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General approach to $\mathfrak{SU}(n)$ quasi-distribution functions

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Online at stacks.iop.org/JPhysA/43/402001**Abstract**

We propose an operational form for the kernel of a mapping between an operator acting in a Hilbert space of a quantum system with an $\mathfrak{SU}(n)$ symmetry group and its symbol in the corresponding classical phase space. For symmetric irreps of $\mathfrak{SU}(n)$, this mapping is bijective. We briefly discuss complications that will occur in the general case.

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1. Introduction

There is renewed interest in physical systems involving higher symmetries, motivated in part by recent experimental and theoretical results on physical systems involving such higher symmetries. Examples include work on atomic and molecular systems [1], n -qubits in symmetric $SU(2^n)$ state space, three-well Bose–Einstein condensates [2] and general qudit systems, such as collections of distinguishable d -level atoms.

These developments motivate a proper formulation of phase-space methods adapted to higher symmetries. Phase-space methods in quantum mechanics were pioneered by Wigner [3], and his work has been the seed for several hugely successful approaches having the common objective of mapping quantum-mechanical operators, defined in an abstract Hilbert space, to complex-valued functions in a classical phase space appropriate for the system under consideration [4–11]. Several different methods for Wigner-like mapping, applicable to a wide class of continuous (Lie type) and discrete groups, have also appeared in the recent literature [12–15].

Considerable insight into the possible mappings is provided by the axiomatic Stratonovich–Weyl approach [16–18], in which a one-to-one correspondence between an operator \hat{X} and its phase-space symbol W_X is established by restricting mappings to those having ‘reasonable’ physical properties: (1) covariance, (2) hermiticity, (3) traciality and (4) normalization.

Within this framework, and using general ideas first advanced by Berezin, an elegant form of the mapping kernel construction was proposed in [13], where an explicit form for the s -ordered kernel $\hat{w}^{(s)}(\Omega)$, automatically satisfying the conditions above, was also constructed. The apparent simplicity of the formulation of [13] hides several technical complications. The most unfortunate is that for higher groups, harmonic functions typically depend on additional parameters: in $\mathfrak{SU}(3)$ for instance, harmonic functions contain five parameters [19] so one must be eliminated ‘by hand’.

In this paper we propose an alternate but operational form of the s -ordered kernel, valid for $\mathfrak{SU}(n)$ irreducible representations (irreps) of the type $(\lambda, 0, \dots, 0)$. The particular cases of $\mathfrak{SU}(2)$ and $\mathfrak{SU}(3)$ are given in enough details for the generalization to any symmetric representation of $\mathfrak{SU}(n)$.

2. General setup and comments for the $(\lambda, 0, \dots, 0)$ irreps

2.1. Notation

Throughout this paper we will use the shorthand λ to mean the irrep $(\lambda, 0, \dots, 0)$ of $\mathfrak{SU}(n)$. Exceptions to this abuse of notation will be noted explicitly or will be clear from context.

Suppose the Hilbert space \mathbb{H} for a quantum system carries a unitary irreducible representation λ of a compact Lie group \mathfrak{G} , which we take henceforth to be $\mathfrak{SU}(n)$. $\Lambda(g)$ is the matrix realization for irrep λ of the element $g \in \mathfrak{G}$, so that $\Lambda(g)$ acts by linear transformations on \mathbb{H} .

For $\mathfrak{SU}(n)$ irreps of the type $(\lambda, 0, \dots, 0)$, states will be written as $|\lambda; \nu\rangle$, where the i th component of the weight $\nu = (\nu_1, \dots, \nu_{n-1})$ is the eigenvalue of the i th Cartan element on the state:

$$\hat{h}_i |\lambda; \nu\rangle = \nu_i |\lambda; \nu\rangle. \quad (1)$$

The highest weight $|\lambda; \text{h.w.}\rangle$ of irrep λ is invariant under a subgroup \mathfrak{H} of \mathfrak{G} , and the phase space for the corresponding classical system is isomorphic to the coset $\mathfrak{M} = \mathfrak{G}/\mathfrak{H}$ [21]. \mathfrak{G} acts on \mathfrak{M} by canonical transformations.

Operators acting on the Hilbert space carrying irrep λ will transform according to the irrep $\lambda \otimes \lambda^*$, where λ^* is the irrep conjugate to λ .

The product $\lambda \otimes \lambda^*$ is reducible so, in addition to the labeling of states in irrep λ , we must consider the labeling of tensors from a general irrep $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{n-1})$. A tensor $\hat{T}_{\sigma; \alpha I_\alpha}^\lambda$ is given explicitly by

$$\hat{T}_{\sigma; \gamma I_\gamma}^\lambda = \sum_{\alpha \beta} |\lambda; \alpha\rangle \langle \lambda; \beta^* | \tilde{C}_{\lambda \alpha; \lambda^* \beta^*}^{\sigma \gamma I_\gamma}, \quad (2)$$

where λ^* is the irrep conjugate to λ , β^* is the weight conjugate to β and σ is an irrep in the decomposition of $\lambda \otimes \lambda^*$. Note that, for $\lambda \equiv (\lambda, 0, \dots, 0)$, σ occurs at most once in $\lambda \otimes \lambda^*$. The coefficients $\tilde{C}_{\lambda \alpha; \lambda^* \beta^*}^{\sigma \nu I_\nu}$ satisfy the orthogonality relation

$$\sum_{\alpha \beta} (\tilde{C}_{\lambda \alpha; \lambda^* \beta^*}^{\sigma' \nu' I_{\nu'}})^* \tilde{C}_{\lambda \alpha; \lambda^* \beta^*}^{\sigma \nu I_\nu} = \delta_{\sigma \sigma'} \delta_{\nu \nu'} \delta_{I_\nu I_{\nu'}} \quad (3)$$

and are elements of a unitary matrix. The tensors satisfy

$$[\hat{h}_i, \hat{T}_{\sigma; \alpha I_\alpha}^\lambda] = \alpha_i \hat{T}_{\sigma; \alpha I_\alpha}^\lambda. \quad (4)$$

