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Correspondence rules for Wigner functions over $SU(3)/U(2)$

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Abstract

We present results on the $\star$ product for $SU(3)$ Wigner functions over $SU(3)/U(2)$. In particular, we present a form of the so-called correspondence rules, which provide a differential form of the $\star$ product $\hat{A}\star\hat{B}$ and $\hat{B}\star\hat{A}$ when $\hat{A}$ is an $\mathfrak{su}(3)$ generator. For the $\mathfrak{su}(3)$ Wigner map, these rules must contain second order derivatives and thus substantially differ from the rules of other known cases.

Keywords: Wigner functions, semiclassical dynamics, $SU(3)$, star product

1. Introduction

The possibility of 'switching on' our classical intuition in the analysis of quantum systems is an important advantage of the phase space approach to quantum mechanics [1]. Given this, the phase-space approach has been directly and efficiently applied to 'large' (semi-classical) quantum systems with Heisenberg–Weyl $HW(n)$, rotation $SO(3)$ or Euclidean $E(2)$ dynamical symmetries. The corresponding classical manifolds ($2n$-dimensional plane, S^2 sphere and 2-dimensional cylinder) are easy to visualize, and quantum states $\hat{\rho}$ can be conveniently mapped to corresponding (quasi-) distribution functions $W_\rho(\Omega)$, where Ω is a phase-space point.

For quantum systems with observables in the algebra of the $SU(n)$ group acting in a Hilbert space that carries a symmetric unitary irreducible representation $(\lambda, 0..., 0)$, there is a systematic construction [2] of phase-space functions satisfying the basic Stratanovich–Weyl requirements [3–5] (see also [6]).

In this paper, we concentrate on the Wigner representation [3–5], where an operator $\hat{f} \mapsto W_{\hat{f}}(\Omega)$ is mapped into a self-dual symbol $W_{\hat{f}}(\Omega)$,

$$\hat{f} \Leftrightarrow W_{\hat{f}}(\Omega), \quad \Omega \in \mathfrak{M}, \tag{1}$$

where $\mathfrak{M} = SU(n)/U(n-1)$ is a symplectic manifold corresponding to a classical phase space [7], intimately related to the set of orbit-type coherent states [8] generated from a highest weight state of the corresponding Hilbert space. Of the several types of existing phase-space maps, the Wigner representation is of particular interest as not only sensitive to the interference pattern but also endowed with certain important dynamical properties [1, 3, 9, 10].

The map of equation (1) has a relatively simple form for the fundamental representation of $SU(n)$, but its explicit construction for arbitrary symmetric representations of $SU(n)$ requires detailed knowledge of the appropriate Clebsch–Gordan technology and is more involved [2, 10, 11]. In spite of this, the limit of large dimensions where $\lambda \gg 1$ is of immediate interest to this work as it is related to the semi-classical description of quantum systems, and is especially important for the analysis of the evolution of macroscopic systems since λ is often identified with the number of particle constituents in the system. Moreover, whereas the dimension of the symmetric irrep of $SU(n)$ grows with λ like λ^{n-1}, the dimension of the classical phase-space $SU(n)/U(n-1)$ is $2(n-1)$ and thus independent of λ and grows linearly with n; phase-space methods thus become increasingly efficient as λ increases. In the limit $\lambda \gg 1$, the evolution equation for the Wigner distribution admits a natural expansion in a single semi-classical parameter ϵ, which scales as $\lambda^{-(n-1)/2}$, as can be seen for $SU(3)$ in equation (48). This classical limit of large dimensions is well-studied in the case of $SU(2)$, describing very pictorially spin-like systems [10] as quasi-probability distributions on the sphere S^2, but the situation is far from transparent for higher-rank unitary groups.

The phase-space picture is complete when a sensible expression for the star-product $\star$, defining the composition map $\hat{f}\hat{g} \mapsto W_{\hat{f}}(\Omega) \star W_{\hat{g}}(\Omega)$, is found [12–14]. The (non-commutative) $\star$-product operation is essential to capture the non-commutative nature of quantum mechanical operators and can be used to map the Schrödinger equation into a Liouville-type equation of motion for the Wigner function $W_{\hat{\rho}}(\Omega)$ [15]. Our experience with $SU(2)$ systems [16] suggests that this Liouville-type equation can be efficiently expanded in powers of the semiclassical parameter, even for systems with $SU(n)$ symmetry. In this expansion, the leading term is a first-order differential operator describing the classical dynamics of the Wigner distribution and the first-order corrections to the classical motion vanish. For Hamiltonians polynomial in the generators $\hat{C}_\alpha$ of the corresponding $\mathfrak{su}(n)$ Lie algebra the star-product can be replaced by the more specialized correspondence rules (or Bopp operators) [17], that establish particular maps

$$\hat{C}_\alpha \hat{f} \to \hat{\mathfrak{C}}^L_\alpha W_{\hat{f}}(\Omega), \qquad \hat{f}\hat{C}_\alpha \to \hat{\mathfrak{C}}^R_\alpha W_{\hat{f}}(\Omega) \tag{2}$$

where $\hat{\mathfrak{C}}^L_\alpha$ and $\hat{\mathfrak{C}}^R_\alpha$ are some differential operators.

