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Polynomial intelligent states

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Abstract
The construction of $su(2)$ intelligent states is simplified using a polynomial representation of $su(2)$. The cornerstone of the new construction is the diagonalization of a 2×2 matrix. The method is sufficiently simple to be easily extended to $su(3)$, where one is required to diagonalize a single 3×3 matrix. For two perfectly general $su(3)$ operators, this diagonalization is technically possible but the procedure loses much of its simplicity owing to the algebraic form of the roots of a cubic equation. Simplified expressions can be obtained by specializing the choice of $su(3)$ operators. This simpler construction will be discussed in detail.

Keywords: intelligent states, polynomial states, uncertainty relations

1. Introduction

The purpose of this communication is to illustrate a simple and powerful method for constructing intelligent states. The method largely circumvents the use of recursion relations [1, 2] or special functions [3] and produces, with near triviality, all intelligent states for a set of two operators. Because the method depends on diagonalizing the smallest matrix representations of two operators, it is well suited to analytical results if the dimension of the system is small or if the operators have symmetries that allow analytical diagonalization.

First, recall that, given two self-adjoint operators A and B and a normalized state $|\psi\rangle$, one can obtain, using the Cauchy–Schwartz inequality, the uncertainty relation

$$\Delta A \Delta B \geqslant \tfrac{1}{2} |\langle [A, B] \rangle| \,, \qquad (1)$$

where the variance is defined by

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \qquad (2)$$

and where every expectation value is calculated using $|\psi\rangle$:, *v.g.*

$$\langle A \rangle = \langle \psi | A | \psi \rangle \,. \qquad (3)$$

A state is called intelligent if it satisfies the strict equality in equation (1). If the operators A and B act in a finite dimensional space, and if their eigenvalues are finite, their eigenstates are normalizable; the equality of equation (1) has a trivial lower bound of zero, obtained by selecting for $|\psi\rangle$ an eigenstate of either A or B. We will, instead, be concerned with states $|\psi\rangle$ which are not eigenstates of either A or B, and which are intelligent.

It is well known [4] that states are intelligent if they satisfy the eigenvalue equation

$$(A - \mathrm{i}\alpha B) |\psi\rangle = \lambda |\psi\rangle \,, \qquad (4)$$

where α is a real parameter. Our strategy is to first diagonalize a specific realization of A and B as $d \times d$ matrices, thereby obtaining the intelligent states $\{|\psi_i(\alpha)\rangle, i = 1, 2, \ldots, d\}$. Using an equivalent representation of A and B as differential operators allows us to express $|\psi_i(\alpha)\rangle$ as a polynomial in some dummy variables. The intelligent states of A and B for higher dimensional representation are then found by simply taking powers of those 'fundamental' polynomials; there is no need to solve a recursion relation.

2. Angular momentum intelligent states

Recall first that the angular momentum algebra is spanned by three Hermitian operators $\{L_x, L_y, L_z\}$, with cyclic commutation relations given by

$$[L_x, L_y] = \mathrm{i}L_z, [L_y, L_z] = \mathrm{i}L_x, [L_z, L_x] = \mathrm{i}L_y, \qquad (5)$$

where we set $\hbar = 1$ for convenience.

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2.1. Solving the eigenvalue equation for intelligence

Consider the problem of finding states such that

$$\Delta L_x \Delta L_y = \tfrac{1}{2}\,|\langle L_z\rangle|\,. \tag{6}$$

To proceed, we first set up the problem as a diagonalization of the 2 × 2 matrix

$$L_x - \mathrm{i}\alpha L_y \rightarrow \begin{pmatrix} 0 & 1-\alpha \\ 1+\alpha & 0 \end{pmatrix}. \tag{7}$$