In general, some weights in σ will occur multiple times; I_α distinguishes between multiple occurrences of the weight α . Equation (3) implies that the tensors are trace orthogonal over σ , α and I_α .

2.2. Construction of the kernel

According to the Stratonovich–Weyl method, we construct a Hermitian kernel $\hat{w}^{(s)}(\Omega)$, $\Omega \in \mathfrak{M}$, that implements a mapping between operators \hat{X} acting in \mathbb{H} and symbols $W_X \in \mathfrak{M}$ via

$$\begin{aligned} \hat{X} &\xrightarrow{\hat{w}^{(s)}(\Omega)} W_X^{(s)}(\Omega) = \text{Tr}(\hat{X} \hat{w}^{(s)}(\Omega)), \\ W_X^{(s)}(\Omega) &\xrightarrow{\hat{w}^{(-s)}(\Omega)} \hat{X} = \int d\Omega \hat{w}^{(-s)}(\Omega) W_X^{(s)}(\Omega). \end{aligned} \quad (5)$$

Here, s is a (continuous) ordering parameter, which takes values $s = -1, 0$ and 1 for the normal, symmetric and anti-normal ordering of operators, respectively.

The kernel $\hat{w}^{(s)}(\Omega)$ can always be written in the form

$$\hat{w}^{(s)}(\Omega) = \Lambda(\Omega) \hat{P}^{(s)} \Lambda^\dagger(\Omega), \quad \Omega \in \mathfrak{M}. \quad (6)$$

The essential information about the mapping is contained in the operator $\hat{P}^{(s)}$, which we write in the integral form

$$\hat{P}^{(s)} = \int d\omega e^{i\omega \hat{h}_1} f^{(s)}(\omega), \quad (7)$$

with $f^{(s)}(\omega)$ being a scalar function to be determined and \hat{h}_1 the \mathfrak{H} -invariant Cartan element:

$$\Lambda(\bar{g}) \hat{h}_1 \Lambda^\dagger(\bar{g}) = \hat{h}_1, \quad \bar{g} \in \mathfrak{H}. \quad (8)$$

This form for $\hat{P}^{(s)}$ guarantees the covariance of $\hat{w}^{(s)}(\Omega)$ under transformations from the coset $\mathfrak{G}/\mathfrak{H}$:

$$\Lambda(\tilde{\Omega}) \hat{w}^{(s)}(\Omega) \Lambda^\dagger(\tilde{\Omega}) = \hat{w}^{(s)}(\tilde{\Omega}\Omega), \quad \tilde{\Omega} \in \mathfrak{G}/\mathfrak{H}. \quad (9)$$

To determine $f^{(s)}$ we proceed as follows. For symmetric irreps of $\mathfrak{SU}(n)$, the highest weight vector is $\mathfrak{U}(n-1)$ invariant. This forces \hat{h}_1 (in the defining $n \times n$ irrep) to be

$$\hat{h}_1 \equiv \text{diag}(n-1, -1, \dots, -1). \quad (10)$$

Using the tensors defined in equation (2), one then expands $e^{i\omega \hat{h}_1}$ as

$$e^{i\omega \hat{h}_1} = \sum_{\sigma} \tilde{\chi}_{\sigma}^{\lambda}(\omega) \hat{T}_{\sigma; \mathbf{00}}^{\lambda}, \quad \tilde{\chi}_{\sigma}^{\lambda}(\omega) = \text{Tr}[e^{i\omega \hat{h}_1} (\hat{T}_{\sigma; \mathbf{00}}^{\lambda})^\dagger] \quad (11)$$

of zero-weight tensors $\hat{T}_{\sigma; \mathbf{00}}^{\lambda}$ that must carry the trivial $I_0 \equiv 0$ irrep of $\mathfrak{H} = \mathfrak{U}(n-1)$.

For later convenience, we note that one can also expand $\Lambda(\Omega) \hat{T}_{\sigma; \mathbf{00}}^{\lambda} \Lambda^\dagger(\Omega)$ to get

$$\hat{w}^{(s)}(\Omega) = \sum_{\sigma} F_{\sigma}^{(s)} \sum_{\beta I_{\beta}} D_{\beta I_{\beta}; \mathbf{00}}^{\sigma}(\Omega) \hat{T}_{\sigma; \beta I_{\beta}}^{\lambda}, \quad (12)$$

$$F_{\sigma}^{(s)} = \int d\omega \tilde{\chi}_{\sigma}^{\lambda}(\omega) f^{(s)}(\omega), \quad (13)$$

with

$$D_{\beta I_{\beta}; \mathbf{00}}^{\sigma}(\Omega) \equiv \langle \sigma; \beta I_{\beta} | \Lambda(\Omega) | \sigma; \mathbf{00} \rangle, \quad (14)$$

an $\mathfrak{SU}(n)$ group function for the irrep σ [20].

