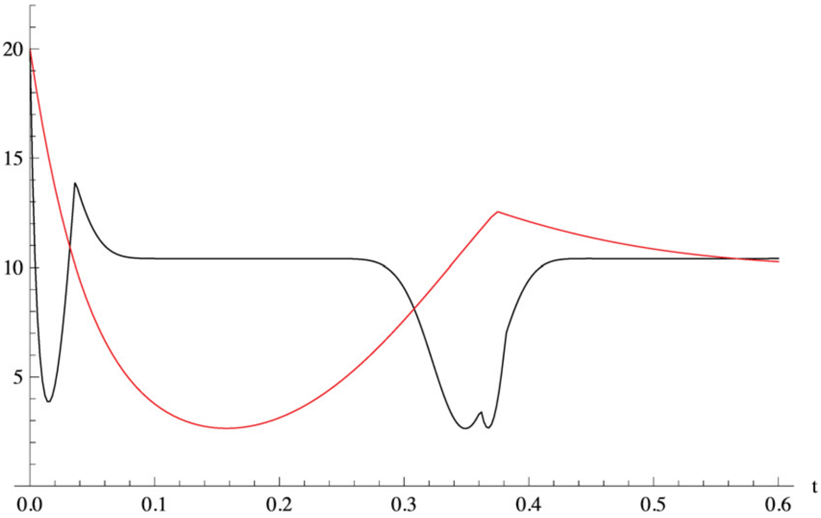

Figure 3. The time-evolution of squeezing for $\lambda = 20$. Black: $\hat{H} = \hat{H}_1$; Red: $\hat{H} = \hat{H}_2$.

Table 1. The scaling behaviour of the optimal time and depth as a function of λ for various Hamiltonians.

$\hat{H}$	$\hat{H}_1$ Pure SU(3)	$\hat{H}_2$ SU(2)	$\hat{S}_z^2$
Optimal t	$\lambda^{-4/5}$	$\lambda^{-2/3}$	$j^{-2/3}$
Max squeezing	$\lambda^{2/3}$	$\lambda^{0.37}$	$j^{0.35}$

3. *SU*(*n*) symmetry

We now extend our discussion to $SU(n)$. Throughout this section we will use the shorthand λ to mean the irrep $(\lambda, 0, \ldots, 0)$ of $SU(n)$.

3.1. General remarks

Again we use a realization in terms of harmonic oscillator creation and destruction operators. With $\hat{C}_{ij} = a_i^\dagger a_j$, the algebra $su(n)$ is spanned by the $n^2 - n$ ladder operators $\hat{C}_{ij}$, $i \neq j = 1, 2, \ldots, n$ and $n - 1$ Cartan elements. For convenience, we choose them as

$$\hat{h}_1 = (n-1)\hat{C}_{11} - \sum_{k=2}^{n} \hat{C}_{kk} = \mathrm{diag}(n-1, -1, \ldots, -1), \tag{47}$$

$$\hat{h}_2 = (n-2)\hat{C}_{22} - \sum_{k=3}^{n} \hat{C}_{kk} = \mathrm{diag}(0, n-2, -1, \ldots, -1), \tag{48}$$

$$\vdots$$

$$\hat{h}_{n-1} = \hat{C}_{n-1,n-1} - \hat{C}_{n,n} = \mathrm{diag}(0, \ldots, 0, 1, -1). \tag{49}$$

These operators act naturally on the set $\{|\nu_1, \ldots, \nu_n\rangle, \nu_1 + \nu_2 + \cdots + \nu_n = \lambda\}$ of n-dimensional harmonic oscillator states. This set is a basis for the irrep λ of $SU(n)$.

$\hat{h}_2$ is invariant under $SU(n-1)$ transformations block diagonal in the last $(n-1) \times (n-1)$ entries. h_3 is invariant under $SU(n-2)$ transformations block diagonal in the last $(n-2) \times (n-2)$ entries and so forth.

