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SU(3) phase states and finite Fourier transform

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Abstract
We describe the construction of SU(3) phase operators using a Fourier-like transform on a hexagonal lattice. The advantages and disadvantages of this approach are contrasted with other results, in particular with the more traditional approach based on polar decomposition of operators.

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(Some figures may appear in colour only in the online journal)

1. Introduction: complementarity and the Fourier transform

The idea of complementarity in quantum mechanics goes back to Bohr and his attempt to explain wave–particle duality. The concept was sharpened by Pascual Jordan, who has stated [1] that:

For a given value of x, all values of p are equally possible.

This formulation automatically singles out the Fourier transform connecting operators such as $\hat{x}$ and $\hat{p}$ as their respective (generalized) eigenstates satisfy

$$\langle x|\,p\rangle \sim e^{ixp/\hbar} \quad\Rightarrow\quad |\langle x|\,p\rangle|^2 = \text{constant}. \tag{1}$$

The concept is not limited to continuous systems but also exists in finite dimensions. In this paper, we will discuss the construction of SU(3) phase operators, which are expected to be complementary to number operators. This paper emphasizes the importance of the finite Fourier transform and in particular investigates a new type of generalization of the Fourier transform that is constructed to preserve the symmetry of a hexagonal lattice, which is the natural (discrete) lattice to describe states appropriate for the description of a collection of three-level systems. Our approach should be contrasted with the approach of Dirac [2] which emphasizes polar decompositions and which has been applied to SU(2) and other systems in [3, 4].

2. Two examples

Consider first a spin-$\frac{1}{2}$ system, taking as operators the Pauli matrices σ_x, σ_y and σ_z. The eigenstates $\{|+\rangle_z, |-\rangle_z\}$ of σ_z and the eigenstates $\{|+\rangle_x, |-\rangle_x\}$ of σ_x are complementary:

$$|{}_x\langle +|+\rangle_z|^2 = |{}_x\langle -|+\rangle_z|^2 = |{}_x\langle +|-\rangle_z|^2 = |{}_x\langle -|-\rangle_z|^2 = \tfrac{1}{2} \\ = \text{constant}. \tag{2}$$

The eigenstates of σ_z and σ_x are related by a *finite* Fourier transform:

$$F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3}$$

The operators σ_x and σ_z are said to be complementary. The same property holds for the pair σ_y and σ_z and for the pair σ_y and σ_x. The transformation matrix connecting any two sets of eigenstates of the Pauli operators remains a finite Fourier transform.

A similar construction exists for a three-level system (or qutrit). Defining

$$\hat{Z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}, \quad \omega = e^{2\pi i/3}, \tag{4}$$

and writing their respective eigenstates as $\{|0_z\rangle, |1_z\rangle, |2_z\rangle\}$ and $\{|0_x\rangle, |1_x\rangle, |2_x\rangle\}$ we find for instance $|\langle 1_x\,|\,0_z\rangle|^2 = |\langle 2_x\,|\,2_z\rangle|^2 = \frac{1}{3}$ with all other such overlaps constant. Here again, the eigenstates of $\hat{X}$ and $\hat{Z}$ are related by a finite

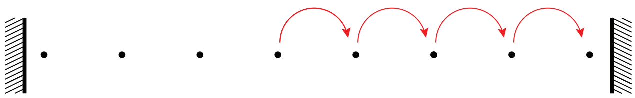

Figure 1. The shift action of the su(2) raising operator $\hat{S}_+$.

Fourier transform:

$$F = \frac{1}{\sqrt{3}}\begin{pmatrix} \omega & 1 & \omega^2 \\ \omega & \omega^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{5}$$

This is the right time to mention some of the properties of the finite Fourier matrix F. It is unitary, which implies

$$\sum_i F^*_{ki} F_{ij} = \delta_{ij}. \tag{6}$$

(This would be orthogonality under integration in the continuous case.) $F^4 = \mathbb{1}$, and its entries are characters of finite Abelian groups. In dimension n:

$$F_{jk} = \mathrm{e}^{2\pi ijk/n}/\sqrt{n}. \tag{7}$$

Finally, $|F_{ij}|$ have constant magnitude, connecting with Jordan's definition of complementarity.