To accommodate the requirement that Hermitian operators are mapped into real functions, we demand that $\hat{w}^{(s)}(\Omega)$ be Hermitian, which in turn implies

$$\hat{w}^{(s)}(\Omega) = (\hat{w}^{(s)}(\Omega))^\dagger \Rightarrow (f^{(s)}(\omega))^* = f^{(s)}(-\omega). \quad (15)$$

To guarantee the invertibility of the map, we impose a *traciality* condition on $\hat{w}^{(s)}(\Omega)$, expressed by

$$\text{Tr}(\hat{w}^{(s)}(\Omega')^\dagger \hat{w}^{(-s)}(\Omega)) = N^\lambda \Delta(\Omega', \Omega), \tag{16}$$

where N^λ is the proportionality constant and $\Delta(\Omega', \Omega)$ is the self-reproducing kernel

$$\int d\Omega z(\Omega) \Delta(\Omega', \Omega) = z(\Omega'), \quad \Omega, \Omega' \in \mathfrak{M}, \tag{17}$$

valid for any function z on \mathfrak{M} .

The traciality condition can be expanded to produce

$$N^\lambda \Delta(\Omega', \Omega) = \sum_{\sigma} (F_{\sigma}^{(s)})^* F_{\sigma}^{(-s)} \sum_{\alpha I_{\alpha}} (D_{\alpha I_{\alpha}, \mathbf{00}}^{\sigma}(\Omega'))^* D_{\alpha I_{\alpha}, \mathbf{00}}^{\sigma}(\Omega), \tag{18}$$

where the orthogonality condition (3) has been used. This is a restriction on $F_{\sigma}^{(s)}$ and thus on the functions $f^{(s)}(\omega)$, which expand $F_{\sigma}^{(s)}$ via equation (13).

Since the functions $\{D_{\beta I_{\beta}, \mathbf{00}}^{\sigma}(\Omega)\}$ are complete and orthogonal under integration over the coset, the reproducing kernel can be written as

$$\Delta(\Omega', \Omega) = \sum_{\sigma \alpha I_{\alpha}} \left(\frac{\text{vol}(\mathfrak{M})}{\text{dim}(\sigma)} \right) (D_{\alpha I_{\alpha}, \mathbf{00}}^{\sigma}(\Omega'))^* D_{\alpha I_{\alpha}, \mathbf{00}}^{\sigma}(\Omega), \tag{19}$$

where $\text{vol}(\mathfrak{M})$ is related to the volume of \mathfrak{M} calculated using the invariant measure $d\Omega$ and $\text{dim}(\sigma)$ is the dimension of irrep σ .

Combining equations (18) and (19), and observing that the irrep σ occurs at most once in $\lambda \otimes \lambda^*$, we find a cross-condition on dual $F_{\sigma}^{(s)}$'s:

$$N^\lambda \frac{\text{vol}(\mathfrak{M})}{\text{dim}(\sigma)} = (F_{\sigma}^{(s)})^* F_{\sigma}^{(-s)}. \tag{20}$$

The normalization property is simply

$$\text{Tr}(\hat{w}^{(s)}(\Omega)) = 1 = F_0^{(s)} \sqrt{\text{dim}(\lambda)}, \tag{21}$$

as all tensors but the $\sigma = 0$ (scalar) representation in $\lambda \otimes \lambda^*$ are traceless. We can use this in equation (20) to obtain

$$N^\lambda = \frac{1}{\text{vol}(\mathfrak{M}) \text{dim}(\lambda)}. \tag{22}$$

and feed this back in equation (20) to obtain more generally

$$(F_{\sigma}^{(s)})^* F_{\sigma}^{(-s)} = \frac{1}{\text{dim}(\lambda) \text{dim}(\sigma)}. \tag{23}$$

Next, we make the important observation that, for $s = -1$, the usual Q -function kernel should be recovered:

$$\hat{w}^{(-1)}(\Omega) = \Lambda(\Omega)|\lambda; \text{h.w.}\rangle \langle \lambda; \text{h.w.}| \Lambda^\dagger(\Omega). \tag{24}$$

This implies the ‘boundary condition’

$$\hat{P}^{(-1)} = |\lambda; \text{h.w.}\rangle \langle \lambda; \text{h.w.}| = \int d\omega e^{i\omega \hat{h}_1} f^{(-1)}(\omega) = \sum_{\sigma} F_{\sigma}^{(-1)} \hat{T}_{\sigma; \mathbf{00}}^{\lambda}, \tag{25}$$

from which

$$F_{\sigma}^{(-1)} = \tilde{C}_{\lambda \text{h.w.}; \lambda^* \text{h.w.}^*}^{\sigma \mathbf{00}} = \langle \lambda; \text{h.w.}| \hat{T}_{\sigma; \mathbf{00}}^{\lambda} |\lambda; \text{h.w.}\rangle. \tag{26}$$