The highest weight state $|\lambda 0 \ldots 0\rangle$ is stable under the subgroup of transformation $\mathcal{H} = U(n-1) = SU(n-1) \otimes U(1)$, with the $SU(n-1)$ subgroup acting on modes 2 to n of the oscillator kets. This highest weight state can be constructed by tensoring λ copies of the highest weight of the fundamental $(1, 0, \ldots, 0)$ representation:

$$|\lambda\rangle = |1, 0, \ldots, 0\rangle_1 \otimes |1, 0, \ldots, 0\rangle_2 \otimes \cdots \otimes |1, 0, \ldots, 0\rangle_\lambda. \tag{50}$$

Because $|\lambda\rangle$ is a permutation-symmetric factorized product of 'single particle' states, so is the coherent state $|\omega\rangle$ defined by

$$|\omega\rangle \equiv D(\omega)|\lambda\rangle, \qquad \omega \in SU(n)/U(n-1). \tag{51}$$

As such, $|\omega\rangle$ displays maximal *classical* correlations. We can always find, for a given coherent state, a parametrized family of operators, written as linear combination of generators, for which the fluctuations of any element in the set will be invariant under transformations generated by the stationary subgroup $\mathcal{H}$. Moreover, the fluctuations of this operator reach their minimum possible value, determined by the dimension of the representation space $\mathbb{H}$, when evaluated using coherent states.

Using ω as a shorthand for $(\alpha_1, \beta_1, \ldots, \alpha_{n-1}, \beta_{n-1})$, a convenient realization of the coset elements ω is given in terms of the sequence of subgroup transformations

$$D(\omega) = R_{n-1,n}(\alpha_1, \beta_1, -\alpha_1) \ldots R_{23}(\alpha_2, \beta_2, -\alpha_2) R_{12}(\alpha_{n-1}, \beta_{n-1}, -\alpha_{n-1}), \tag{52}$$

where $R_{ij}(\gamma, \tau, \zeta)$ is a transformation from the $SU_{ij}(2)$ subgroup with algebra spanned by $\hat{C}_{ij}, \hat{C}_{ji}, [\hat{C}_{ij}, \hat{C}_{ji}]$. R_{ij} subgroup transformation only mixes modes i and j.

3.2. Possible types of correlations

Start with

$$|\omega\rangle = R_{n-1,n}(\omega_1) \ldots R_{23}(\omega_{n-2})[R_{12}(\omega_{n-1})|\lambda, 0, \ldots, 0\rangle] \tag{53}$$

$$= \sum_{p_1} [R_{n-1,n}(\omega_1) \ldots R_{23}(\omega_{n-2})|\lambda - p_1, p_1, 0, \ldots, 0\rangle] D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda - p_1, \frac{1}{2}\lambda}(\omega_{n-1}). \tag{54}$$

The states

$$|\omega_1 \omega_2 \ldots \omega_{n-2}; p_1\rangle = R_{n-1,n}(\omega_1) \ldots R_{23}(\omega_{n-2})|\lambda - p_1, p_1, 0, \ldots, 0\rangle \tag{55}$$

are $SU(n-1)_{2,n}$ coherent states for each different p_1. Thus, we can write

$$|\omega\rangle = \sum_{p_1=0}^{\lambda} |\omega_1 \omega_2 \ldots \omega_{n-2}; p_1\rangle D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda - p_1, \frac{1}{2}\lambda}(\omega_{n-1}) \tag{56}$$

as a superposition of $SU(n-1)$ coherent states.

Alternatively, we also have

$$|\omega\rangle = R_{n-1,n}(\alpha_1, \beta_1, -\alpha_1) \ldots R_{34}(\alpha_{n-3}, \beta_{n-3}, -\alpha_{n-3})|\omega_{n-1}; \omega_{n-2}\rangle$$

$$|\omega_{n-1}; \omega_{n-2}\rangle = R_{23}(\omega_{n-2}) R_{12}(\omega_{n-1})|\lambda\rangle$$

$$= \sum_{p_1 p_2} D^{\frac{1}{2}p_1}_{\frac{1}{2}p_1 - p_2, \frac{1}{2}p_1}(\omega_{n-2}) D^{\lambda/2}_{\frac{1}{2}\lambda - p_1, \frac{1}{2}\lambda}(\omega_{n-1})|\lambda - p_1, p_1 - p_2, p_2, \ldots, 0\rangle \tag{57}$$

with $|\omega_{n-1}; \omega_{n-2}\rangle$ an $SU(3)$ coherent state, and so forth.