The eigenstates can be expressed either as column vectors or as linear combinations of spin-$\frac{1}{2}$ states:

$$|\psi_1(\mu)\rangle = \frac{1}{\sqrt{1+|\mu|^2}}\begin{pmatrix} 1 \\ \mu \end{pmatrix} = \frac{1}{\sqrt{1+|\mu|^2}}\left(\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \mu\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\right), \tag{8}$$

$$|\psi_2(\mu)\rangle = \frac{1}{\sqrt{1+|\mu|^2}}\begin{pmatrix} 1 \\ -\mu \end{pmatrix} = \frac{1}{\sqrt{1+|\mu|^2}}\left(\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle - \mu\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\right) \tag{9}$$

with eigenvalues $\lambda_1 = \sqrt{1-\alpha^2}$, $\lambda_2 = -\sqrt{1-\alpha^2}$, respectively, and where the combination

$$\mu = \sqrt{\frac{1-\alpha^2}{1-\alpha}} \tag{10}$$

has been introduced. Note that μ is real if $|\alpha| \leqslant 1$. Otherwise, μ is purely imaginary. The eigenstates $|\psi_i(\alpha)\rangle$ are not orthogonal, a result associated with the non-Hermiticity of $L_x - \mathrm{i}\alpha L_y$.

We now introduce the well known differential representation

$$L_x \mapsto \frac{1}{2}\left(\xi\frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial\xi}\right), \tag{11}$$

$$L_y \mapsto \frac{1}{2\mathrm{i}}\left(\xi\frac{\partial}{\partial\eta} - \eta\frac{\partial}{\partial\xi}\right), \tag{12}$$

$$L_z \mapsto \frac{1}{2}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right). \tag{13}$$

The operators act on functions of the dummy variables η, ξ. In particular, angular momentum kets $|\ell, m\rangle_{\ell m}$ and bras ${}_{\ell m}\langle\ell, m|$ are mapped to the polynomials and differentials

$$|\ell, m\rangle_{\ell m} \mapsto \frac{\xi^{\ell+m}\eta^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}}, \tag{14}$$

$${}_{\ell m}\langle\ell, m\,| \mapsto \frac{1}{\sqrt{(\ell+m)!(\ell-m)!}}\frac{\partial^{\ell+m}}{\partial\xi^{\ell+m}}\frac{\partial^{\ell-m}}{\partial\eta^{\ell-m}}. \tag{15}$$

In terms of these variables, our intelligent states become

$$\langle\xi, \eta\,|\psi_1(\mu)\rangle = f(\xi, \eta; \mu) = \frac{1}{\sqrt{1+|\mu|^2}}(\xi + \mu\eta)\,, \tag{16}$$

$$\langle\xi, \eta\,|\psi_2(\mu)\rangle = g(\xi, \eta; \mu) = \frac{1}{\sqrt{1+|\mu|^2}}(\xi - \mu\eta)\,. \tag{17}$$

This is almost all that is required to proceed. Observe now that, if $f(\xi, \eta; \mu)$ and $g(\xi, \eta; \mu)$ are eigenstates of

$$L_x - \mathrm{i}\alpha L_y \mapsto (1-\alpha)\,\xi\frac{\partial}{\partial\eta} + (1+\alpha)\,\eta\frac{\partial}{\partial\xi}, \tag{18}$$

then the polynomial

$$H_{x,y}(\xi, \eta; \mu) \equiv [f(\xi, \eta; \mu)]^x[g(\xi, \eta; \mu)]^y \tag{19}$$

is also an eigenstate of equation (18), by inspection. The total degree of H is $x + y$. Let

$$x + y = 2\ell. \tag{20}$$

There are $x+y+1 = 2\ell+1$ polynomials of total degree 2ℓ, and the representation of $su(2)$ of dimension ℓ has precisely $2\ell+1$ states. Thus, the set of all $H_{x,y}(\xi, \eta; \alpha)$ such that the integers x, y satisfy $x + y = 2\ell$ must be the set of all eigenvectors of equation (18). In others words, we can construct the complete set of (un-normalized) intelligent states of angular momentum ℓ by simply constructing, in sequence, the set of all $H_{x,y}(\xi, \eta; \alpha)$ such that the integers x, y satisfy $x + y = 2\ell$.