Combining equations (23) and (26) yields

$$F_{\sigma}^{(1)} = \frac{1}{\dim(\lambda) \dim(\sigma) (\tilde{C}_{\lambda \text{ h.w.}; \lambda^* \text{ h.w.}^*}^{\sigma \mathbf{00}})^*}. \quad (27)$$

We can interpolate to arbitrary s by defining

$$F_{\sigma}^{(s)} = \frac{(\tilde{C}_{\lambda \text{ h.w.}; \lambda^* \text{ h.w.}^*}^{\sigma \mathbf{00}})^{-s}}{[\dim(\lambda) \dim(\sigma)]^{(s+1)/2}}. \quad (28)$$

We are now in a position to determine $f^{(s)}$. If we suppose

$$f^{(-1)}(\omega) = \sum_{\sigma} c_{\sigma}^{(-1)} (\tilde{\chi}_{\sigma}^{\lambda}(\omega))^*, \quad (29)$$

then

$$\tilde{C}_{\lambda \text{ h.w.}; \lambda^* \text{ h.w.}^*}^{\sigma \mathbf{00}} = \sum_{\sigma'} g_{\sigma \sigma'} c_{\sigma'}^{(-1)}, \quad g_{\sigma \sigma'} = \int d\omega \tilde{\chi}_{\sigma}^{\lambda}(\omega) (\tilde{\chi}_{\sigma'}^{\lambda}(\omega))^*. \quad (30)$$

Since

$$\int d\omega (f^{(-1)}(\omega))^* f^{(-1)}(\omega) = \sum_{\sigma, \sigma'} c_{\sigma}^{(-1)} g_{\sigma \sigma'} (c_{\sigma'}^{(-1)})^* > 0, \quad (31)$$

the overlap matrix $g_{\sigma \sigma'}$ is necessarily invertible. Writing $g^{\mu\nu}$ as $(g)_{\mu\nu}^{-1}$ we can solve for $c_{\sigma'}^{(-1)}$ and, extending the expansion of f to any s , obtain

$$f^{(s)}(\omega) = \sum_{\sigma} c_{\sigma}^{(s)} (\tilde{\chi}_{\sigma}^{\lambda}(\omega))^*, \quad c_{\sigma}^{(s)} = \sum_{\mu} g^{\sigma\mu} F_{\mu}^{(s)}. \quad (32)$$

We can combine all the relevant equations and obtain the following simplified form for $\hat{P}^{(s)}$:

$$\hat{P}^{(s)} = \int d\omega e^{i\omega \hat{h}_1} \left(\sum_{\sigma'} c_{\sigma'}^{(s)} (\tilde{\chi}_{\sigma'}^{\lambda}(\omega))^* \right) \quad (33)$$

$$= \sum_{\sigma \sigma'} \hat{T}_{\sigma; \mathbf{00}}^{\lambda} \left(\int d\omega \tilde{\chi}_{\sigma}^{\lambda}(\omega) (\tilde{\chi}_{\sigma'}^{\lambda}(\omega))^* \right) c_{\sigma'}^{(s)} \quad (34)$$

$$= \sum_{\sigma \sigma'} \hat{T}_{\sigma; \mathbf{00}}^{\lambda} g_{\sigma \sigma'} \sum_{\mu} g^{\sigma'\mu} F_{\mu}^{(s)} \quad (35)$$

$$= \sum_{\sigma \mu} \hat{T}_{\sigma; \mathbf{00}}^{\lambda} \left(\sum_{\sigma'} g_{\sigma \sigma'} g^{\sigma'\mu} \right) F_{\mu}^{(s)} \quad (36)$$

$$= \sum_{\sigma} \hat{T}_{\sigma; \mathbf{00}}^{\lambda} \frac{(\tilde{C}_{\lambda \text{ h.w.}; \lambda^* \text{ h.w.}^*}^{\sigma \mathbf{00}})^{-s}}{[\dim(\lambda) \dim(\sigma)]^{(s+1)/2}}. \quad (37)$$

3. The case of $\mathfrak{SU}(2)$

Basis states for the irrep j are taken as usual to be $\{|jm\rangle\}$, such that

$$\hat{S}_z |jm\rangle = m |jm\rangle. \quad (38)$$

The highest weight of irrep j is $|jj\rangle$ and invariant (up to a phase) under transformations of the form $e^{i\omega\hat{S}_z}$. Thus, \mathfrak{H} is the $\mathfrak{U}(1)$ subgroup generated by $e^{i\omega\hat{S}_z}$ and $\mathfrak{M} = \mathfrak{SU}(2)/\mathfrak{U}(1) \sim S^2$. Coset representatives are taken to be

$$\Lambda(\Omega) = R_z(\alpha)R_y(\beta) \equiv e^{i\alpha\hat{S}_z} e^{i\beta\hat{S}_y}, \quad (39)$$

and $\hat{P}^{(s)}$ is written simply as the integral

$$\hat{P}^{(s)} = \int d\omega e^{i\omega\hat{S}_z} f^{(s)}(\omega). \quad (40)$$

