

Multicomplementary operators via finite Fourier transform

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Abstract

A complete set of $d + 1$ mutually unbiased bases exists in a Hilbert space of dimension d , whenever d is a power of a prime. We discuss a simple construction of $d + 1$ disjoint classes (each one having $d - 1$ commuting operators) such that the corresponding eigenstates form sets of unbiased bases. Such a construction works properly for prime dimension. We investigate an alternative construction in which the real numbers that label the classes are replaced by a finite field having d elements. One of these classes is diagonal, and can be mapped to cyclic operators by means of the finite Fourier transform, which allows one to understand complementarity in a similar way as for the position–momentum pair in standard quantum mechanics. The relevant examples of two and three qubits and two qutrits are discussed in detail.

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1. Introduction

The concept of complementarity is a direct non-trivial consequence of the superposition principle and distinguishes purely quantum systems from those that may be accurately treated classically. Therefore, a thorough understanding of this idea is of fundamental importance for a correct interpretation of quantum mechanics [1].

In short, Bohr's idea of complementarity could be loosely formulated by stating that, in order to understand a quantum phenomenon completely we need a combination of mutually exclusive properties: the precise knowledge of one of them implies that all possible outcomes in the other are equally probable.

Perhaps the best textbook illustration of complementarity is that the observation of interference and the knowledge of the path followed by the interfering particle are mutually exclusive. This is often expressed as the statement that all quantum objects exhibit particle-like or wave-like behaviour under different experimental conditions. Standard examples, such

as the Einstein recoiling slit [2], Feynman light-scattering arrangement [3] or Heisenberg microscope [4], are always explained in terms of position and momentum. It is thus natural that links between canonical conjugacy and complementarity have been fully explored [5–7].

However, most if not all of the recent examples of complementarity involve finite-dimensional systems (for a complete and up-to-date review, see [8]). Note that, contrary to what one would expect, the finite dimensionality of the system space introduces difficulties when dealing with uncertainty relations, because the commutators between complementary observables are operators instead of c -numbers, which makes the analysis of the problem more involved. The situation has been recently addressed for two-dimensional spaces [9]; however, there are subtle aspects that are not fully revealed by these easy-to-understand systems.

In finite-dimensional systems, complementarity is tantamount to *unbiasedness* [10, 11]: each eigenstate of any measurement is an equal-magnitude superposition of the eigenstates of any of the complementary measurements. This leads naturally to the concept of mutually unbiased bases (MUBs), which have recently been considered with an increasing interest because of the central role they play not only in understanding complementarity [12–14], but also in specific quantum information tasks, such as protocols of quantum cryptography [15, 16], Wigner functions in discrete phase spaces [17, 18] or the so-called Mean King problem [19–23].

For a d -dimensional system it has been found that the maximum number of MUBs cannot be greater than $d + 1$ and this limit is reached if d is prime [24] or power of prime [25, 26]. Remarkably though, there is no known answer for any other values of d , not even for $d = 6$. Recent works have suggested that the answer to this question may well be related with the non-existence of finite projective planes of certain orders [27, 28] or with the problem of mutually orthogonal Latin squares in combinatorics [29, 30].

Quite recently, a number of papers have addressed the explicit construction of MUBs for dimensions that are prime or composite (i.e., power of a prime), exploiting different algebraic properties [31–38]. In this paper, we give an explicit construction with a different method that resorts to elementary notions of finite field theory. We wish to emphasize the distinct features of our approach: first, we recall that complementarity for the position–momentum pair is implemented by the Fourier transform, which exchanges both operators. Therefore, we construct classes of maximally commuting operators and map them using the finite Fourier transform and an additional diagonal operator. In consequence, we obtain in a systematic way the whole family of complementary operators and not merely MUBs. Additionally, the final expression for these MUBs is compact and can be immediately expressed in different bases, in some of which they appear as tensor products of generalized Pauli matrices. In summary, we hope that our unified construction provides a simple picture of complementarity for both prime and composite dimensions.

2. Complementary operators in prime dimension

We consider a system living in a Hilbert space \mathcal{H}_d , whose dimension d is a prime number. It is useful to choose a computational basis $|n\rangle$ (where $n = 0, \dots, d - 1$) in \mathcal{H}_d and introduce the basic operators

$$X|n\rangle = |n + 1\rangle, \quad Z|n\rangle = \omega^n|n\rangle, \quad (2.1)$$

where

$$\omega = \exp(2\pi i/d) \quad (2.2)$$

is a d th root of the unity and addition and multiplication must always be understood modulo d . These operators X and Z , which are generalizations of the Pauli matrices, were studied by

Patera and Zassenhaus [39] in connection with additive quantum numbers, and have been used recently by many authors in a variety of applications [40–42]. They generate a group under multiplication known as the generalized Pauli group and obey

$$ZX = \omega XZ, \tag{2.3}$$

which is the finite-dimensional version of the Weyl form of the commutation relations.

According to the ideas in [31], we can find $d + 1$ disjoint classes (each one having $d - 1$ commuting operators) such that the corresponding eigenstates form sets of MUBs. The explicit construction starts with the following sets of operators:

$$\begin{aligned} \{Z^k\}, & \quad k = 1, \dots, d - 1, \\ \{(XZ^m)^k\}, & \quad k = 1, \dots, d - 1, \quad m = 0, \dots, d - 1. \end{aligned} \tag{2.4}$$

One can easily check that

$$\begin{aligned} \text{Tr}(Z^k Z^{k'\dagger}) &= d\delta_{kk'}, & \text{Tr}(X^k X^{k'\dagger}) &= d\delta_{kk'}, \\ \text{Tr}[(XZ^m)^k (XZ^{m'})^{k'\dagger}] &= d\delta_{kk'}\delta_{mm'}. \end{aligned} \tag{2.5}$$

These pairwise orthogonality relations indicate that, for every value of m , we generate a maximal set of $d - 1$ commuting operators and that all these sets are disjoint. In addition, the common eigenstates of each set m form different elements of unbiased bases. We shall refer to these classes as multicomplementary.