We may now envisage generating correlations with evolutions of the form

$$U_k(\xi_k) = \exp(-\mathrm{i}\xi_k H_k(\hat{h}_k)) \tag{58}$$

for some polynomial H_k at least quadratic in $\hat{h}_k$.

The operator $\hat{h}_1$ commutes with all $SU(2)$ transformations of the type $R_{k-1,k}(\omega)$ except R_{12}. Thus, for Hamiltonians of the type $H(\hat{h}_1)$ polynomial in $\hat{h}_1$, one will produce 'pure' $SU(n)$ correlations, generalizing equation (19) to

$$U_1(\xi_1)|\omega\rangle = \sum_{p_1}[R_{n-1,n}(\omega_{n-1,n})\ldots R_{23}(\omega_{23})U_1(\xi_1)|\lambda - p_1, p_1, 0, \ldots, 0\rangle]D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda-p,\frac{1}{2}\lambda}(\omega_1), \tag{59}$$

$$= \sum_{p_1}[\mathrm{e}^{-\mathrm{i}\xi_1 H_1((n-1)\lambda - np)}|\omega_1\omega_2\ldots\omega_{n-2}; p_1\rangle]D^{\frac{1}{2}\lambda}_{\frac{1}{2}\lambda-p,\frac{1}{2}\lambda}(\omega_1), \tag{60}$$

with $|\omega_1\omega_2\ldots\omega_{n-2}; p_1\rangle$ given in equation (55) .

This kind of squeezing can be detected by using a family of observables of the form $\mathcal{K}(\eta) = T(\eta)(\hat{C}_{1n} + \hat{C}_{n1})T^{-1}(\eta)$ with $T(\eta) \in \mathcal{H} = U(n-1)$. When evaluated using the highest weight state $|\lambda\rangle$, the fluctuations of $\mathcal{K}(\eta)$ are independent of the parameters η and equal λ. Thus, the shifted operator

$$\mathcal{K}(\omega; \eta) = D(\omega)\mathcal{K}(\eta)D^{-1}(\omega) \tag{61}$$

will have fluctuations independent of η when evaluated in the coherent state $D(\omega)|\lambda\rangle$. A state $|\psi\rangle$ will be *purely $SU(n)$-squeezed* if there is some η^* for which $(\Delta\mathcal{K}(\omega; \eta^*))^2 < \lambda$ when evaluated using $|\psi\rangle$.

For Hamiltonians polynomial in $\hat{h}_2$, one will produce $SU(n-1)$-type correlations, and so recursively various types of squeezing can be achieved. In general, two Hamiltonians $\hat{H}_i$ and $\hat{H}_j$, invariant under different transformations, will induce inequivalent motions in the $SU(n)/U(n-1)$ phase space, isomorphic to the sphere $S^{2(n-1)}$, and thus inequivalent deformations of the initial distribution.

4. Conclusion

In this paper we have shown that correlations generated by Hamiltonians with different invariance properties under subgroup transformations have an essentially different nature. The differences clearly appear in the scaling behaviour for optimal squeezing times. For systems with $SU(n)$ symmetries, the fastest squeezing will occur when correlating $SU(n-1)$ coherent states.

Our idea is that, in systems with higher symmetries, correlations between different types of states are possible. We have shown how the symmetry of the Hamiltonian is related to the type of correlations it generates. These differences are due to impossibility of transforming the Hamiltonians from one to the other using unitary transformations.

In the case of $SU(n)$ systems, we can identify different inequivalent deformations of phase-space distributions with specific type of correlations between states labelled using the canonical subgroup chain of equation (1), all the way from correlating coherent states of $SU(n-1)$ to correlating individual states labelled by all links in the subgroup chain. In the examples of this paper, using our parametrization of the S^4 sphere, different deformations generated by $\hat{h}_1^2$ and $\hat{h}_2^2$ (up to rigid translations), identified with different types of possible correlations, correspond to obviously distinct trajectories along different azimuthal directions only.