The trace-orthogonal $\mathfrak{SU}(2)$ tensor operators are

$$\hat{T}_{LM}^j = \sum_{mm'} |jm\rangle \langle jm'| C_{jm;j-m'}^{LM} (-1)^{j-m}, \quad (41)$$

where $C_{jm;j-m'}^{LM}$ is the usual Clebsch–Gordan (CG) coefficient for $\mathfrak{SU}(2)$. We expand

$$e^{i\omega\hat{S}_z} = \sum_L \tilde{\chi}_L^j(\omega) \hat{T}_{L0}^j, \quad \tilde{\chi}_L^j(\omega) = \sum_m (C_{jm;j-m}^{L0} (-1)^{j-m}) e^{i\omega m}, \quad (42)$$

in terms of the zero-weight tensors \hat{T}_{L0}^j . With this, equations (19) and (22) specialize to

$$\Delta(\Omega', \Omega) = \sum_L \left(\frac{4\pi}{2L+1} \right) (D_{M0}^L(\Omega'))^* D_{M0}^L(\Omega), \quad (43)$$

$$N^j = \frac{1}{4\pi(2j+1)}, \quad (44)$$

with D_{M0}^L being the usual Wigner D -function that is proportional to $Y_{LM}^*(\beta, \alpha)$.

For $\mathfrak{SU}(2)$, the generalized characters $\tilde{\chi}_L^j(\omega)$ satisfy the orthogonality relation $g_{L/L} = 2\pi \delta_{L/L}$. A closed form solution is possible for any j since, by equation (32), $c_L^{(s)} = (2\pi)^{-1} F_L^{(s)}$, and

$$F_L^{(s)} = \frac{(C_{jm,j-m}^{L0})^{-s}}{[(2j+1)(2L+1)]^{(s+1)/2}}, \quad (45)$$

according to equation (28). Hence, the final expression for $f^{(s)}$ is

$$f^{(s)}(\omega) = \frac{1}{2\pi} \sum_L F_L^{(s)} (\tilde{\chi}_L^j(\omega))^*. \quad (46)$$

4. The case of $\mathfrak{SU}(3)$ irreps of the type $(\lambda, 0)$

We label states of the $\mathfrak{SU}(3)$ irrep $(\lambda, 0)$ by $|\lambda; \nu\rangle$. The weight $\nu \equiv (\nu_1 - \nu_2, \nu_2 - \nu_3)$ of the state is extracted from a triple $[\nu_1, \nu_2, \nu_3]$ on non-negative integers constrained by $\nu_1 + \nu_2 + \nu_3 = \lambda$. The irrep $(\lambda, 0)$ does not have weight multiplicities, i.e. each weight ν occurs at most once. In $(\lambda, 0)$, the multiplicity label $I \equiv I_{23}$ of [20] is redundant but given by $I_{23} = \frac{1}{2}(\nu_2 + \nu_3)$.

We denote by $(\lambda, 0)^* \equiv (0, \lambda)$ the irrep conjugate to $(\lambda, 0)$ and recall

$$(\lambda, 0) \otimes (\lambda, 0)^* = (0, 0) \oplus (1, 1) \oplus \dots \oplus (\lambda, \lambda) = \sum_{\sigma=0}^{\lambda} (\sigma, \sigma). \quad (47)$$

Following the template of [20], one shows that $\mathfrak{SU}(3)$ transformations can be factorized as

$$R(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma_1, \gamma_2) = R_{23}(\alpha_1, \beta_1, -\alpha_1) \cdot R_{12}(\alpha_2, \beta_2, -\alpha_2) \cdot R_{23}(\alpha_3, \beta_3, -\alpha_3) \cdot e^{-i\gamma_1(C_{11}-C_{22})} e^{-i\gamma_2(C_{22}-C_{33})}. \quad (48)$$

The highest weight is $|(\lambda, 0)(\lambda, 0)\rangle$. It is an $I_{23} = 0$ state invariant (up to a phase) under transformations of the subgroup $\mathfrak{H} = \mathfrak{U}_{23}(2)$ generated by $R_{23}(\alpha_3, \beta_3, -\alpha_3) e^{-i\gamma_1(C_{11}-C_{22})} e^{-i\gamma_2(C_{22}-C_{33})}$. In the fundamental representation $(1, 0)$, the matrices of $\mathfrak{U}_{23}(2)$ are of the form

$$\mathfrak{U}_{23}(2) \sim \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad (49)$$

with $*$ denoting a non-zero entry. Elements Ω in the coset $\mathfrak{SU}(3)/\mathfrak{U}_{23}(2)$ correspond to transformations of the form

$$\Lambda(\Omega) = R_{23}(\alpha_1, \beta_1, -\alpha_1)R_{12}(\alpha_2, \beta_2, -\alpha_2), \quad (50)$$

with parameter range $0 \leq \alpha_1, \alpha_2 \leq 2\pi, 0 \leq \beta_1, \beta_2 \leq \pi$. The measure on the coset is

$$d\Omega = \sin(\beta_1) \cos\left(\frac{1}{2}\beta_2\right) \sin^3\left(\frac{1}{2}\beta_2\right) d\alpha_1 d\beta_1 d\alpha_2 d\beta_2, \quad (51)$$

and the coset volume is $4\pi^2$.