We would now like to make the very important observation that, starting from Z , it is possible to obtain any element of the form $(XZ^m)^k$ by using a combination of only two operators F and V defined as follows: F is the finite Fourier transform [43]

$$F = \frac{1}{\sqrt{d}} \sum_{n,n'=0}^{d-1} \omega^{nn'} |n\rangle\langle n'|, \tag{2.6}$$

and V is the diagonal transformation (assuming d is odd)

$$V = \sum_{n=0}^{d-1} \omega^{-(n^2-n)(d+1)/2} |n\rangle\langle n|. \tag{2.7}$$

Indeed this is the case, since one easily verifies that

$$X = F^\dagger Z F, \tag{2.8}$$

much in the spirit of the standard way of looking at complementary variables in the infinite-dimensional Hilbert space: the position and momentum eigenstates are Fourier transform one of the other. On the other hand, the diagonal transformation V acts as a Z -right shift:

$$XZ^m = V^{\dagger m} X V^m. \tag{2.9}$$

The case $d = 2$ needs minor modifications. In fact, it turns out that one cannot find a diagonal unitary transformation V such that $X \mapsto XZ$. For this reason, instead of XY the matrix Y is defined as iXZ , so that $Y = V^\dagger X V$, where V is

$$V = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \tag{2.10}$$

The construction of multicomplementary operators is otherwise identical to the case where d is an odd prime.

3. Complementary operators in composite dimensions

3.1. Constructing multicomplementary operators

For all its simplicity, the construction of the previous section fails if the dimension of the system is a power of a prime. A simple illustration of this is obtained in dimension $4 = 2^2$. According to equation (2.1), the operators X and Z are simply

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \quad (3.1)$$

Then, for instance, $X^2 = -(XZ^2)^2$ and (XZ^3) is proportional to XZ , so operators constructed following (2.4) no longer form disjoint sets. The root of this failure can be traced to the fact that \mathbb{Z}_4 , the set of integers modulo 4, does not form an algebraic field. The same failure generally occurs for any composite dimension $d = p^n$, where p is a prime and n is an integer. In short, the construction of multicomplementary operators cannot proceed by simply taking powers of some basic elements.

However, when $d = p^n$, we know there exists (up to isomorphisms) exactly one field, written as \mathbb{F}_d , with d elements. If $d = p$ is prime, the field essentially coincides with \mathbb{Z}_p . We briefly recall the minimum background needed to proceed. For more details, the reader is referred to the pertinent literature [44]. The field \mathbb{F}_d can be represented as the field of equivalence classes of polynomials whose coefficients belong to \mathbb{Z}_p . The product in the multiplicative group \mathbb{F}_d^* (i.e. excluding the zero) is defined as the product of the corresponding polynomials modulo a primitive polynomial of degree n , irreducible in \mathbb{Z}_p . In fact, \mathbb{F}_d^* is a cyclic group of order $d - 1$: it is generated by powers of a primitive element α , which is a monic irreducible polynomial of degree n . This establishes a natural order for the field elements, and we use this order to label the elements of a basis in \mathcal{H}_d as follows:

$$\{|0\rangle, |\alpha\rangle, |\alpha^2\rangle, \dots, |\alpha^{d-1}\rangle\}. \quad (3.2)$$

Our solution to the problem of MUBs in composite dimension consists in using elements of \mathbb{F}_d , instead of natural numbers, to label the classes of complementary operators.

Next we define the trace of a field element $\theta \in \mathbb{F}_d$ as

$$\text{tr}(\theta) = \theta + \theta^p + \theta^{p^2} + \dots + \theta^{p^{n-1}}. \quad (3.3)$$

Note that we distinguish it from the trace of an operator by the lower case tr . The trace has remarkably simple properties, the most important being that it is linear and that it is always an element of the prime field \mathbb{Z}_p .

For the additive group in the field \mathbb{F}_d we can introduce additive characters as a map that fulfils

$$\chi(\theta_1)\chi(\theta_2) = \chi(\theta_1 + \theta_2), \quad \theta_1, \theta_2 \in \mathbb{F}_d. \quad (3.4)$$

These additive characters have the form

$$\chi(\theta) = \exp\left[\frac{2\pi i}{p} \text{tr}(\theta)\right]. \quad (3.5)$$

In subsequent calculations we shall need the property

$$\sum_{\theta \in \mathbb{F}_d} \chi(\theta) = 0, \quad (3.6)$$

which leads to the relation

$$\sum_{k=0}^{d-2} \chi(\alpha^k \theta) = d\delta_{\theta,0} - 1. \tag{3.7}$$

We start by introducing diagonal operators with respect to the basis (3.2) as follows:

$$Z_q = |0\rangle\langle 0| + \sum_{k=1}^{d-1} \chi(\alpha^{q+k})|\alpha^k\rangle\langle \alpha^k|, \quad q = 0, \dots, d - 2. \tag{3.8}$$

This definition implies

$$Z_q|\alpha^k\rangle = \chi(\alpha^{q+k})|\alpha^k\rangle, \tag{3.9}$$

and the combination property

$$Z_q Z_{q'} \equiv Z_{(q)+(q')} = |0\rangle\langle 0| + \sum_{k=1}^{d-1} \chi(\alpha^{q+k} + \alpha^{q'+k})|\alpha^k\rangle\langle \alpha^k|. \tag{3.10}$$

These are quite natural generalizations of the properties of matrices in the class $\{Z^k\}$ in (2.4), since the $|\alpha^k\rangle$ are eigenstates of Z_q .