It remains to normalize the states. This is done using a Gram–Schmidt procedure and equation (15). After some effort, the result is the normalized intelligent state $h_{x,y}(\xi, \eta; \mu)$ given by

$$h_{x,y}(\xi, \eta; \mu) = \frac{H_{x,y}(\xi, \eta; \mu)}{\sqrt{x!y!p(x, y; \mu)}} \tag{21}$$

$$p(x, y; \mu) = \frac{1}{(1+|\mu|^2)^{2y}} \times \sum_{k=0}^{y} 4^{y-k}\,|\mu|^{2(y-k)}\,(1-|\mu|^2)^{2k}\binom{y}{k}\binom{x+k}{k}. \tag{22}$$

Thus, for instance,

$$h_{2,2}(\xi, \eta; \mu) = \frac{\xi^4 - 2\mu^2\xi^2\eta^2 + \mu^4\eta^4}{2\sqrt{6+4|\mu|^4+6|\mu|^8}}. \tag{23}$$

Using the correspondence

$$|2, 2\rangle_{\ell m} \mapsto \frac{\xi^4}{\sqrt{4!}}, \qquad |2, 0\rangle_{\ell m} \mapsto \frac{\xi^2\eta^2}{\sqrt{2!2!}}, \qquad |2, -2\rangle_{\ell m} = \frac{\eta^4}{\sqrt{4!}}, \tag{24}$$

we can immediately write $h_{2,2}(\xi, \eta; \mu)$ as a sum of $\ell = 2$ angular momentum states:

$$h_{2,2}(\xi, \eta; \mu) \mapsto \frac{\sqrt{6}\,|2, 2\rangle_{\ell,m} - 2\,\mu^2\,|2, 0\rangle_{\ell,m} + \sqrt{6}\mu^4|2, -2\rangle_{\ell,m}}{\sqrt{6+4\,|\mu|^4+6\,|\mu|^8}}. \tag{25}$$

The calculation of the product $\Delta L_x \Delta L_y$ is now semi–automatic. One can use either the angular momentum expansion or the polynomial form of the states and the operator L_z.

The set of all normalized intelligent states of angular momentum $\ell = 2$ is given in table 1.

Table 1. The list of normalized intelligent states $h_{x,y}(\xi, \eta; \mu)$ for angular momentum $\ell = 2$.

x	y	$p(x, y; \mu)$	$h_{x,y}(\xi, \eta; \mu)$
0	4	1	$\dfrac{(\xi - \mu\eta)^4}{2\sqrt{6}\,(1+\lvert\mu\rvert^2)^2}$
1	3	$\dfrac{4\left(1-\lvert\mu\rvert^2+\lvert\mu\rvert^4\right)}{\left(1+\lvert\mu\rvert^2\right)^2}$	$\dfrac{(\xi + \mu\eta)\,(\xi - \mu\eta)^3}{2\sqrt{6}\sqrt{1+\lvert\mu\rvert^2+\lvert\mu\rvert^6+\lvert\mu\rvert^8}}$
2	2	$\dfrac{6+4\lvert\mu\rvert^4+6\lvert\mu\rvert^8}{\left(1+\lvert\mu\rvert^2\right)^4}$	$\dfrac{(\xi + \mu\eta)^2\,(\xi - \mu\eta)^2}{2\sqrt{6+4\lvert\mu\rvert^4+6\lvert\mu\rvert^8}}$
3	1	$\dfrac{4\left(1-\lvert\mu\rvert^2+\lvert\mu\rvert^4\right)}{\left(1+\lvert\mu\rvert^2\right)^2}$	$\dfrac{(\xi + \mu\eta)^3\,(\xi - \mu\eta)}{2\sqrt{6}\sqrt{1+\lvert\mu\rvert^2+\lvert\mu\rvert^6+\lvert\mu\rvert^8}}$
4	0	1	$\dfrac{(\xi + \mu\eta)^4}{2\sqrt{6}\,(1+\lvert\mu\rvert^2)^2}$

2.2. *Sample results*

We present in figures 1–3 some sample curves of ΔL_x, ΔL_y and $\Delta L_x \Delta L_y$ as a function of the control parameter α. We have chosen, for illustrative purposes, to focus on the set of intelligent states with angular momentum $\ell = 2$.