The operator $\hat{P}^{(s)}$ of equation (7) is now

$$\hat{P}^{(s)} = \int d\omega e^{i\omega\hat{h}_1} f^{(s)}(\omega). \quad (52)$$

In the fundamental $(1, 0)$ irrep, we have

$$e^{i\omega\hat{h}_1} = \text{diag}(e^{2i\omega}, e^{-i\omega}, e^{-i\omega}). \quad (53)$$

The expansion of $e^{i\omega\hat{h}_1}$ will be a sum of the diagonal terms of the form

$$e^{i\omega\hat{h}_1} = \sum_{\sigma=0}^{\lambda} \tilde{\chi}_{(\sigma,\sigma)}^{(\lambda,0)}(\omega) \hat{T}_{(\sigma,\sigma),\mathbf{00}}^{(\lambda,0)}, \quad (54)$$

with $\mathbf{0}$ being the zero weight; there can be more than one copy of this weight in (σ, σ) , but only $I_{23} = 0$ appear so as to ensure \mathfrak{U}_{23} -invariance of $e^{i\omega\hat{h}_1}$. The coset functions are

$$D_{\nu I; \mathbf{00}}^{(\sigma,\sigma)}(\Omega) \equiv \langle(\sigma, \sigma)\nu I | R_{23}(\alpha_1, \beta_1, -\alpha_1)R_{12}(\alpha_2, \beta_2, -\alpha_2) |(\sigma, \sigma)\mathbf{00}\rangle \quad (55)$$

with Ω now in $\mathfrak{M} = \mathfrak{SU}(3)/\mathfrak{U}(2)$.

Since $\dim(\lambda, 0) = \frac{1}{2}(\lambda+1)(\lambda+2)$ and $\dim(\sigma, \sigma) = (\sigma+1)^3$, equations (19) and (22) now specialize, for $\mathfrak{SU}(3)$ irreps of the type $(\lambda, 0)$, to

$$\Delta(\Omega', \Omega) = \sum_{\nu I \sigma} \frac{4\pi^2}{(\sigma+1)^3} (D_{\nu I; \mathbf{00}}^{(\sigma,\sigma)}(\Omega'))^* D_{\nu I; \mathbf{00}}^{(\sigma,\sigma)}(\Omega), \quad (56)$$

$$N^\lambda = \frac{1}{4\pi^2 \frac{1}{2}(\lambda+1)(\lambda+2)}. \quad (57)$$

In the case of $\mathfrak{SU}(3)$, the various $\mathfrak{U}_{23}(2)$ subspaces will typically contain more than one weight. This in turn leads to non-trivial overlap relations $g_{\sigma\sigma'}$ between various $\tilde{\chi}_{(\sigma,\sigma)}^{(\lambda,0)}(\omega)$'s, defined by equation (54).

4.1. The case of $(1, 0)$

Direct calculation shows the zero-weight tensors related to uncoupled projectors by

$$\begin{pmatrix} \hat{T}_{(0,0); \mathbf{00}}^{(1,0)} \\ \hat{T}_{(1,1); \mathbf{01}}^{(1,0)} \\ \hat{T}_{(1,1); \mathbf{00}}^{(1,0)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} |(1, 0)(10)\rangle \langle(1, 0)(10)| \\ |(1, 0)(-11)\rangle \langle(1, 0)(-11)| \\ |(1, 0)(0-1)\rangle \langle(1, 0)(0-1)| \end{pmatrix}, \quad (58)$$

with $\mathbf{0} = (00)$ denoting the zero weight. The highest weight projector is given by $|(1, 0)(10)\rangle\langle(1, 0)(10)|$, and equation (28) gives

$$F_0^{(s)} = \frac{(\sqrt{1/3})^{-s}}{3^{(s+1)/2}}, \quad F_1^{(s)} = \frac{(\sqrt{2/3})^{-s}}{(3 \times 8)^{(s+1)/2}}. \quad (59)$$

Moreover, one rapidly verifies that for $(1, 0)$,

$$e^{i\omega\hat{h}_1} = e^{2i\omega} |(1, 0)(10)\rangle\langle(1, 0)(10)| + e^{-i\omega} [| (1, 0)(-1 - 1)\rangle\langle(1, 0)(-1 - 1)| + |(1, 0)(0 - 1)\rangle\langle(1, 0)(0 - 1)|] \quad (60)$$

$$= \frac{1}{\sqrt{3}}(2e^{-i\omega} + e^{2i\omega})T_{(0,0);00}^{(1,0)} + \sqrt{\frac{2}{3}}(-e^{-i\omega} + e^{2i\omega})T_{(1,1);00}^{(1,0)}, \quad (61)$$

so that

$$\tilde{\chi}_{(0,0)}^{(1,0)}(\omega_1) = \frac{1}{\sqrt{3}}(2e^{-i\omega_1} + e^{2i\omega_1}), \quad \chi_{(1,1)}^{(1,0)}(\omega_1) = \sqrt{\frac{2}{3}}(-e^{-i\omega_1} + e^{2i\omega_1}). \quad (62)$$

With these we can form the overlap matrix

$$g_{\sigma\sigma'} = \begin{pmatrix} \frac{10\pi}{3} & -\frac{2\sqrt{2}\pi}{3} \\ -\frac{2\sqrt{2}\pi}{3} & \frac{8\pi}{3} \end{pmatrix}. \quad (63)$$

From the inverse $g^{\sigma\sigma'}$, the final form of the expansion coefficients $c_\sigma^{(s)}$ is thus

$$c_0^{(s)} = \frac{8 + 4^{-s}}{24\sqrt{3}\pi}, \quad c_1^{(s)} = \frac{4 + 5 \times 4^{-s}}{24\sqrt{6}\pi}. \quad (64)$$

For the special cases $s = -1, 0, 1$, we obtain

$$f^{(-1)}(\omega) = \frac{e^{-2i\omega}}{2\pi}, \quad f^{(0)}(\omega) = \frac{e^{i\omega} + 2e^{-2i\omega}}{8\pi}, \quad f^{(1)}(\omega) = \frac{5e^{i\omega} + 6e^{-2i\omega}}{32\pi}. \quad (65)$$

The operator $\hat{P}^{(0)}$ has the form $\text{diag}(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. This case has also been investigated from a different approach in [22].