Although Hamiltonians invariant under different subgroups in principle produce different types of trajectories in phase-space, the number of azimuthal angles needed in our parametrization to described the classical orbits generated by such Hamiltonians strongly depends on the position of the initial state. In the explicit example discussed in section 2.4.3, the orbits should involve both azimuthal angles but reduce to one-dimensional motion because of the choice of initial state.

In order to observe the effect of two-dimensional evolution in $SU(3)$ systems we consider the dynamics generated by the sum of Hamiltonians $H_1 + H_2$. The phase space symbol of the Hamiltonian is just $W_{H_1+H_2}$ and the classical trajectories are

$$\alpha_1(t) = \alpha_1 - \sqrt{(\lambda-1)(\lambda+4)}\cos\beta_1 \sin^2\left(\tfrac{1}{2}\beta_2\right) t, \tag{62}$$

$$\alpha_2(t) = \alpha_2 - \tfrac{1}{20}\sqrt{(\lambda-1)(\lambda+4)}[38 - 5\cos\beta_1 + 5\cos\beta_2(36+\cos\beta_1)]t. \tag{63}$$

The minimum of $W_{H_1+H_2}$ occurs at $B_1 = \frac{1}{2}\pi$ and $B_2 = \arccos(-19/90\ )$ therefore

$$|\omega\rangle = R_{23}(0, B_1, 0)R_{12}(0, B_2, 0)|\lambda 00\rangle, \tag{64}$$

$$\begin{aligned}\cos\bar{\beta}_2 = \tfrac{1}{4}\big[&2\sqrt{2}\left(\cos\alpha_2\cos\tfrac{1}{2}\beta_1 + \cos(\alpha_1+\alpha_2)\sin\tfrac{1}{2}\beta_1\right)\sin\beta_2\sin B_2 \\ &+ 4\cos\alpha_1\sin\beta_1\sin^2\left(\tfrac{1}{2}\beta_2\right)\sin^2\left(\tfrac{1}{2}B_2\right) + \cos\beta_2 + \cos B_2 + 3\cos\beta_2\cos B_2 - 1\big]\end{aligned} \tag{65}$$

$$W_{\rho_\omega}(\Omega) = W_\lambda(\bar{\beta}_2). \tag{66}$$

Here the time-evolved Wigner function is obtained by the replacements $\alpha_1 \to \alpha_1(t)$ and $\alpha_2 \to \alpha_2(t)$ in the relation for $\cos\bar{\beta}_2$. In this case the initial state is a true $SU(3)$ coherent state whereas in the H_1 and H_2 cases the initial state is an $SU(2)$ coherent state.

We have exemplified our analysis by selecting only diagonal Hamiltonians; if more general Hamiltonians are used, both types of squeezing will simultaneously occur and the description of the trajectories in terms of the angles on S^4 is more complicated, but of course our conclusions are unaltered.

Acknowledgments

The work of ABK is partially supported by the grant 106525 of CONACyT (Mexico). The work of HDG and HTD is partially supported by NSERC of Canada.

Appendix. An closed form expression for $\tilde{C}^{(\sigma\sigma)(\sigma\sigma\sigma)0}_{\nu_1\nu_2\nu_3;\nu_1\nu_2\nu_3}$

Here we provide an explicit expression for the coefficients $\tilde{C}^{(\sigma\sigma)(\sigma\sigma\sigma)0}_{\nu_1\nu_2\nu_3;\nu_1\nu_2\nu_3}$ of equation (28). This coefficient is, up to a phase, the $SU(3)$ Clebsch–Gordan coefficient

$$\tilde{C}^{(\sigma\sigma)(\sigma\sigma\sigma)0}_{\nu_1\nu_2\nu_3;\nu_1\nu_2\nu_3} \sim \left\langle \begin{smallmatrix}(\lambda,0)\\(\nu_1\nu_2\nu_3)I_\nu\end{smallmatrix};\ \begin{smallmatrix}(0,\lambda)\\(\nu_1\nu_2\nu_3)^*I_\nu\end{smallmatrix}\ \middle|\ \begin{smallmatrix}(\sigma,\sigma)\\(\sigma\sigma\sigma)0\end{smallmatrix}\right\rangle_{\mathrm{SU}(3)} \tag{A.1}$$

with $(\nu_1\nu_2\nu_3)^* \equiv (\lambda-\nu_1, \lambda-\nu_2, \lambda-\nu_3)$.