In the $SU(2)$ case [10, 18, 19] the operators $\hat{\mathfrak{C}}^{L,R}_\alpha$ have the form of first-order differential operators multiplied by functions of the Casimir operators [9, 20]. This form of $\hat{\mathfrak{C}}^{L,R}_\alpha$ operators is related to the simple structure of the classical phase-space isomorphic to the coset $SU(2)/U(1)$ and closely connected to $SU(2)$ coherent states [8].

For systems with higher unitary symmetries, the explicit differential realizations of the $\star$-product and of the operators $\hat{\mathfrak{C}}^{L,R}_\alpha$ is still an open question. Indeed for systems with symmetries beyond $SU(2)$, $E(2)$ or $HW(n)$, the situation with phase-space mapping becomes significantly more abstract [2, 21]. Even in the simple case of a system transforming by a symmetric irreducible representation $(\lambda, 0)$ of the $SU(3)$ group, the Wigner map to the symplectic manifold $\mathfrak{M} = SU(3)/U(2)$ is quite involved [2]. Some relevant information from four-dimensional quasi-distribution functions can be extracted by appropriately projecting them into two-dimensional submanifolds, as was done in [22]. Nevertheless it is remarkable that a single semi-classical parameter, easily related to the square root of the eigenvalue of quadratic Casimir operator of $\mathfrak{su}(3)$ acting in the representation $(\lambda, 0)$, still appears very naturally in the Wigner map [10].

In this paper we obtain the correspondence rules for the $SU(3)$ Wigner function defined on the phase space $SU(3)/U(2)$, appropriate to the most physically relevant case of the symmetric representation $(\lambda, 0)$ of $SU(3)$. We find that, contrary to the $SU(2)$ and other known cases, the correspondence rules require second order derivatives (in addition to the differential action of the $\mathfrak{su}(3)$ Casimir operator), highlighting a fundamental difference between the present work and all previous results [3]. We ascribe this difference to the representation theory of the $U(2)$ subgroup that leaves invariant the highest weights of $SU(3)$ irreps of the type $(\lambda, 0)$; this subgroup can accommodate representations of dimension greater than 1. We apply the correspondence rules to the equations of motion for some non-linear Hamiltonians quadratic in the $\mathfrak{su}(3)$ generators. We find the form of the correspondence rules in the limit of large dimension of the representations, which allows us to obtain the semiclassical expansion of the equations of motion for Hamiltonian quadratic and more generally polynomial in the $\mathfrak{su}(3)$ generators.

2. The *SU*(3) Wigner function

The group $SU(3)$ is defined as the set of unitary 3×3 matrices with determinant 1. A convenient way to parametrize this set is by using a slight variation of the result given in [23] (see also [24]) and write an element $g \in SU(3)$ as a sequence of block $SU(2)$ matrices [10]:

$$g = \hat{R}_{23}(\alpha_1, \beta_1, -\alpha_1)\hat{R}_{12}(\alpha_2, \beta_2, -\alpha_2)\left[\hat{R}_{23}(\alpha_3, \beta_3, -\alpha_3)\Phi(\gamma_1, \gamma_2)\right] \tag{3}$$

where Φ is a diagonal matrix with determinant 1 containing two independent phases. Note that the last bracketed factors

$$\left[\hat{R}_{23}(\alpha_3, \beta_3, -\alpha_3)\Phi(\gamma_1, \gamma_2)\right] \tag{4}$$

are elements of an $U(2) \sim SU(2) \otimes U(1)$ subgroup. The corresponding $\mathfrak{su}(3)$ Lie algebra is spanned by the eight generators

$$\begin{aligned}
&\hat{C}_{12}, \hat{C}_{23}, \hat{C}_{13}\,, && \text{3 raising operators}\,,\\
&\hat{h}_1 = \hat{C}_{22} - \hat{C}_{33}\,, \hat{h}_2 = 2\hat{C}_{11} - \hat{C}_{22} - \hat{C}_{33}\,, && \text{2 Cartan generators}\,,\\
&\hat{C}_{21}, \hat{C}_{32}, \hat{C}_{31}\,, && \text{3 lowering operators}\,,
\end{aligned} \tag{5}$$

with the basic commutation relations

$$\left[\hat{C}_{ij}, \hat{C}_{k\ell}\right] = \delta_{jk}\hat{C}_{i\ell} - \delta_{i\ell}\hat{C}_{kj}. \tag{6}$$