3. SU(2) phase states

Following Dirac [2] and others [3], phase operators in su(2) (and other) systems are constructed by writing the matrix for $\hat{S}_+$ (or $\hat{S}_-$) in polar form, namely

$$\hat{S}_+ \mapsto \sum_{m=0}^{j-1} c_m |j, m+1\rangle\langle j, m| = E \cdot D, \tag{8}$$

where D is diagonal and E is a 'phase' part, containing entries that produce the shifting action of S_+ on the basis states. The operator E is expected on physical grounds to be complementary to the diagonal operator $\hat{S}_z$.

Geometrically, the set of eigenvalues $\{m; m = -j, -j+1, \ldots, j-1, j\}$ of $\hat{S}_z$ acting on number states $|jm\rangle$ are equidistant points on a line and the action of the ladder operators $\hat{S}_\pm$ takes a point m to its neighbor $m \pm 1$. The action of $\hat{S}_+$ is pictorially represented in figure 1.

Because $|jj\rangle$ is killed by $\hat{S}_+$, the rank of $\hat{S}_+$ is one less than the dimension of the system, so E is not completely defined: we can adjust the entries in one line. The usual choice makes E cyclic ($E^{2j+1} = \mathbb{1}$), so it generates an Abelian group of order $2j+1$:

$$E = \mathrm{e}^{\mathrm{i}\varphi} \mapsto \begin{pmatrix} 0 & 1 & \ldots & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix} = \sum_{m=-j}^{j-1} |m+1\rangle\langle m| + |-j\rangle\langle j|. \tag{9}$$

This E is unitary and can be written in the form $E = \mathrm{e}^{\mathrm{i}\hat{\varphi}}$, with $\hat{\varphi}$ being the putative Hermitian phase operator. The eigenvectors of E are eigenvectors of $\hat{\varphi}$ and defined to be the SU(2) phase states. The components of the mth eigenvector $|\varphi_m\rangle$ are just elements of a Fourier matrix F. Thus, the phase eigenstate $|\varphi_m\rangle$ is given by

$$|\varphi_m\rangle = \sum_k F_{mk}|jk\rangle = \frac{1}{\sqrt{2j+1}} \sum_k \mathrm{e}^{2\pi ikm/(2j+1)}|jk\rangle. \tag{10}$$

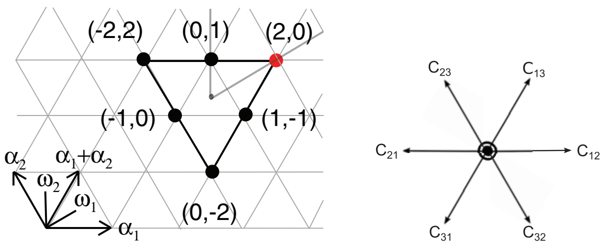

Figure 2. Left: population differences $(n_1 - n_2, n_2 - n_3)$ for the states $|n_1n_2n_3\rangle$ with $n_1 + n_2 + n_3 = 2$. (a, b) is located at $a\omega_1 + b\omega_2$ with ω_1, ω_2 being the lattices vectors. Right: graphical representation of the shift action of the ladder generators of su(3) on the hexagonal lattice. The ringed dot at the center represents the two diagonal population difference operators $\hat{h}_1$ and $\hat{h}_2$, which do not shift the basis states.

4. SU(3) and SU(3) phase states

4.1. Geometry of SU(3) states

The algebra su(3) appears naturally in the construction of number-preserving transition operators for three-level systems. There are six transition operators, usually denoted by $\hat{C}_{ij} = a_i^\dagger a_j$ for $i \neq j = 1, 2, 3$, and two population differences $\hat{h}_1 = a_1^\dagger a_1 - a_2^\dagger a_2$ and $\hat{h}_2 = a_2^\dagger a_2 - a_3^\dagger a_3$. The states $|200\rangle$ and $|110\rangle$, for instance, respectively correspond to the pairs (2, 0) and (0, 1) of population differences. Pairs are located on a hexagonal lattice with basis vectors ω_1 and ω_2 as illustrated on the left of figure 2.