In a similar fashion, the operators X_q are defined as

$$X_q = \sum_{k=1}^{d-1} |\alpha^k + \alpha^q\rangle\langle \alpha^k| + |\alpha^q\rangle\langle 0|, \quad q = 0, \dots, d - 2, \tag{3.11}$$

so they act as operators shifting $|\alpha^k\rangle$ to $|\alpha^k + \alpha^q\rangle$ and satisfy the combination rule

$$X_q X_{q'} = X_{(q)+(q')} = |\alpha^q + \alpha^{q'}\rangle\langle 0| + \sum_{k=1}^{d-1} |\alpha^k + \alpha^q + \alpha^{q'}\rangle\langle \alpha^k|. \tag{3.12}$$

Because elements of the field close under addition, $\alpha^k + \alpha^q$ is another element in the field: it must be that there is some number $L(n)$, called the Jacobi logarithm, such that

$$\alpha^k + \alpha^q = \alpha^{k+L(q-k)}, \tag{3.13}$$

whenever $\alpha^k + \alpha^q \neq 0$. In applications, $\alpha^k + \alpha^q$ can be found from the summation table of \mathbb{F}_d .

The finite Fourier transform F , when expressed in terms of the basis $|\alpha^k\rangle$, takes the form

$$F = \frac{1}{\sqrt{d}} \left[|0\rangle\langle 0| + \sum_{k,k'=1}^{d-1} \chi(\alpha^{k'+k})|\alpha^{k'}\rangle\langle \alpha^k| + \sum_{k=1}^{d-1} (|0\rangle\langle \alpha^k| + |\alpha^k\rangle\langle 0|) \right]. \tag{3.14}$$

This definition satisfies the natural property

$$F^\dagger F = F F^\dagger = I. \tag{3.15}$$

The operator F is set up to transform the operators Z_q into X_q :

$$X_q = F^\dagger Z_q F. \tag{3.16}$$

The Weyl form of the canonical commutation relations in \mathbb{F}_d is now

$$Z_q X_{q'} = \chi(\alpha^{q+q'}) X_{q'} Z_q. \tag{3.17}$$

The operators Z_q and X_q have been designed to be d -‘periodic’, in the sense that

$$Z_d = Z_0, \quad X_d = X_0. \tag{3.18}$$

In full analogy with the sets in (2.4), we can generate operators from X_q and Z_q ; they will be of the form $X_q Z_r$. Linear independence and orthogonality are guaranteed, in the sense that (compare equation (2.5))

$$\begin{aligned} \text{Tr}(Z_q Z_{q'}^\dagger) &= d\delta_{qq'}, & \text{Tr}(X_q X_{q'}^\dagger) &= d\delta_{qq'}, \\ \text{Tr}[(X_q Z_r)(X_{q'} Z_{r'})^\dagger] &= d\delta_{qq'}\delta_{rr'}. \end{aligned} \tag{3.19}$$

The commutation relations read

$$[X_q Z_r, X_{q'} Z_{r'}] = X_{(q)+(q')} Z_{(r)+(r')} [\chi(\alpha^{q'+r}) - \chi(\alpha^{q+r'})]. \tag{3.20}$$

It is clear from (3.19) and (3.20) that the sets (compare equation (2.4))

$$\begin{aligned} \{Z_q\}, & & q &= 0, \dots, d-2, \\ \{X_q Z_{q+r}\}, & & q, r &= 0, \dots, d-2, \end{aligned} \tag{3.21}$$

are disjoint and that every element of a set with a fixed value r commutes with every other element in the same set: they define multicomplementary operators.

Finally, let us consider the form of the diagonal operators similar to (2.9) transforming X_q to $X_q Z_r$. If we restrict to odd dimensions, we have

$$V_q^{(r)} = |0\rangle\langle 0| + \sum_{k=1}^{d-1} \bar{\chi}(2^{-1}\alpha^{r+2k-q})|\alpha^k\rangle\langle\alpha^k|, \quad q, r = 0, \dots, d-2, \tag{3.22}$$

where $\bar{\chi}$ means conjugate character and 2^{-1} is an element of \mathbb{Z}_p ; in particular, if $p = 2N + 1$ we have $2^{-1} = N + 1$. In this way, one can check that

$$V_{q+r}^{(q)\dagger} X_q V_{q+r}^{(q)} = \chi(2^{-1}\alpha^{2q+r}) X_q Z_{q+r}. \tag{3.23}$$