4.2. The case of $(2, 0)$

The transformation between zero-weight tensors and diagonal projectors is given by

$$\begin{pmatrix} \hat{T}_{(0,0);00}^{(2,0)} \\ \hat{T}_{(1,1);01}^{(2,0)} \\ \hat{T}_{(1,1);00}^{(2,0)} \\ \hat{T}_{(2,2);02}^{(2,0)} \\ \hat{T}_{(2,2);01}^{(2,0)} \\ \hat{T}_{(2,2);00}^{(2,0)} \end{pmatrix} = U \begin{pmatrix} |(2, 0)(20)\rangle\langle(2, 0)(20)| \\ |(2, 0)(01)\rangle\langle(2, 0)(01)| \\ |(2, 0)(1 - 1)\rangle\langle(2, 0)(1 - 1)| \\ |(2, 0)(-22)\rangle\langle(2, 0)(-22)| \\ |(2, 0)(-10)\rangle\langle(2, 0)(-10)| \\ |(2, 0)(0 - 2)\rangle\langle(2, 0)(0 - 2)| \end{pmatrix}, \quad (66)$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & -\sqrt{\frac{2}{5}} & 0 & \sqrt{\frac{2}{5}} \\ 2\sqrt{\frac{2}{15}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & -\sqrt{\frac{2}{15}} & -\sqrt{\frac{2}{15}} & -\sqrt{\frac{2}{15}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\sqrt{\frac{2}{5}} & \sqrt{\frac{2}{5}} & \frac{1}{\sqrt{10}} & 0 & -\frac{1}{\sqrt{10}} \\ \sqrt{\frac{3}{10}} & -\sqrt{\frac{3}{10}} & -\sqrt{\frac{3}{10}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{pmatrix}.$$

For (2, 0), the highest weight projector is $|(2, 0)(20)\rangle\langle(2, 0)(20)|$; equation (28) this time gives

$$F_0^{(s)} = \frac{(\sqrt{6})^s}{6^{(s+1)/2}}, \quad F_1^{(s)} = \frac{2^{-s} \left(\sqrt{\frac{15}{2}}\right)^s}{(6 \cdot 8)^{(s+1)/2}}, \quad F_2^{(s)} = \frac{\left(\sqrt{\frac{10}{3}}\right)^s}{(6 \cdot 27)^{(s+1)/2}}. \quad (67)$$

The expansion of $e^{i\omega_1 \hat{h}_1}$ contains three terms as follows:

$$e^{i\omega \hat{h}_1} = \frac{1}{\sqrt{6}}(e^{4i\omega} + 2e^{i\omega} + 3e^{-2i\omega})\hat{T}_{(0,0);00}^{(2,0)} + \sqrt{\frac{2}{15}}(2e^{4i\omega} + e^{i\omega} - 3e^{-2i\omega})\hat{T}_{(1,1);00}^{(2,0)} + \sqrt{\frac{3}{10}}(e^{4i\omega} - 2e^{i\omega} - 3e^{-2i\omega})\hat{T}_{(2,2);00}^{(2,0)}. \quad (68)$$

The overlap matrix is found to be

$$g_{\sigma\sigma'} = \begin{pmatrix} \frac{14\pi}{3} & -\frac{2\sqrt{5}\pi}{3} & 0 \\ -\frac{2\sqrt{5}\pi}{3} & \frac{56\pi}{15} & -\frac{6\pi}{5} \\ 0 & -\frac{6\pi}{5} & \frac{18\pi}{5} \end{pmatrix}. \quad (69)$$

The expansion coefficients for $f^{(s)}$ are then

$$\begin{pmatrix} c_0^{(s)} \\ c_1^{(s)} \\ c_2^{(s)} \end{pmatrix} = \begin{pmatrix} \frac{1}{4\pi} & \frac{1}{4\sqrt{5}\pi} & \frac{1}{12\sqrt{5}\pi} \\ \frac{1}{4\sqrt{5}\pi} & \frac{7}{20\pi} & \frac{7}{60\pi} \\ \frac{1}{12\sqrt{5}\pi} & \frac{7}{60\pi} & \frac{19}{60\pi} \end{pmatrix} \begin{pmatrix} F_0^{(s)} \\ F_1^{(s)} \\ F_2^{(s)} \end{pmatrix}. \quad (70)$$