The action of $\hat{C}_{ij}$ on lattice points is illustrated on the right of figure 2. Basis vectors α_1 and α_2 associated with the operators $\hat{C}_{12}$ and $\hat{C}_{23}$ are dual (reciprocal) to the lattice vectors ω_1 and ω_2, respectively, as illustrated. Using the hexagonal geometry, two points (a, b) and (c, d) corresponding to two pairs of population differences differ by an integer combination of the vectors α_1 and α_2. The action of $\hat{C}_{ij}$ on the state $|n_1n_2n_3\rangle$ is to translate the point $(n_1 - n_2, n_2 - n_3)$ by the vector associated with $\hat{C}_{ij}$ to the point $(n_1' - n_2', n_2' - n_3')$, so that, for instance,

$$\hat{C}_{12}|110\rangle \sim |200\rangle \Rightarrow (0, 1) \mapsto (2, 0) = \alpha_1 + (0, 1). \tag{11}$$

The central ringed dot represents the two diagonal population difference operators $\hat{h}_1$ and $\hat{h}_2$. There should be one phase operator conjugate to each $\hat{h}_i$.

4.2. Two solutions: boundaries

If we approach the construction of SU(3) phase states using polar decompositions, we are faced with an interesting problem. Because there are two basic shift directions, α_1 and α_2, each one of $\hat{C}_{12}$ and $\hat{C}_{23}$ will come with its own set of not necessarily mutually compatible boundary conditions.

In the simplest case of the states $|100\rangle$, $|010\rangle$ and $|001\rangle$, the shift matrices E_{12} and E_{23} that enter in the decompositions

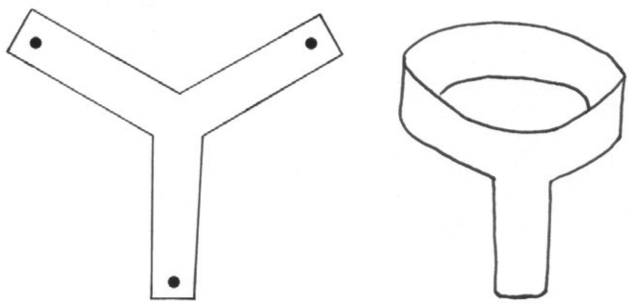

Figure 3. A graphical representation of how cyclic boundary conditions can be imposed to complete the matrix E_{12} to yield equation (12).

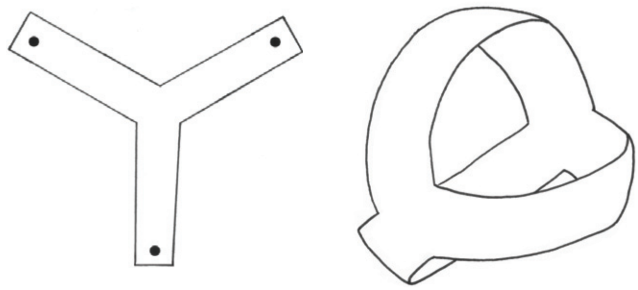

Figure 4. A graphical representation of how cyclic boundary conditions can be imposed to complete the matrices E_{12} and E_{23} to yield equation (15).

of $\hat{C}_{12}$ and $\hat{C}_{23}$, respectively, contain two lines that cannot be uniquely determined. These matrices can be completed in two different ways. First, we can write

$$E_{12}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}=[|010\rangle\langle 100|+|100\rangle\langle 010|]+|001\rangle\langle 001|, \tag{12}$$

$$E_{23}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}=|100\rangle\langle 100|+[|010\rangle\langle 001|+|001\rangle\langle 010|]. \tag{13}$$

This kind of solution also exists for more general cases where $n_1+n_2+n_3>1$. It consists in considering subsets of states with the same value for n_3—such subsets of states fall on lines parallel to the α_1 direction—and following the procedure of SU(2) on each line to obtain E_{12}. Similarly, by considering subsets of states with the same value of n_1—these states now lie on lines parallel to the α_2 direction—we can follow the SU(2) procedure for each line and obtain E_{23}. However, one feature of this solution, already present in equations (12) and (13), is that the resulting phase operators do not commute:

$$[E_{12}, E_{23}]\neq 0 \quad\Rightarrow \mathrm{e}^{\mathrm{i}\theta\hat{\varphi}_{12}}\,\mathrm{e}^{\mathrm{i}\gamma\hat{\varphi}_{23}}\neq \mathrm{e}^{\mathrm{i}(\theta\hat{\varphi}_{12}+\gamma\hat{\varphi}_{23})}. \tag{14}$$

This in turn implies that the phases are not additive.