Using (3.14) and (3.22) we can generate all the complementary bases. Indeed, if the vectors $|\vec{\alpha}\rangle_q$ are the eigenstates of Z_0 , then the whole set of complementary bases can be obtained as follows:

$$V_q^{(0)\dagger} F^\dagger |\vec{\alpha}\rangle_q. \tag{3.24}$$

3.2. Complementary operators for two qubits

We illustrate our approach with the simplest case of a quantum system of composite dimension: two qubits described in a four-dimensional Hilbert space \mathcal{H}_4 . To construct multicomplementary operators, we start from the field \mathbb{F}_4 containing four elements. The polynomial

$$\theta^2 + \theta + 1 = 0 \tag{3.25}$$

is irreducible in \mathbb{Z}_2 and the primitive element α is defined as a root of (3.25). In consequence the four elements of \mathbb{F}_4 as in (3.2) can be written as

$$\{0, 1, \alpha, \alpha + 1\}, \tag{3.26}$$

where we have taken into account arithmetic modulo 2 and the fact that if α satisfies equation (3.25), then we have the relations

$$\alpha^2 = \alpha + 1, \quad \alpha^3 = 1. \tag{3.27}$$

A direct application of the definition (3.5) gives

$$\chi(0) = 1, \quad \chi(\alpha) = -1, \quad \chi(\alpha^2) = -1, \quad \chi(\alpha^3) = 1. \tag{3.28}$$

Using the representation where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\alpha\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\alpha^2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\alpha^3\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (3.29)$$

the matrices Z_q are

$$Z_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (3.30)$$

$$Z_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The matrix realization of the Fourier transform is

$$F = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad (3.31)$$

and the matrices X_q are

$$X_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.32)$$

$$X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The rest of the sets are routinely obtained according to (3.21). For completeness we quote all of them for this example:

$$X_0 Z_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad X_1 Z_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.33)$$

$$X_2 Z_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
 X_0 Z_1 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & X_1 Z_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
 X_2 Z_0 &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},
 \end{aligned} \tag{3.34}$$

$$\begin{aligned}
 X_0 Z_2 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & X_1 Z_0 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
 X_2 Z_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
 \end{aligned} \tag{3.35}$$

3.3. Complementary operators for three qubits

Our next example is the case of three qubits, the Hilbert space of which is eight dimensional. For the field \mathbb{F}_8 the primitive element α is a root of the following irreducible polynomial on \mathbb{F}_2 :

$$\theta^3 + \theta + 1 = 0. \tag{3.36}$$

In consequence, the elements of \mathbb{F}_8 according to (2.4) are

$$\{0, 1, \alpha, \alpha^2, \alpha + 1, \alpha^2 + \alpha, \alpha^2 + \alpha + 1, \alpha^2 + 1\}, \tag{3.37}$$

where we have taken into account that

$$\alpha^3 = \alpha + 1, \quad \alpha^4 = \alpha^2 + \alpha, \quad \alpha^5 = \alpha^2 + \alpha + 1, \quad \alpha^6 = \alpha^2 + 1, \quad \alpha^7 = 1. \tag{3.38}$$

One can obtain again the additive characters in a straightforward way

$$\begin{aligned}
 \chi(0) = 1, & \quad \chi(\alpha) = 1, & \quad \chi(\alpha^2) = 1, & \quad \chi(\alpha^3) = -1, \\
 \chi(\alpha^4) = 1, & \quad \chi(\alpha^5) = -1, & \quad \chi(\alpha^6) = -1, & \quad \chi(\alpha^7) = -1.
 \end{aligned} \tag{3.39}$$

Using the basis labelling as in (3.29), the matrix of the Fourier transform is

$$F = \frac{1}{\sqrt{2^3}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix}. \tag{3.40}$$

The matrices Z_q ($q = 0, \dots, 6$) are

$$\begin{aligned} Z_0 &= \text{diag}(1, 1, 1, -1, 1, -1, -1, -1), & Z_1 &= \text{diag}(1, 1, -1, 1, -1, -1, -1, 1), \\ Z_2 &= \text{diag}(1, -1, 1, -1, -1, -1, 1, 1), & Z_3 &= \text{diag}(1, 1, -1, -1, -1, 1, 1, -1), \\ Z_4 &= \text{diag}(1, -1, -1, -1, 1, 1, -1, 1), & Z_5 &= \text{diag}(1, -1, -1, 1, 1, -1, 1, -1), \\ Z_6 &= \text{diag}(1, -1, 1, 1, -1, 1, -1, -1), \end{aligned} \tag{3.41}$$

and, for instance,

$$X_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.42}$$

The rest of them can be easily worked out.

3.4. Complementary operators for two qutrits

Our final example is the case of two qutrits, described in the nine-dimensional Hilbert space \mathcal{H}_9 . The primitive element of the field \mathbb{F}_9 is a root of the following irreducible polynomial on \mathbb{F}_3 :

$$\theta^2 + \theta + 2 = 0, \tag{3.43}$$

so that the elements of \mathbb{F}_9 are

$$\{0, 1, 2, \alpha, 2\alpha + 1, 2\alpha + 2, 2, 2\alpha, \alpha + 2, \alpha + 1, \}, \tag{3.44}$$

and the additive characters are

$$\begin{aligned} \chi(0) &= 1, & \chi(\alpha) &= \bar{\omega}, & \chi(\alpha^2) &= 1, & \chi(\alpha^3) &= \bar{\omega}, & \chi(\alpha^4) &= \omega, \\ \chi(\alpha^5) &= \omega, & \chi(\alpha^6) &= 1, & \chi(\alpha^7) &= \omega, & \chi(\alpha^8) &= \bar{\omega}, \end{aligned} \tag{3.45}$$

where $\omega = e^{2\pi i/3}$ and the bar denotes complex conjugation. Using the same basis as (3.29), we have that the matrix of the Fourier transform is

$$F = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \bar{\omega} & \omega & \omega & 1 & \omega & \bar{\omega} & \bar{\omega} \\ 1 & \bar{\omega} & \omega & \omega & 1 & \omega & \bar{\omega} & \bar{\omega} & 1 \\ 1 & \omega & \omega & 1 & \omega & \bar{\omega} & \bar{\omega} & 1 & \bar{\omega} \\ 1 & \omega & 1 & \omega & \bar{\omega} & \bar{\omega} & 1 & \bar{\omega} & \omega \\ 1 & 1 & \omega & \bar{\omega} & \bar{\omega} & 1 & \bar{\omega} & \omega & \omega \\ 1 & \omega & \bar{\omega} & \bar{\omega} & 1 & \bar{\omega} & \omega & \omega & 1 \\ 1 & \bar{\omega} & \bar{\omega} & 1 & \bar{\omega} & \omega & \omega & 1 & \omega \\ 1 & \bar{\omega} & 1 & \bar{\omega} & \omega & \omega & 1 & \omega & \bar{\omega} \end{pmatrix}. \tag{3.46}$$