With this we can recall some elementary facts on the construction of the symmetric $SU(3)$ Wigner function for $SU(3)$ irreps of the type $(\lambda, 0)$ [2, 23]. Basis states in the corresponding $\frac{1}{2}(\lambda+1)(\lambda+2)$ dimensional Hilbert space

$$|(\lambda, 0)\nu I\rangle := |(\lambda, 0)\nu_1\nu_2\nu_3; I\rangle \quad , \tag{7}$$

are labeled by the triple $\nu = (\nu_1, \nu_2, \nu_3)$ of nonnegative occupation numbers, subject to the constraints $\nu_1 + \nu_2 + \nu_3 = \lambda$. The weight of the state with occupation numbers (ν_1, ν_2, ν_3) is

$[\nu_1-\nu_2,\nu_2-\nu_3]$. For a general (p,q) irrep, a given weight can occur more than once; in such cases an $\mathfrak{su}(2)$ label I is enough to uniquely identify a set of basis states with identical weights. For states in $(\lambda,0)$ I is completely determined by the occupation numbers: $I=\frac{1}{2}(\nu_2+\nu_3)$; as a result this I index can sometimes be conveniently omitted for states in $(\lambda,0)$.

The appropriate phase-space is isomorphic to the coset $SU(3)/U(2)\sim\mathbb{CP}^2$, where the displacement operator

$$\hat{R}(\Omega)=\hat{R}_{23}(\alpha_1,\beta_1,-\alpha_1)\hat{R}_{12}(\alpha_2,\beta_2,-\alpha_2)\hat{R}_{23}(\alpha_1,-\beta_1,-\alpha_1), \tag{8}$$

depends on $\Omega\in SU(3)/U(2)$, with angles in the ranges

$$0\leqslant\alpha_{1,2}\leqslant 2\pi,\quad 0\leqslant\beta_{1,2}\leqslant\pi, \tag{9}$$

and where $\hat{R}_{ij}$ is an $SU(2)$ subgroup rotations, generated by $\{\hat{C}_{ij},\hat{C}_{ji},[\hat{C}_{ij},\hat{C}_{ji}]\}$.

If states transform by the irrep $(\lambda,0)$, operators acting on these states will be of the generic form $|(\lambda,0)\nu;I\rangle\langle(\lambda,0)\nu';I'|$ and will transform by the irrep $(\lambda,0)\otimes(0,\lambda)$, where $(0,\lambda)$ is the irrep conjugate to $(\lambda,0)$. This representation is reducible and decomposes in the direct sum [25]

$$(\lambda,0)\otimes(0,\lambda)=\sum_{\sigma=0}^{\lambda}(\sigma,\sigma)\,, \tag{10}$$

with (σ,σ) irreducible. The Wigner symbol of an arbitrary operator $\hat{A}$ acting on $|\lambda;\nu\rangle$ states is then defined as

$$W_{\hat{A}}(\Omega)=\mathrm{Tr}(\hat{w}_\lambda(\Omega)\hat{A}) \tag{11}$$

where the kernel $\hat{w}_\lambda(\Omega)$ has the form

$$\hat{w}_\lambda(\Omega)=\sum_{\sigma=0}^{\lambda}F_\sigma^\lambda\sum_{\nu J}D^{(\sigma,\sigma)}_{\nu J;(\sigma\sigma\sigma)0}(\Omega)\hat{T}^\lambda_{\sigma;\nu J}, \tag{12}$$

$$F_\sigma^\lambda=\sqrt{\frac{2(\sigma+1)^3}{(\lambda+1)(\lambda+2)}}\,, \tag{13}$$

with

$$\hat{T}^\lambda_{\sigma;\gamma I_\gamma}=\sum_{\nu\beta}|(\lambda,0)\nu;I_\nu\rangle\langle(\lambda,0)\beta;I_\beta|\tilde{C}^{\sigma\gamma I_\gamma}_{\lambda\nu I_\nu;\lambda^*\beta^* I_\beta}\,, \tag{14}$$

$$I_\nu=\frac{1}{2}(\lambda-\nu_1) \tag{15}$$

a component of the irreducible tensor operator transforming by (σ,σ), and where coset (harmonic) functions, invariant under the $U(2)$ subgroup of equation (4), are obtained as matrix elements of (8):

$$D^{(\sigma,\sigma)}_{\nu I;(\sigma\sigma\sigma)0}(\Omega)\equiv\langle(\sigma,\sigma)\nu I|\hat{R}(\Omega)|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle. \tag{16}$$

The coefficients $\tilde{C}^{\sigma\gamma I_\gamma}_{\lambda\alpha I_\alpha;\lambda^*\beta^* I_{\beta^*}}$ are up to a phase, $\mathfrak{su}(3)$ Clebsch–Gordan coefficients; their expressions are given in appendix A.1. In particular, the symbol of a tensor operator $\hat{T}^\lambda_{\sigma;\mu J}$ is

Table 1. Wigner symbols of the $su(3)$ generators.