For the case of the states $|100\rangle$, $|010\rangle$ and $|001\rangle$, it is possible to find shift matrices E_{12} and E_{23} compatible with the polar decomposition of the respective operators so that $[E_{12}, E_{23}]=0$ (figure 4). These matrices are

$$E_{12}=\begin{pmatrix}0&1&0\\0&0&\omega\\\omega^2&0&0\end{pmatrix},\quad E_{23}=\begin{pmatrix}0&\omega^2&0\\0&0&1\\\omega&0&0\end{pmatrix},\quad \omega=\mathrm{e}^{2\pi\mathrm{i}/3}. \tag{15}$$

Note that E_{12} is just the operator $\hat{X}$ of equation (4), while $E_{23}=\hat{X}^2$. Clearly, E_{12} and E_{23} commute. However, we have not been able to find similar solutions for sets of states with $n_1+n_2+n_3>1$.

4.3. Finite Fourier transform on a hexagonal lattice

As an alternative to the construction based on polar decomposition, we look for a finite Fourier transform (FFT) adapted to the discrete hexagonal symmetry natural to SU(3) states. Such an FFT was proposed in [5] and will be adapted to our needs.

We start with the physical states $|n_1n_2n_3\rangle$. The procedure of [5] requires that the 'data points' be in the first hextant of the lattice, so we find a rigid displacement of the set of population differences (n_1-n_2, n_2-n_3) corresponding to the physical states so that every pair (n_1-n_2, n_2-n_3) is mapped to a single point in the first hextant (figure 4). One can show that the rigid displacement is a linear transformation comprising a translation, a rotation and a change of scale of the original pairs of points. The final result of the sequence is

$$|n_1n_2n_3\rangle\mapsto(n_1-n_2, n_2-n_3)\mapsto(n_1, n_2). \tag{16}$$

An example of the result is given in figure 5. We obtain for each point (a, b) in the first extant its orbit, i.e. the set of points obtained by considering reflections of (a, b) through mirrors perpendicular to α_1 and α_2. Depending on the value of a and b, an orbit may contain 1, 3 or 6 points. The orbits for the points (2, 0) and (1, 1) are illustrated in figure 6. Each orbit is labeled by its starting point (a, b) in the first extant. There is the same number of orbits as the number of states. Each orbit is used to construct a so-called orbit function

$$\begin{aligned}\chi_{(a,b)}(n_1, n_2)&\sim\omega^{(2a+b)n_1+(a+2b)n_2}+\omega^{(b-a)n_1+(a+2b)n_2}\\&\quad+\omega^{(2a+b)n_1+(a-b)n_2}+\omega^{-(a-b)n_1-(b+2a)n_2}\\&\quad+\omega^{-(a+2b)n_1+(b-a)n_2}+\omega^{-(2b+a)n_1-(b+2a)n_2},\end{aligned} \tag{17}$$

with (n_1, n_2) points in the first hextant. The functions χ are closely related to characters of elements of finite order of SU(3).

It is *essential* to rigidly translate the population differences of physical states. Two states $|n_1n_2n_3\rangle$, $|n_1'n_2'n_3'\rangle$ that differ only by a permutation of n_1, n_2, n_3 yield population differences (n_1-n_2, n_2-n_3) and $(n_1'-n_2', n_2'-n_3')$ that are *on the same orbit* and so produce identical functions χ. It is only once the population differences have been translated to the first hextant that (n_1, n_2) and (n_1', n_2') will lie on different orbits.