The operators Z_q take the form

$$\begin{aligned} Z_0 &= \text{diag}(1, \bar{\omega}, 1, \bar{\omega}, \omega, \omega, 1, \omega, \bar{\omega}), & Z_1 &= \text{diag}(1, 1, \bar{\omega}, \omega, \omega, 1, \omega, \bar{\omega}, \bar{\omega}), \\ Z_2 &= \text{diag}(1, \bar{\omega}, \omega, \omega, 1, \omega, \bar{\omega}, \bar{\omega}, 1), & Z_3 &= \text{diag}(1, \omega, \omega, 1, \omega, \bar{\omega}, \bar{\omega}, 1, \bar{\omega}), \\ Z_4 &= \text{diag}(1, \omega, 1, \omega, \bar{\omega}, \bar{\omega}, 1, \bar{\omega}, \omega), & Z_5 &= \text{diag}(1, 1, \omega, \bar{\omega}, \bar{\omega}, 1, \bar{\omega}, \omega, \omega), \\ Z_6 &= \text{diag}(1, \omega, \bar{\omega}, \bar{\omega}, 1, \bar{\omega}, \omega, \omega, 1), & Z_7 &= \text{diag}(1, \bar{\omega}, \bar{\omega}, 1, \bar{\omega}, \omega, \omega, 1, \omega). \end{aligned} \tag{3.47}$$

Note that Z_q ($q = 1, \dots, 7$) are obtained from Z_0 by a cyclic permutation of the diagonal elements, except the first element that always remains 1. The X_q are also easily constructed, and we have, e.g.

$$X_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.48}$$

The diagonal operators V_q have the form

$$\begin{aligned} V_0 &= \text{diag}(1, 1, \omega, 1, \bar{\omega}, 1, \omega, 1, \bar{\omega}), & V_1 &= \text{diag}(1, \bar{\omega}, \omega, \omega, \bar{\omega}, \bar{\omega}, \omega, \omega, \bar{\omega}), \\ V_2 &= \text{diag}(1, \omega, 1, \bar{\omega}, 1, \omega, 1, \bar{\omega}, 1), & V_3 &= \text{diag}(1, \omega, \omega, \bar{\omega}, \bar{\omega}, \omega, \omega, \bar{\omega}, \bar{\omega}), \\ V_4 &= \text{diag}(1, 1, \bar{\omega}, 1, \omega, 1, \bar{\omega}, 1, \omega), & V_5 &= \text{diag}(1, \omega, \bar{\omega}, \bar{\omega}, \omega, \omega, \bar{\omega}, \bar{\omega}, \omega), \\ V_6 &= \text{diag}(1, \bar{\omega}, 1, \omega, 1, \bar{\omega}, 1, \omega, 1), & V_7 &= \text{diag}(1, \bar{\omega}, \bar{\omega}, \omega, \omega, \bar{\omega}, \bar{\omega}, \omega, \omega). \end{aligned} \tag{3.49}$$

Note that all the complementary bases can be obtained directly from (3.24); i.e. applying $V_q F$ to the basis (3.44).

4. Multicomplementary operators as tensor products

One can establish an isomorphism between form (3.21) of complementary operators and its representation in terms of direct product of generalized Pauli operators (2.1). This isomorphism can be put forward by showing a one-to-one correspondence between the basis (3.2) and the coefficients of the expansion of the powers of the primitive element on, for instance, the polynomial basis, formed by $(1, \alpha, \alpha^2, \dots, \alpha^{n-1})$. In this way we get

$$\alpha^k \mapsto (c_0^{(k)}, c_1^{(k)}, \dots, c_{n-1}^{(k)}), \quad c_l^{(k)} \in \mathbb{Z}_p, \tag{4.1}$$

where

$$\alpha^k = \sum_{l=0}^{n-1} c_l^{(k)} \alpha^l. \tag{4.2}$$

This allows us to rewrite the basis (3.2) in the equivalent form

$$\{|0\rangle, |c_0^{(k)}, c_1^{(k)}, \dots, c_{n-1}^{(k)}\rangle\} \equiv \{|0\rangle, |c_0^{(k)}\rangle |c_1^{(k)}\rangle \cdots |c_{n-1}^{(k)}\rangle\}, \tag{4.3}$$

and the representation of (3.21) as a tensor product is now possible. This is due to the fact that \mathbb{F}_d is isomorphic to $\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, with n products.

It is worth noting that this isomorphism can also be settled in other bases. For example, the so-called normal basis, obtained finding an element $\beta \in \mathbb{F}_d$ such that

$$\{\beta, \beta^p, \dots, \beta^{p^{n-1}}\} \tag{4.4}$$

is a basis of \mathbb{F}_d , is quite useful in applications, since squaring a field element can be easily accomplished by a right cyclic shift. Of course, different bases lead to different coefficients $c_l^{(k)}$, different sets of commuting operators and different factorizations of these operators in tensor products.