We start by evaluating the Clebsch in two step. First, we note that

$$\begin{aligned} I_1 &= \int \mathrm{d}\Omega D^{(\lambda,0)}_{\nu I_\nu;(\lambda 00)0}(\Omega) D^{(0,\lambda)}_{\nu^* I_\nu;\tau\frac{1}{2}\sigma}(\Omega)\left(D^{(\sigma,\sigma)}_{(\sigma,\sigma,\sigma)0;(2\sigma,\sigma,0)\frac{\sigma}{2}}(\Omega)\right)^* \\ &= \frac{128\pi^5}{(\sigma+1)^3}\left\langle \begin{smallmatrix}(\lambda,0)\\(\nu_1\nu_2\nu_3)I_\nu\end{smallmatrix};\ \begin{smallmatrix}(0,\lambda)\\(\nu_1\nu_2\nu_3)^*I_\nu\end{smallmatrix}\ \middle|\ \begin{smallmatrix}(\sigma,\sigma)\\(\sigma\sigma\sigma)0\end{smallmatrix}\right\rangle_{\mathrm{SU}(3)}\left\langle \begin{smallmatrix}(\lambda,0)\\(\lambda 00)0\end{smallmatrix};\ \begin{smallmatrix}(0,\lambda)\\(\sigma,\lambda,\lambda-\sigma)\frac{1}{2}\sigma\end{smallmatrix}\ \middle|\ \begin{smallmatrix}(\sigma,\sigma)\\(2\sigma,\sigma,0)\frac{1}{2}\sigma\end{smallmatrix}\right\rangle_{\mathrm{SU}(3)} \end{aligned} \tag{A.2}$$

using the usual orthogonality and combination properties of the $SU(3)$ D-functions.

The integral can be evaluated analytically using the explicit expression of those D-functions. We find

$$I_1 = \frac{4(2\pi)^5}{(\sigma+1)^2}\frac{1}{\sqrt{2\sigma+1}}(-1)^{\nu_3}\frac{2}{(\lambda+2)\sqrt{\lambda+1-\nu_1}} \times\Bigg[\sqrt{\lambda+1-\nu_1}\left\langle {\frac{1}{2}\lambda \atop \frac{1}{2}(2\nu_1-\lambda)};{\frac{1}{2}\lambda \atop \frac{1}{2}(\lambda-2\nu_1)}\,\Big|\,{\sigma \atop 0}\right\rangle_{SU(2)}\left\langle {\frac{1}{2}\lambda \atop \frac{1}{2}\lambda};{\frac{1}{2}\lambda \atop \frac{1}{2}(2\sigma-\lambda)}\,\Big|\,{\sigma \atop \sigma}\right\rangle_{SU(2)} - \sqrt{\frac{\nu_1+1}{\lambda+1}}\left\langle {\frac{1}{2}\lambda+1 \atop \frac{1}{2}(2\nu_1-\lambda)};{\frac{1}{2}\lambda \atop \frac{1}{2}(\lambda-2\nu_1)}\,\Big|\,{\sigma \atop 0}\right\rangle_{SU(2)}\left\langle {\frac{1}{2}\lambda+1 \atop \frac{1}{2}\lambda};{\frac{1}{2}\lambda \atop \frac{1}{2}(2\sigma-\lambda)}\,\Big|\,{\sigma \atop \sigma}\right\rangle_{SU(2)}\Bigg] \tag{A.3}$$