Apparently, previous efforts in producing such curves have been limited to $\ell = \frac{1}{2}$ (see [5]), possibly due to the difficulty of extending the results of [1], which are limited to coherent states. In fact, it will be shown in [6] that angular momentum intelligent states of the type $(x, 0)$ or $(0, y)$ are angular momentum coherent states. Thus, the plots for ΔL_x and ΔL_y for figure 1 reproduce the results of [1].

All curves are symmetric w.r.t. α. Curves for the intelligent state (x, y) are identical to curves for (y, x), so only those with $x > y$ need to be presented.

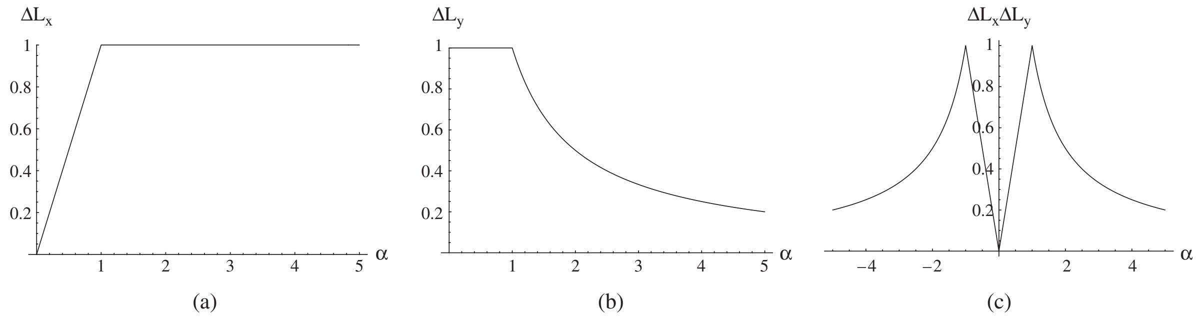


Figure 1. Uncertainty curves as a function of the control parameter α for the intelligent state with $x = 0$, $y = 4$: (a) ΔL_x; (b) ΔL_y; (c) the product $\Delta L_x \Delta L_y$. All curves are symmetric functions of α.

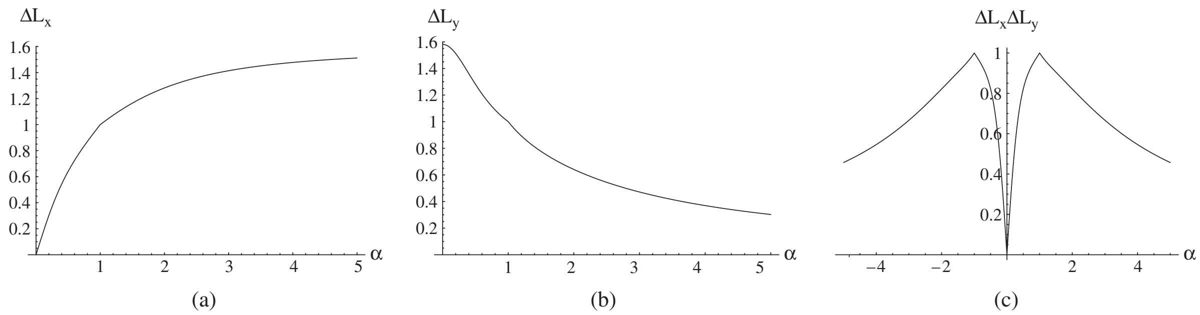


Figure 2. Uncertainty curves as a function of the control parameter α for the intelligent state with $x = 1$, $y = 3$: (a) ΔL_x; (b) ΔL_y; (c) the product $\Delta L_x \Delta L_y$. All curves are symmetric functions of α.