4.1. Two qubits

To represent (3.30) and (3.32) as a tensor product we follow the described above and write the basis (3.26) in form (4.3). This yields

$$|0\rangle = |00\rangle, \quad |\alpha\rangle = |01\rangle, \quad |\alpha^2\rangle = |11\rangle, \quad |\alpha^3\rangle = |10\rangle. \quad (4.5)$$

Then,

$$\begin{aligned} Z_0 &\mapsto |00\rangle\langle 00| - |01\rangle\langle 01| - |11\rangle\langle 11| + |10\rangle\langle 10| = I \otimes \mathcal{Z}, \\ Z_1 &\mapsto |00\rangle\langle 00| - |01\rangle\langle 01| + |11\rangle\langle 11| - |10\rangle\langle 10| = \mathcal{Z} \otimes \mathcal{Z}, \\ Z_2 &\mapsto |00\rangle\langle 00| + |01\rangle\langle 01| - |11\rangle\langle 11| - |10\rangle\langle 10| = \mathcal{Z} \otimes I, \end{aligned} \quad (4.6)$$

where $\mathcal{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$. In a similar way the representation of the X_k is

$$\begin{aligned} X_0 &\mapsto |11\rangle\langle 01| + |01\rangle\langle 11| + |00\rangle\langle 10| + |10\rangle\langle 00| = \mathcal{X} \otimes I, \\ X_1 &\mapsto |00\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| + |01\rangle\langle 00| = I \otimes \mathcal{X}, \\ X_2 &\mapsto |10\rangle\langle 01| + |00\rangle\langle 11| + |01\rangle\langle 10| + |11\rangle\langle 00| = \mathcal{X} \otimes \mathcal{X}, \end{aligned} \quad (4.7)$$

where $\mathcal{X} = |0\rangle\langle 1| + |1\rangle\langle 0|$, so that $\mathcal{Z}\mathcal{X} = -\mathcal{X}\mathcal{Z}$.

Note that such an asymmetrical correspondence is a result of the Fourier transform (3.31), which is not factorized into a direct product of two Fourier operators. The other complementary operators are obtained as a simple product of (4.6) and (4.7):

$$\begin{aligned} (X_0Z_0, X_1Z_1, X_2Z_2) &\mapsto (\mathcal{X} \otimes \mathcal{Z}, \mathcal{Z} \otimes \mathcal{Y}, \mathcal{Y} \otimes \mathcal{X}), \\ (X_0Z_1, X_1Z_2, X_2Z_0) &\mapsto (\mathcal{Y} \otimes \mathcal{Z}, \mathcal{Z} \otimes \mathcal{X}, \mathcal{X} \otimes \mathcal{Y}), \\ (X_0Z_2, X_1Z_0, X_2Z_1) &\mapsto (\mathcal{Y} \otimes I, I \otimes \mathcal{Y}, \mathcal{Y} \otimes \mathcal{Y}), \end{aligned} \quad (4.8)$$

where $\mathcal{Y} = \mathcal{X}\mathcal{Z}$.

We complete this example using the normal basis $\{\beta, \beta^2\}$, with $\alpha = \beta, \alpha^2 = \beta^2, \alpha^3 = \beta + \beta^2$. Then

$$|0\rangle = |00\rangle, \quad |\alpha\rangle = |10\rangle, \quad |\alpha^2\rangle = |01\rangle, \quad |\alpha^3\rangle = |11\rangle, \quad (4.9)$$

and the new tensor product representatives are

$$\begin{aligned} Z_0 &\mapsto \mathcal{Z} \otimes \mathcal{Z}, & X_0 &\mapsto \mathcal{X} \otimes \mathcal{X}, \\ Z_1 &\mapsto \mathcal{Z} \otimes I, & X_1 &\mapsto \mathcal{X} \otimes I, \\ Z_2 &\mapsto I \otimes \mathcal{Z}, & X_2 &\mapsto I \otimes \mathcal{X}, \end{aligned} \quad (4.10)$$

while the rest of complementary operators are given by

$$\begin{aligned} (X_0Z_0, X_1Z_1, X_2Z_2) &\mapsto (\mathcal{Y} \otimes \mathcal{Y}, \mathcal{Y} \otimes I, I \otimes \mathcal{Y}), \\ (X_0Z_1, X_1Z_2, X_2Z_0) &\mapsto (\mathcal{Y} \otimes \mathcal{Z}, \mathcal{Z} \otimes \mathcal{X}, \mathcal{X} \otimes \mathcal{Y}), \\ (X_0Z_2, X_1Z_0, X_2Z_1) &\mapsto (\mathcal{Z} \otimes \mathcal{X}, \mathcal{Y} \otimes \mathcal{X}, \mathcal{X} \otimes \mathcal{Z}). \end{aligned} \quad (4.11)$$

The symmetrical aspect of (4.10) is a consequence of the factorization of Fourier transform in this basis. In fact, we have

$$F = F_2 \otimes F_2, \quad (4.12)$$

with

$$F_2 = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|). \quad (4.13)$$

4.2. Three qubits

Mapping to tensor product form is obtained for this case in a manner similar to \mathbb{F}_4 , by establishing a correspondence between the bases (3.2) and (4.3). Taking into account (3.37) we obtain (by expanding over the polynomial basis $\{1, \alpha, \alpha^2\}$):

$$\begin{aligned} |0\rangle &= |000\rangle, & |\alpha\rangle &= |010\rangle, & |\alpha^2\rangle &= |001\rangle, & |\alpha^3\rangle &= |110\rangle, \\ |\alpha^4\rangle &= |011\rangle, & |\alpha^5\rangle &= |111\rangle, & |\alpha^6\rangle &= |101\rangle, & |\alpha^7\rangle &= |100\rangle. \end{aligned} \quad (4.14)$$