$$= \frac{4(2\pi)^5}{(\sigma+1)^2}(-1)^{\nu_3}\frac{2}{(\lambda+2)}\sqrt{\frac{\lambda!(2\sigma)!}{(\lambda+1-\nu_1)(\lambda+\sigma+1)!\sigma!}} \times\Bigg[\sqrt{\frac{\sigma(\nu_1+1)(\lambda-\sigma+1)}{(\lambda+\sigma+2)(\sigma+1)}}\left\langle {\frac{1}{2}\lambda+1 \atop \frac{1}{2}(2\nu_1-\lambda)};{\frac{1}{2}\lambda \atop -\frac{1}{2}(2\nu_1-\lambda)}\,\Big|\,{\sigma \atop 0}\right\rangle_{SU(2)} + \sqrt{\lambda+1-\nu_1}\left\langle {\frac{1}{2}\lambda \atop \frac{1}{2}(2\nu_1-\lambda)};{\frac{1}{2}\lambda \atop -\frac{1}{2}(2\nu_1-\lambda)}\,\Big|\,{\sigma \atop 0}\right\rangle_{SU(2)}\Bigg]. \tag{A.4}$$

We can obtain the square of the CG $\left\langle {(\lambda,0) \atop (\lambda 00)};{(0,\lambda) \atop (\sigma,\lambda,\lambda-\sigma)}\,\Big|\,{(\sigma,\sigma) \atop (2\sigma,\sigma,0)\frac{1}{2}\sigma}\right\rangle_{SU(3)}$ from the integral

$$I_2 = \int d\Omega D^{(\lambda,0)}_{(\lambda,0,0)0;(\lambda,0,0)0}(\Omega)D^{(0,\lambda)}_{\tau I_\tau;\tau I_\tau}(\Omega)\left(D^{(\sigma,\sigma)}_{(2\sigma,\sigma,0)\frac{\sigma}{2};(2\sigma,\sigma,0)\frac{\sigma}{2}}(\Omega)\right)^* = \frac{128\pi^5}{(\sigma+1)^3}\left|\left\langle {(\lambda,0) \atop (\lambda 00)};{(0,\lambda) \atop (\sigma,\lambda,\lambda-\sigma)}\,\Big|\,{(\sigma,\sigma) \atop (2\sigma,\sigma,0)\frac{1}{2}\sigma}\right\rangle_{SU(3)}\right|^2, \tag{A.5}$$

where $\tau = (\sigma,\lambda,\lambda-\sigma)$, $I_\tau = \frac{\sigma}{2}$. We find, upon insertion of the explicit expressions for the D-functions, that the resulting expression can be eventually simplified to

$$I_2 = \frac{256\pi^5\lambda!(2\sigma+1)!}{(\lambda+\sigma+2)!\sigma!(\sigma+1)^2}. \tag{A.6}$$

Using the phase convention that $\left\langle {(\lambda,0) \atop (\lambda 00)};{(0,\lambda) \atop (\sigma,\lambda,\lambda-\sigma)}\,\Big|\,{(\sigma,\sigma) \atop (2\sigma,\sigma,0)\frac{1}{2}\sigma}\right\rangle_{SU(3)}$ is positive, we thus find

$$\left\langle {(\lambda,0) \atop (\nu_1\nu_2\nu_3)I_\nu};{(0,\lambda) \atop (\nu_1\nu_2\nu_3)^*I_\nu}\,\Big|\,{(\sigma,\sigma) \atop (\sigma\sigma\sigma)0}\right\rangle_{SU(3)} = \sqrt{\frac{\sigma!(\lambda+\sigma+2)!}{(2\lambda!(2\sigma+1)!(\sigma+1)}}\frac{I_1}{128\pi^5(\sigma+1)^3}. \tag{A.7}$$

Finally, the coefficient $\tilde{C}^{\lambda(\sigma,\sigma)}_{\nu I_\nu}$ differs in general from the CG in equation (A.7) by a phase. Direct calculation of $\tilde{C}^{\lambda(\sigma,\sigma)}_{\nu I_\nu}$ for the first few λs and σs shows that

$$\tilde{C}^{\lambda(\sigma,\sigma)}_{\nu I_\nu} = (-1)^{\nu_2}\left\langle {(\lambda,0) \atop (\nu_1\nu_2\nu_3)I_\nu};{(0,\lambda) \atop (\nu_1\nu_2\nu_3)^*I_\nu}\,\Big|\,{(\sigma,\sigma) \atop (\sigma\sigma\sigma)0}\right\rangle_{SU(3)}. \tag{A.8}$$

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