The functions χ need to be properly normalized and weighted as described in [5], but once this is done, they satisfy an orthogonality relation

$$\sum_{n_1,n_2}(\chi_{(a,b)}(n_1, n_2))^*\,\chi_{(a',b')}(n_1, n_2)\sim\delta_{aa'}\delta_{bb'}. \tag{18}$$

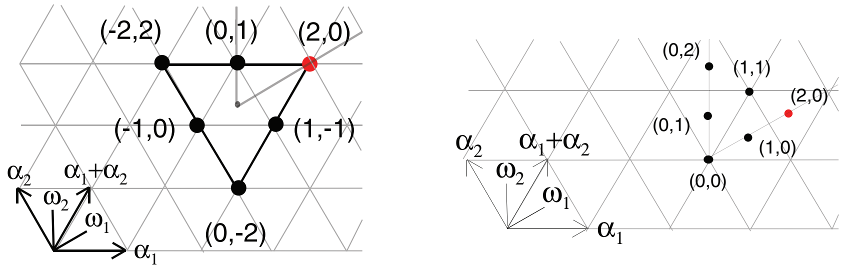

Figure 5. An example of how the pairs of population differences (p, q) obtained from the states $|n_1n_2n_3\rangle$ with $n_1+n_2+n_3$ are mapped to the first hextant.

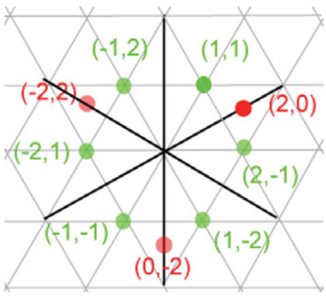

Figure 6. The orbits of the points (2, 0) (in red) and (1, 1) (in green).

The orbit functions can then be used to obtain a Fourier matrix

$$F = \begin{pmatrix} \chi_{a_1,b_1}(s_1,s_2)_1 & \cdots & \chi_{a_1,b_1}(s_1,s_2)_k \\ \vdots & \ddots & \vdots \\ \chi_{a_k,b_k}(s_1,s_2)_1 & \cdots & \chi_{a_k,b_k}(s_1,s_2)_k \end{pmatrix}. \quad (19)$$

So defined, the matrix F immediately satisfies the majority of conditions given at the end of section 2. In particular, for the set of states $\{|100\rangle, |010\rangle, |001\rangle$, the matrix F is exactly the same as that of equation (5). However, for other states with $n_1+n_2+n_3>1$, the matrix F no longer contains entries of the same magnitude. For instance, using the states $|n_1n_2n_3\rangle$ with $n_1+n_2+n_3=2$, we find

$$F = \frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & 1 & 1 & \frac{1}{\sqrt{3}} \\ 1 & -\frac{1}{\sqrt{3}}\,\omega & \omega^2 & -\frac{1}{\sqrt{3}}\,\omega^2 & \frac{-1}{\sqrt{3}} & \omega \\ \frac{1}{\sqrt{3}} & \omega^2 & \frac{1}{\sqrt{3}}\,\omega & \omega & 1 & \frac{1}{\sqrt{3}}\,\omega \\ 1 & -\frac{1}{\sqrt{3}} & \omega & -\frac{1}{\sqrt{3}}\,\omega & \frac{-1}{\sqrt{3}} & \omega^2 \\ 1 & \frac{-1}{\sqrt{3}} & 1 & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 1 \\ \frac{1}{\sqrt{3}} & \omega & \frac{1}{\sqrt{3}}\,\omega^2 & \omega^2 & 1 & \frac{1}{\sqrt{3}}\,\omega \end{pmatrix}, \quad (20)$$

$$\omega = \mathrm{e}^{2\pi\mathrm{i}/3}.$$

4.4. SU(3) phase states

Now *define* SU(3) phase states as transforms of the shifted population difference eigenstates:

$$|\eta_1,\eta_2\rangle_{(n_1,n_2)} \equiv \sum_{t_1,t_2} F_{(n_1,n_2),(t_1,t_2)}|t_1,t_2\rangle. \quad (21)$$

Phase 'operators' are conjugate to population difference operators:

$$\hat{\eta}_1 = F\hat{h}_1 F^{-1}, \quad \hat{\eta}_2 = F\hat{h}_2 F^{-1}. \quad (22)$$

Since $[\hat{h}_1, \hat{h}_2]=0$, we recover $[\hat{\eta}_1, \hat{\eta}_2]=0$: phases commute.