In this way one gets

$$\begin{aligned} Z_0 &\mapsto Z \otimes I \otimes I, & X_0 &\mapsto X \otimes I \otimes I; \\ Z_1 &\mapsto I \otimes I \otimes Z, & X_1 &\mapsto I \otimes X \otimes I; \\ Z_2 &\mapsto I \otimes Z \otimes I, & X_2 &\mapsto I \otimes I \otimes X; \\ Z_3 &\mapsto Z \otimes I \otimes Z, & X_3 &\mapsto X \otimes X \otimes I; \\ Z_4 &\mapsto I \otimes Z \otimes Z, & X_4 &\mapsto I \otimes X \otimes X; \\ Z_5 &\mapsto Z \otimes Z \otimes Z, & X_5 &\mapsto X \otimes X \otimes X; \\ Z_6 &\mapsto Z \otimes Z \otimes I, & X_6 &\mapsto X \otimes I \otimes X, \end{aligned} \quad (4.15)$$

where the operators Z and X are those of the two-qubit example.

The other sets of commutative operators are as follows:

$$\begin{aligned} \{X_q Z_q\} &\mapsto (\mathcal{Y}II, I\mathcal{X}Z, IZ\mathcal{X}, \mathcal{Y}\mathcal{X}Z, I\mathcal{Y}\mathcal{Y}, \mathcal{Y}\mathcal{Y}\mathcal{Y}, \mathcal{Y}Z\mathcal{X}), \\ \{X_q Z_{q+1}\} &\mapsto (\mathcal{X}IZ, I\mathcal{Y}I, ZI\mathcal{Y}, \mathcal{X}\mathcal{Y}Z, Z\mathcal{Y}\mathcal{Y}, \mathcal{Y}\mathcal{Y}\mathcal{X}, \mathcal{Y}I\mathcal{X}), \\ \{X_q Z_{q+2}\} &\mapsto (\mathcal{X}ZI, Z\mathcal{X}Z, IZ\mathcal{Y}, \mathcal{Y}\mathcal{Y}Z, Z\mathcal{Y}\mathcal{X}, \mathcal{Y}\mathcal{X}\mathcal{X}, \mathcal{X}I\mathcal{Y}), \\ \{X_q Z_{q+3}\} &\mapsto (\mathcal{Y}IZ, I\mathcal{Y}Z, ZZ\mathcal{Y}, \mathcal{Y}\mathcal{Y}I, Z\mathcal{X}\mathcal{X}, \mathcal{X}\mathcal{X}\mathcal{Y}, \mathcal{X}Z\mathcal{X}), \\ \{X_q Z_{q+4}\} &\mapsto (\mathcal{X}ZZ, Z\mathcal{Y}Z, ZZ\mathcal{X}, \mathcal{Y}\mathcal{X}I, I\mathcal{X}\mathcal{Y}, \mathcal{X}\mathcal{Y}\mathcal{X}, \mathcal{Y}I\mathcal{Y}), \\ \{X_q Z_{q+5}\} &\mapsto (\mathcal{Y}ZZ, Z\mathcal{Y}I, ZI\mathcal{X}, \mathcal{X}\mathcal{X}Z, I\mathcal{Y}\mathcal{X}, \mathcal{Y}\mathcal{X}\mathcal{Y}, \mathcal{X}Z\mathcal{Y}), \\ \{X_q Z_{q+6}\} &\mapsto (\mathcal{Y}ZI, Z\mathcal{X}I, I\mathcal{Y}\mathcal{Y}, \mathcal{X}\mathcal{Y}I, Z\mathcal{X}\mathcal{Y}, \mathcal{X}\mathcal{Y}\mathcal{Y}, \mathcal{Y}Z\mathcal{Y}), \end{aligned} \quad (4.16)$$

where $q = 0, \dots, 6$ and we have omitted the symbol \otimes to simplify the writing. Again the Fourier transform is not factorized in this basis, but it is possible to find a different basis where F factorizes.

4.3. Two qutrits

The representation of Z_q and X_q operators in terms of tensor product is obtained using the expansion in the polynomial basis $\{1, \alpha\}$:

$$\begin{aligned} |0\rangle &= |00\rangle, & |\alpha\rangle &= |01\rangle, & |\alpha^2\rangle &= |12\rangle, & |\alpha^3\rangle &= |22\rangle, & |\alpha^4\rangle &= |20\rangle, \\ |\alpha^5\rangle &= |02\rangle, & |\alpha^6\rangle &= |21\rangle, & |\alpha^7\rangle &= |11\rangle, & |\alpha^8\rangle &= |10\rangle, \end{aligned} \quad (4.17)$$

which leads to

$$\begin{aligned} Z_0 &\mapsto Z^2 Z^2; & X_0 &\mapsto X I; \\ Z_1 &\mapsto Z^2 I, & X_1 &\mapsto I X; \\ Z_2 &\mapsto I Z^2, & X_2 &\mapsto X X^2; \\ Z_3 &\mapsto Z^2 Z, & X_3 &\mapsto X^2 X^2; \\ Z_4 &\mapsto Z Z, & X_4 &\mapsto X^2 I; \\ Z_5 &\mapsto Z I, & X_5 &\mapsto I X^2; \\ Z_6 &\mapsto I Z, & X_6 &\mapsto X^2 X; \\ Z_7 &\mapsto Z Z^2, & X_7 &\mapsto X X, \end{aligned} \quad (4.18)$$

where now

$$\mathcal{Z} = |0\rangle\langle 0| + \omega|1\rangle\langle 1| + \bar{\omega}|2\rangle\langle 2|, \quad \mathcal{X} = |1\rangle\langle 0| + |2\rangle\langle 1| + |0\rangle\langle 2|, \quad (4.19)$$

so (2.1) holds, since $\mathcal{Z}\mathcal{X} = \omega\mathcal{X}\mathcal{Z}$.