For the special cases of $s = -1, 0, 1$, we find

	$s = -1$	$s = 0$	$s = 1$
$c_0^{(s)} =$	$\frac{1}{2\sqrt{6}\pi}$	$\frac{90\sqrt{6+2\sqrt{10+9\sqrt{15}}}}{2160\pi}$	$\frac{8051}{31104\sqrt{6}\pi}$
$c_1^{(s)} =$	$\frac{\sqrt{2}}{\sqrt{15}\pi}$	$\frac{14\sqrt{2+63\sqrt{3}+18\sqrt{30}}}{2160\pi}$	$\frac{9701}{31104\sqrt{30}\pi}$
$c_2^{(s)} =$	$\frac{1}{2\pi} \sqrt{\frac{3}{10}}$	$\frac{38\sqrt{2+21\sqrt{3}+6\sqrt{30}}}{2160\pi}$	$\frac{3767}{31104\sqrt{30}\pi}$

(71)

5. Discussion: the general case

We now briefly discuss the case of general irreps through the example of $\mathfrak{SU}(3)$ irreps of the type (λ, μ) . In such cases, the highest weight projector is invariant under $\mathfrak{U}(1) \otimes \mathfrak{U}(1)$ transformations: the coset space is the six-dimensional space $\mathfrak{M} = \mathfrak{SU}(3)/[\mathfrak{U}(1) \otimes \mathfrak{U}(1)]$.

The situation here is fundamentally different from the $(\lambda, 0)$ case as the decomposition of $(\lambda, \mu) \otimes (\mu, \lambda)$ will contain some irreps more than once [23]. These multiple copies of irreps are completely identical, with the results that two distinct operators may be mapped to the same phase-space symbol, and it becomes impossible to unambiguously reconstruct an operator from its symbol.

For instance, suppose we are working in a Hilbert space that carries the irrep (1, 1) of $\mathfrak{SU}(3)$. One can verify that

$$(1, 1) \otimes (1, 1) = (2, 2) \oplus (3, 0) \oplus (0, 3) \oplus (1, 1) \oplus (1, 1) \oplus (0, 0); \quad (72)$$

the irrep (1, 1) occurs twice in the decomposition.

The highest weight projector is written in terms of tensors as

$$\begin{aligned} & |(1, 1)11; \frac{1}{2}\rangle\langle(1, 1)11; \frac{1}{2}| \\ &= \frac{1}{\sqrt{8}}\hat{T}_{(0,0)00}^{(1,1)} - \frac{1}{2\sqrt{3}}\hat{T}_{(3,0)01}^{(1,1)} - \frac{1}{2\sqrt{3}}\hat{T}_{(0,3)01}^{(1,1)} + \sqrt{\frac{3}{14}}\hat{T}_{(1,1)01}^{(1,1)} - \frac{1}{\sqrt{14}}\hat{T}_{(1,1)00}^{(1,1)} \\ &+ \sqrt{\frac{2}{105}}\hat{T}_{(1,1)01}^{(1,1)} + 2\sqrt{\frac{2}{35}}\hat{T}_{(1,1)00}^{(1,1)} + \frac{1}{\sqrt{10}}\hat{T}_{(2,2)01}^{(1,1)} - \frac{1}{2}\sqrt{\frac{3}{10}}\hat{T}_{(2,2)00}^{(1,1)}. \end{aligned} \quad (73)$$

The tensors $\hat{T}_{(1,1)_1;01}^{(1,1)}$ and $\hat{T}_{(1,1)_2;01}^{(1,1)}$ transform identically under $\mathfrak{SU}(3)$, as do the tensors $\hat{T}_{(1,1)_1;00}^{(1,1)}$ and $\hat{T}_{(1,1)_2;00}^{(1,1)}$. There is no choice of basis in the decomposition that will make one copy of $(1, 1)$ disappear. Thus, tensors such as $\hat{T}_{(1,1)_1;01}^{(1,1)}$ and $\hat{T}_{(1,1)_2;01}^{(1,1)}$, even though they are orthogonal, will be mapped to identical phase-space symbols. In particular, the Q -symbols $Q_{\hat{T}_{(1,1)_1;01}^{(1,1)}} = \langle (1, 1)11; \frac{1}{2} | \Lambda^\dagger(\Omega) \hat{T}_{(1,1)_1;01}^{(1,1)} \Lambda(\Omega) | (1, 1)11; \frac{1}{2} \rangle$ will be proportional to $Q_{\hat{T}_{(1,1)_2;01}^{(1,1)}}$.

6. Conclusion

This paper presents an algorithm to find a kernel $w^{(s)}(\Omega)$ that allows one-to-one mapping between operators in the Hilbert space carrying an irrep $(\lambda, 0, \dots, 0)$ of $\mathfrak{SU}(n)$ and c -functions defined on the corresponding phase space $\mathfrak{SU}(n)/\mathfrak{U}(n-1)$.

For irreps of the $(\lambda, 0, \dots, 0)$ type, our classical manifold is the ‘canonical’ one identified from the properties of the coherent states, as suggested by [21]. For the more general representations of the mapping, no such mapping to the canonical phase space is possible and different Wigner-like distributions should be constructed (as for instance in [14, 15]).

Beyond the obvious computationally attractive extension of the P -, Q - and Wigner functions and their possible applications to describe collection of indistinguishable n -level atoms and BEC [2], our approach could also be the pathway toward understanding the asymptotic forms and contraction limits of these objects, much as was done in [24]. Our prescription can also be the starting point to develop the \star -product for $\mathfrak{SU}(3)$.

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