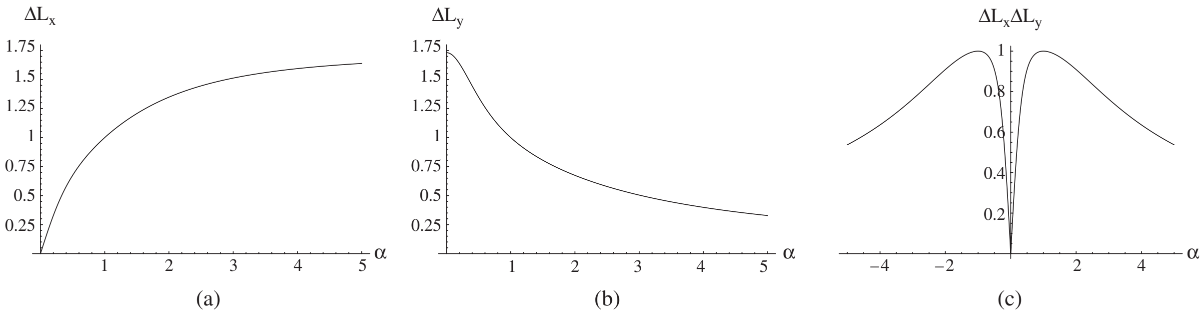


Figure 3. Uncertainty curves as a function of the control parameter α for the intelligent state with $x = 2$, $y = 2$: (a) ΔL_x; (b) ΔL_y; (c) the product $\Delta L_x \Delta L_y$. All curves are symmetric functions of α.

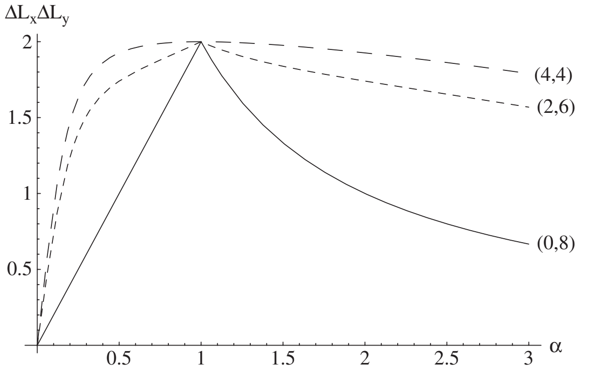


Figure 4. Uncertainty curves as a function of the control parameter α for the intelligent state of angular momentum $\ell = 4$ with $(x, y) = (4, 4)$, $(2, 6)$ and $(0, 8)$. This illustrates that the intelligent state $(0, 8)$ produces the smallest uncertainty in the product $\Delta L_x \Delta L_y$. All curves are symmetric functions of α, so only the $\alpha > 0$ portion is shown. The curves for $(1, 7)$ and $(3, 5)$ are not shown for clarity.

The plots of ΔL_x and ΔL_y are all discontinuous at $\alpha = \pm 1$. This is where the parameter μ, which is real for $|\alpha| \leqslant 1$, becomes purely imaginary for $|\alpha| > 1$. The plots for the products $\Delta L_x \Delta L_y$ are, in general, discontinuous at $\alpha = \pm$, except when $x = y$. Furthermore, one observes that the points $\alpha = \pm 1$ are always maxima of uncertainty.

For the values $\alpha = \pm 1$, the non-Hermitian operator $L_x - \mathrm{i}\alpha L_y$ collapses to a ladder operator L_- or L_+. For $\alpha = 0$, the eigenvalue equation returns the eigenstates of L_x: ΔL_x is therefore trivially zero. For large values of $|\alpha|$, the curves must also go to zero as the eigenvalue equation is dominated by L_y.

Taken together, figures 1(c)–3(c) show that, for any value of α, the intelligent state of the type $(0, 4)$ has the lowest uncertainty product $\Delta L_x \Delta L_y$ of all intelligent states of angular momentum $\ell = 2$. This conclusion can be extended to other values of ℓ: see figure 4, where we present on a single graph the uncertainty curves as a function of α for the intelligent states $(0, 8)$, $(2, 6)$ and $(4, 4)$ having angular momentum $\ell = 4$. The curves for $(1, 7)$ and $(3, 5)$ have been left out for clarity. It is again clear that the intelligent state of type $(0, 8)$, which is an angular momentum coherent state, produces the smallest product $\Delta L_x \Delta L_y$ for a given α. We have not attempted, at this stage, to prove this observation rigorously.