The sets of commutative operators are

$$\begin{aligned} \{X_q Z_q\} &\mapsto (\mathcal{W}\mathcal{Z}^2, \mathcal{Z}^2\mathcal{X}, \mathcal{X}\mathcal{Y}^2, \mathcal{Y}^2\mathcal{W}^2, \mathcal{W}^2\mathcal{Z}, \mathcal{Z}\mathcal{X}^2, \mathcal{X}^2\mathcal{Y}, \mathcal{Y}\mathcal{W}), \\ \{X_q Z_{q+1}\} &\mapsto (\mathcal{W}I, I\mathcal{W}, \mathcal{W}\mathcal{W}^2, \mathcal{W}^2\mathcal{W}^2, \mathcal{W}^2I, I\mathcal{W}^2, \mathcal{W}^2\mathcal{W}, \mathcal{W}\mathcal{W}), \\ \{X_q Z_{q+2}\} &\mapsto (\mathcal{X}\mathcal{Z}^2, \mathcal{Z}^2\mathcal{Y}, \mathcal{Y}\mathcal{W}^2, \mathcal{W}^2\mathcal{X}^2, \mathcal{X}^2\mathcal{Z}, \mathcal{Z}\mathcal{Y}^2, \mathcal{Y}^2\mathcal{W}, \mathcal{W}\mathcal{X}), \\ \{X_q Z_{q+3}\} &\mapsto (\mathcal{W}\mathcal{Z}, \mathcal{Z}\mathcal{Y}, \mathcal{Y}\mathcal{X}^2, \mathcal{X}^2\mathcal{W}^2, \mathcal{W}^2\mathcal{Z}^2, \mathcal{Z}^2\mathcal{Y}^2, \mathcal{Y}^2\mathcal{X}, \mathcal{X}\mathcal{W}), \\ \{X_q Z_{q+4}\} &\mapsto (\mathcal{Y}\mathcal{Z}, \mathcal{Z}\mathcal{X}, \mathcal{X}\mathcal{W}^2, \mathcal{W}^2\mathcal{Y}^2, \mathcal{Y}^2\mathcal{Z}^2, \mathcal{Z}^2\mathcal{X}^2, \mathcal{X}^2\mathcal{W}, \mathcal{W}\mathcal{Y}), \\ \{X_q Z_{q+5}\} &\mapsto (\mathcal{Y}I, I\mathcal{Y}, \mathcal{Y}\mathcal{Y}^2, \mathcal{Y}^2\mathcal{Y}^2, \mathcal{Y}^2I, I\mathcal{Y}^2, \mathcal{Y}^2\mathcal{Y}, \mathcal{Y}\mathcal{Y}), \\ \{X_q Z_{q+6}\} &\mapsto (\mathcal{X}\mathcal{Z}, \mathcal{Z}\mathcal{W}, \mathcal{W}\mathcal{Y}^2, \mathcal{Y}^2\mathcal{X}^2, \mathcal{X}^2\mathcal{Z}^2, \mathcal{Z}^2\mathcal{W}^2, \mathcal{W}^2\mathcal{Y}, \mathcal{Y}\mathcal{X}), \\ \{X_q Z_{q+7}\} &\mapsto (\mathcal{Y}\mathcal{Z}^2, \mathcal{Z}^2\mathcal{W}, \mathcal{W}\mathcal{X}^2, \mathcal{X}^2\mathcal{Y}^2, \mathcal{Y}^2\mathcal{Z}, \mathcal{Z}\mathcal{W}^2, \mathcal{W}^2\mathcal{X}), \end{aligned} \quad (4.20)$$

where $q = 0, \dots, 7$, $\mathcal{Y} = \mathcal{X}\mathcal{Z}$, $\mathcal{W} = \mathcal{X}\mathcal{Z}^2$ and we have omitted global phases that appear in the product of operators. Note that the above sets of commuting operators are different from those listed in [34].

5. Concluding remarks

In this paper, we have solved the problem of existence and construction of sets of MUBs in composite dimensions. Inspired by the the approach developed in [31], in which an interesting explicit construction was shown for prime dimension, we have used algebraic field extensions to produce a solution for composite dimensions. Although other constructive algorithms for solving the MUBs problem in composite dimension have appeared, the one presented here does not resort to dual basis, and so is completely analogous to the prime-dimensional case. We have also provided a simple scheme to cast the sets of MUBs observables as tensor products of generalized Pauli matrices. These tensor products can be quite different when we represent the finite field in different bases.

Another major advantage of our approach relies on the use of the finite Fourier transform and a diagonal shift operator as maps between maximal classes of commuting operators. This is in full agreement with our understanding of complementarity in the infinite-dimensional case.

Of course, there are still open questions: the non-prime dimensional case is a challenging issue, or what happens in the limit of high dimensions. In any case, the picture developed in this paper is a valuable tool to deal with concepts such as entanglement or separability in finite-dimensional systems.

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