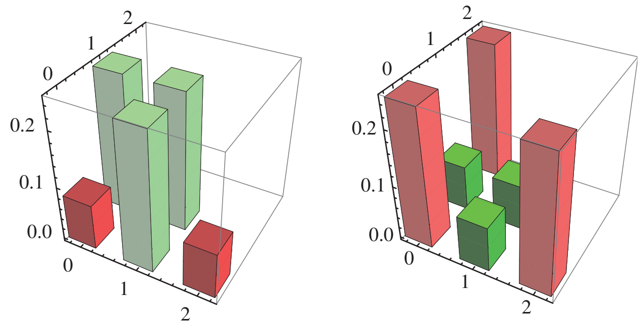

Figure 7. Left: probability histogram for any one of the input states $|200\rangle$, $|020\rangle$ and $|002\rangle$. Right: probability histogram for any one of the input states $|110\rangle$, $|101\rangle$ and $|011\rangle$. Columns of the same color have the same height.

4.5. Complementarity and number difference distributions

To get insight into what the phase states 'look like', we consider the probabilities $|\langle n_1,n_2\,|\,\eta_1,\eta_2\rangle|^2$. Recall the correspondences $|n_1n_2n_3\rangle \leftrightarrow (n_1,n_2)$ between physical states and their translated population difference. Thus

$$|100\rangle \leftrightarrow (1,0), \quad |010\rangle \leftrightarrow (0,1), \quad |001\rangle \leftrightarrow (0,0). \quad (23)$$

For these states, we find, for every (n_1,n_2) and every (a,b),

$$|\langle(n_1,n_2)|\,[F|(a,b)\rangle]\,|^2 = \tfrac{1}{3}. \quad (24)$$

This is no surprise, as $|F_{(n_1n_2),(a,b)}|^2 = \frac{1}{3}$ for this case.

For $n_1+n_2+n_3=2$, it is convenient to construct probability histograms for points in the first hextant. Using as the input state any one of the states $|200\rangle \leftrightarrow (2,0)$, $|020\rangle \leftrightarrow (0,2)$ and $|002\rangle \leftrightarrow (0,0)$, we find two and only two possible amplitudes, as illustrated with two different colors on the left of figure 7. The corresponding histogram for any one of the input states $|110\rangle \leftrightarrow (1,1)$, $|101\rangle \leftrightarrow (1,0)$ and $|011\rangle \leftrightarrow (0,1)$ is on the right of figure 7.

For a given input state not every Fourier component has the same amplitude: complementarity in the sense of Jordan is lost—as expected since $|F_{(n_1n_2),(a,b)}|^2$ is no longer constant. However, points (n_1,n_2) and (n_1',n_2') with equal amplitudes in the first hextant correspond to physical states $|n_1n_2n_3\rangle$ and $|n_1'n_2'n_3'\rangle$ which differ by a permutation of their entries.

For a generic input state, such as $|0\ 21\ 9\rangle$, the probability landscape is rugged without any special features. However, for input states of the type $|N00\rangle$ or $|0N0\rangle$ or $|00N\rangle$, which are mapped to the corner edges of the first hextant, we find that the