3. Extension to $su(3)$ operators

The method illustrated for $su(2)$ can now easily be extended to $su(3)$: one simply has to solve the 3×3 equivalent of equation (4). From this, one extracts three eigenvectors; when translated as polynomials in the dummies ξ, η, ζ, powers of these will be eigenstates of equation (4), and we will have a list of all intelligent states in the unitary irreducible representation $[\lambda, 0]$ of dimension $\frac{1}{2}(\lambda+1)(\lambda+2)$, where λ is the total power of the polynomial in ξ, η, ζ.

It is always possible to find the eigenvalues and eigenvectors of a 3×3 matrix, so it is always possible to find an analytic expression for $su(3)$ intelligent states. However, it is a heavy burden to solve the problem in its full generality because of the number of parameters required to properly describe two general $su(3)$ operators.

It is nevertheless possible to illustrate the method by selecting two sufficiently general operators. To this end, we start by defining

$$\omega = \mathrm{e}^{2\mathrm{i}\pi/3}, \tag{26}$$

and the two matrices

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{27}$$

These matrices are unitary ($\mathcal{A}^\dagger = \mathcal{A}^{-1}, \mathcal{B}^\dagger = \mathcal{B}^{-1}$) and complementary, in the sense that the eigenstates $\{|\psi_{\mathcal{A}}^k\rangle, k = 1, 2, 3\}$ of $\mathcal{A}$ and the eigenstates $\{|\phi_{\mathcal{B}}^m\rangle, m = 1, 2, 3\}$ of $\mathcal{B}$ satisfy

$$|\langle\psi_{\mathcal{A}}^k|\phi_{\mathcal{B}}^m\rangle|^2 = \tfrac{1}{3}. \tag{28}$$

From $\mathcal{A}$ and $\mathcal{B}$, one easily recovers the Hermitian operators

$$A = \frac{2\pi}{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B = \frac{2\pi\mathrm{i}}{3\sqrt{3}}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \tag{29}$$

such that

$$\mathcal{A} = \mathrm{e}^{\mathrm{i}A}, \qquad \mathcal{B} = \mathrm{e}^{\mathrm{i}B}. \tag{30}$$

The operators A and B are themselves complementary, in the sense that their eigenstates also satisfy equation (28) and thus represent a generalization of the Pauli matrices σ_x and σ_y. The eigenvalues and eigenvectors of equation (4) as a 3×3 matrix problem for our operators A and B are given in table 2.

The complex extension of the $su(3)$ algebra is spanned by the ladder operators $C_{ij}, i \neq j = 1, 2, 3$ and by two diagonal operators $h_1 = C_{11} - C_{22}, h_2 = C_{22} - C_{33}$. The C_{ij} satisfy

$$[C_{ij}, C_{k\ell}] = \delta_{jk} C_{i\ell} - \delta_{i\ell} C_{kj}. \tag{31}$$

A 3×3 matrix realization of C_{ij} is obtained by placing a 1 in the position (i, j), and zeros everywhere else. Thus, for instance,

$$C_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

A more general realization of C_{ij} is obtained in terms of creation and destruction operators for three-dimensional harmonic oscillator states. Thus the map

$$C_{ij} \mapsto a_i^\dagger a_j \tag{33}$$

preserves the commutation relation of equation (31). $su(3)$ states of the representation $[\lambda, 0]$ are realized as occupation-number states $|n_x, n_y, n_z\rangle$ such that $n_x + n_y + n_z = \lambda$.