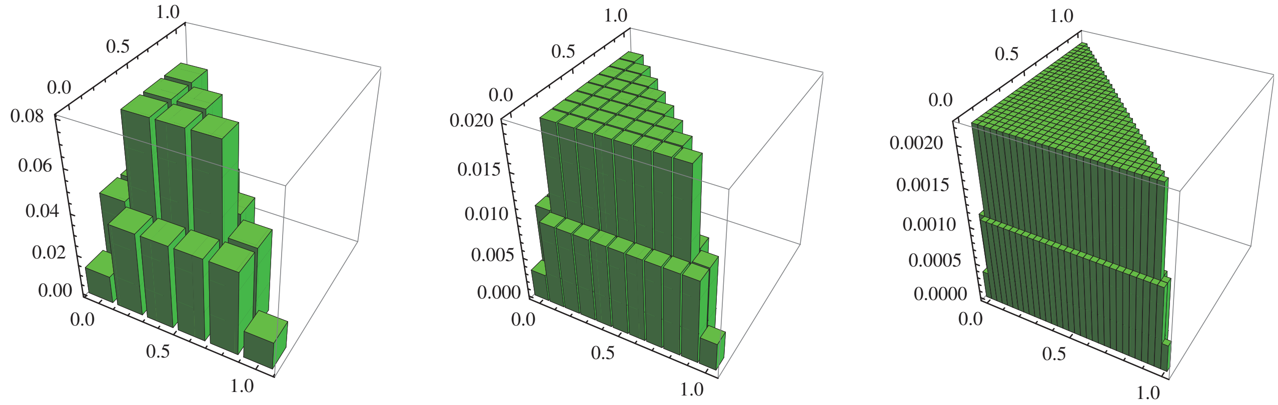

Figure 8. Probability landscapes for input states $|500\rangle$, $|10\,0\,0\rangle$ and $|30\,0\,0\rangle$.

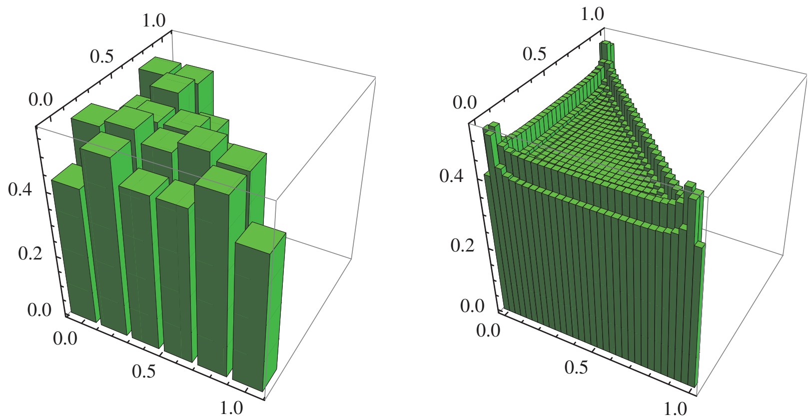

Figure 9. Variance of the phase operator $\hat{\eta}_1$ calculated using the physical states $|n_1n_2n_3\rangle$. Left: $n_1+n_2+n_3=5$. Right: $n_1+n_2+n_3=30$.

probability landscape is remarkably regular. The probability landscape is *asymptotically flat*, meaning that, in the large N limit, the phase states $F|N00\rangle$, etc are *asymptotically conjugate* to the Fock states $|n_1n_2n_3\rangle$.

4.6. su(3) phase operators

The phase operators $\hat{\eta}_1$, $\hat{\eta}_2$ of equation (22) generally have 'complicated' expressions. In spite of this, we have found the following observation to hold. If we evaluate the variances $\Delta\eta_1$ and $\Delta\eta_2$ using the physical states $|n_1n_2n_3\rangle$, the smallest variances always occur for the states $|N00\rangle$, $|0N0\rangle$ or $|00N\rangle$. The landscape of variances of $\hat{\eta}_1$ evaluated in the physical states $|n_1n_2n_3\rangle$ is illustrated in figure 9 for $n_1+n_2+n_3=5$ and 30.

5. Conclusions

The polar decomposition of operators in SU(3) produces phase operators that are ambiguous and not unique: in general, non-commuting raising operators lead to a decomposition that produces non-commuting phase operators. Moreover, this approach produces an 'exponential phase' rather than a phase operator.

We can obtain Hermitian commuting 'phase-like' operators by using symmetry-adapted FFT. The procedure is mathematically systematic but not very intuitive, and we lose the connection with complementarity as defined by Jordon. With this approach the physical states $|N00\rangle$, $|0N0\rangle$ and $|00N\rangle$ stand out as having unexpected properties of asymptotic complementarity. The variances of the phase operators evaluated using those states are always the smallest.

An unanswered question (not discussed in this paper) is the difficulty in imposing correct cyclic boundary conditions on the phase operators themselves once they are exponentiated.

Acknowledgments

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