A polynomial representation is obtained by identifying the states of the $[1, 0]$ representation (of dimension three) with the dummy variables

$$|1, 0, 0\rangle \mapsto \xi, \qquad |0, 1, 0\rangle \mapsto \eta, \qquad |0, 0, 1\rangle \mapsto \zeta, \tag{34}$$

and setting

Table 2. The eigenvalues and associated intelligent states for the 3×3 realization of $A - \mathrm{i}\alpha B$.

λ_i	$\lvert\psi_i(\alpha)\rangle$
0	$(\alpha - \sqrt{3}, \alpha, \alpha)^{\mathrm{T}}$
$-\frac{2\pi}{3}\sqrt{1-\alpha^2}$	$(-2\alpha, \alpha - \sqrt{3(1-\alpha^2)} + \sqrt{3}, \alpha + \sqrt{3(1-\alpha^2)} + \sqrt{3})^{\mathrm{T}}$
$\frac{2\pi}{3}\sqrt{1-\alpha^2}$	$(-2\alpha, \alpha + \sqrt{3(1-\alpha^2)} + \sqrt{3}, \alpha - \sqrt{3(1-\alpha^2)} + \sqrt{3})^{\mathrm{T}}$

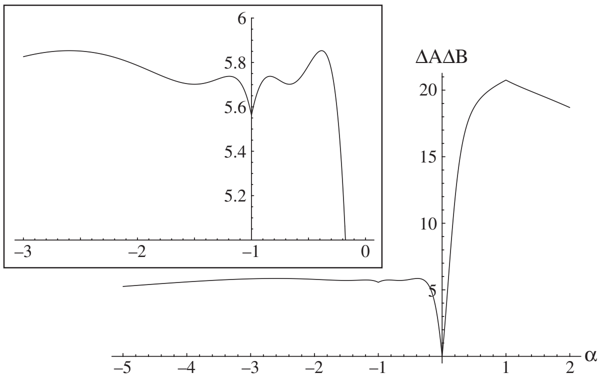


Figure 5. The uncertainty product $\Delta A \Delta B$ as a function of the control parameter α for the $su(3)$ intelligent state with $x = 2$, $y = 3$, $z = 1$. This is one of 28 intelligent states in the representation $(6, 0)$ of $su(3)$. Inset: a segment of the previous curve near $\alpha = -1$, showing that the uncertainty goes through a minimum at that point.

$$
\begin{aligned}
C_{12} &\mapsto \xi \frac{\partial}{\partial \eta}, & C_{21} &\mapsto \eta \frac{\partial}{\partial \xi}, \\
C_{13} &\mapsto \xi \frac{\partial}{\partial \zeta}, & C_{31} &\mapsto \zeta \frac{\partial}{\partial \xi}, \\
C_{23} &\mapsto \eta \frac{\partial}{\partial \zeta}, & C_{32} &\mapsto \zeta \frac{\partial}{\partial \eta}, \\
C_{11} &\mapsto \xi \frac{\partial}{\partial \xi}, \quad C_{22} \mapsto \eta \frac{\partial}{\partial \eta}, & C_{33} &\mapsto \zeta \frac{\partial}{\partial \zeta}.
\end{aligned}
\tag{35}
$$

Following the example from angular momentum, we introduce three functions of the dummies ξ, η, ζ:

$$
f(\xi, \eta, \zeta; \alpha) = (\alpha - \sqrt{3})\xi + \alpha\eta + \alpha\zeta, \tag{36}
$$

$$
\begin{aligned}
g(\xi, \eta, \zeta; \alpha) = {} & -2\alpha\xi + (\alpha - \sqrt{3(1-\alpha^2)} + \sqrt{3})\eta \\
& + (\alpha + \sqrt{3(1-\alpha^2)} + \sqrt{3})\zeta,
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
r(\xi, \eta, \zeta; \alpha) = {} & -2\alpha\xi + (\alpha + \sqrt{3(1-\alpha^2)} + \sqrt{3})\eta \\
& + (\alpha - \sqrt{3(1-\alpha^2)} + \sqrt{3})\zeta,
\end{aligned}
\tag{38}
$$

along with the general intelligent state

$$
\begin{aligned}
& H_{x,y,z}(\xi, \eta, \zeta; \alpha) \\
& \quad = f(\xi, \eta, \zeta; \alpha)^x g(\xi, \eta, \zeta; \alpha)^y r(\xi, \eta, \zeta; \alpha)^z.
\end{aligned}
\tag{39}
$$

It is easily verified that these are eigenstates of $A - \mathrm{i}\alpha B$ in differential form, where the Hermitian operators A and B are obtained from their 3×3 matrix realization as

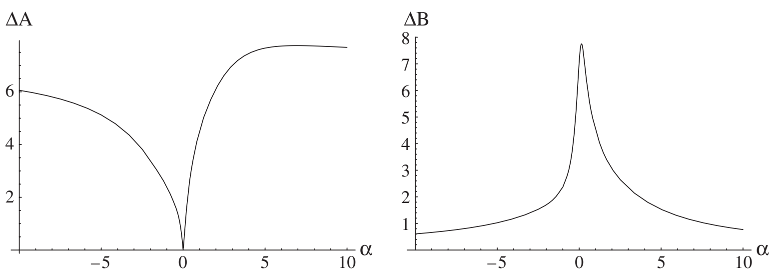
$$
A \mapsto \frac{2\pi}{3}\left(\eta \frac{\partial}{\partial \eta} - \zeta \frac{\partial}{\partial \zeta}\right), \tag{40}
$$

$$
B \mapsto \frac{2\mathrm{i}\pi}{3\sqrt{3}}\left(-\xi \frac{\partial}{\partial \eta} + \xi \frac{\partial}{\partial \zeta} + \eta \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \zeta} - \zeta \frac{\partial}{\partial \xi} + \zeta \frac{\partial}{\partial \eta}\right). \tag{41}
$$

In order to avoid unnecessarily complicated norms, we present only sample uncertainty curves for the product $\Delta A \Delta B$. Normalizations and expectation values were obtained numerically in all cases.

Figure 5 shows the uncertainty curve for the product $\Delta A \Delta B$ as a function of α for the intelligent state with $(x, y, z) = (2, 3, 1)$. It is noteworthy to point out the qualitatively different nature of this curve compared to the angular momentum curves. In particular, the curve is no longer symmetric w.r.t. α and the point $\alpha = -1$ is now a local minimum, as illustrated in figure 6. This feature in not true of all intelligent states of A and B; this is in contrast to our results on angular momentum coherent states, where $\alpha = -1$ was always a maximum of the product $\Delta L_x \Delta L_y$.

Finally, we present in figure 6 the behaviour of ΔA and ΔB as a function of α for the intelligent state with $(x, y, z) = (2, 3, 1)$. Although not obvious from the graph, the curve for ΔA plateaus for large values of $\pm\alpha$; for $\alpha > 10^6$, the values $\Delta A(\alpha) \sim \Delta A(-\alpha)$. This is easily understood as, for large



(a) (b)

Figure 6. (a) $\Delta A(\alpha)$ and (b) $\Delta B(\alpha)$ for the $su(3)$ intelligent state with $x = 2$, $y = 3$, $z = 1$.

$\pm\alpha$, the intelligent state becoming indistinguishable from an eigenstate of B.

4. Conclusion and outlook

The method presented in this article is a simple and straightforward alternative to more complicated methods for constructing intelligent states. In particular, all intelligent states of a given representation of $su(2)$ can be obtained with minimal effort. Intelligent states for $su(3)$ representations of the type $[\lambda, 0]$ are equally easy to obtain, although we have, for clarity, restricted our construction to specific operators. Our method can be advantageously compared to methods based on special functions proposed in [3] for $su(2)$ and extended to an $su(2) \oplus \mathrm{u}(1) \subset su(3)$ in [7] (see also [5]).

Polynomial states also exist for a variety of algebras, and it can be hoped that our results can be adapted to such cases when required.

In a forthcoming publication [6], a more in-depth analysis of the polynomial intelligent states will be provided, along with several other examples of uncertainty curves. In particular, the connection between intelligent and coherent states will be investigated in depth.

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