$\hat{C}_{12}$	$-\frac{N}{2\sqrt{6}}\hat{T}^{\lambda}_{1;(201)\frac{1}{2}}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{\mathrm{i}\alpha_2}\cos(\frac{1}{2}\beta_1)\sin\beta_2$
$\hat{C}_{13}$	$\frac{N}{2\sqrt{6}}\hat{T}^{\lambda}_{1;(210)\frac{1}{2}}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\sin(\frac{1}{2}\beta_1)\sin\beta_2$
$\hat{C}_{23}$	$\frac{N}{2\sqrt{6}}\hat{T}^{\lambda}_{1;(120)1}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{\mathrm{i}\alpha_1}\sin\beta_1\sin^2(\frac{1}{2}\beta_2)$
$\hat{C}_{21}$	$-\frac{N}{2\sqrt{6}}\left(\hat{T}^{\lambda}_{1;(201)\frac{1}{2}}\right)^{\dagger}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{-\mathrm{i}\alpha_2}\cos(\frac{1}{2}\beta_1)\sin\beta_2$
$\hat{C}_{31}$	$\frac{N}{2\sqrt{6}}\left(\hat{T}^{\lambda}_{1;(210)\frac{1}{2}}\right)^{\dagger}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin(\frac{1}{2}\beta_1)\sin\beta_2$
$\hat{C}_{32}$	$\frac{N}{2\sqrt{6}}\left(\hat{T}^{\lambda}_{1;(120)1}\right)^{\dagger}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{-\mathrm{i}\alpha_1}\sin\beta_1\sin^2(\frac{1}{2}\beta_2)$
$\hat{H}_2=\frac{1}{\sqrt{6}}\hat{h}_2$	$\frac{N}{2}\hat{T}^{\lambda}_{1;(111)0}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}(1+3\cos\beta_2)$
$\hat{H}_1=-\frac{1}{\sqrt{2}}\hat{h}_1$	$-\frac{N}{4\sqrt{3}}\hat{T}^{\lambda}_{1;(111)1}$	$\sqrt{\lambda(\lambda+3)}\cos\beta_1\sin^2(\frac{1}{2}\beta_2)$

$$W_{\hat{T}^{\lambda}_{\sigma;\mu J}}(\Omega)=F^{\lambda}_{\sigma}\left(D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\right)^{*}. \tag{17}$$

In table 1 we give symbols of $\mathfrak{su}(3)$ generators of equation (5) and their relations with the $(1,1)$ tensor operators, where

$$N=\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)} \tag{18}$$

is a normalization factor.

Using the transformation property of the tensor operators

$$\hat{R}(\Omega)\hat{T}^{\lambda}_{\sigma;\mu I}\hat{R}^{\dagger}(\Omega)=\sum_{\nu J}D^{(\sigma,\sigma)}_{\nu J;\mu I}(\Omega)\hat{T}^{\lambda}_{\sigma;\nu J} \tag{19}$$

where

$$D^{(\tau,\tau)}_{\nu J;\mu I}(\Omega)=\langle(\sigma,\sigma)\nu I|R(\Omega)|(\sigma,\sigma)\mu J\rangle \tag{20}$$

are $SU(3)$ D-functions, we can represent the quantization kernel $\hat{w}_{\lambda}(\tilde{\Omega})$ in an explicitly covariant form

$$\hat{w}_{\lambda}(\Omega)=\hat{R}(\Omega)\hat{w}_{\lambda}(0)\hat{R}^{\dagger}(\Omega) \tag{21}$$

where $\hat{R}(\Omega)$ is given in equation (8) and $\hat{w}_{\lambda}(0)$ contains only diagonal tensor operators

$$\hat{w}_{\lambda}(0)=\sum_{\sigma=0}^{\lambda}F^{\lambda}_{\sigma}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma)0} \tag{22}$$

and is invariant under $U(2)$ transformations of the type given in equation (4).

3. The correspondence rules

The correspondence rules allow us to represent the symbol of a product of an $\mathfrak{su}(3)$ generator $\hat{C}_{ij}$ of equation (5) with an arbitrary operator $\hat{B}$ in form of a local action, *v.g.*

$$W_{\hat{C}_{ij}\hat{B}}(\Omega)=\mathrm{Tr}\left(\hat{w}_{\lambda}(\Omega)\hat{C}_{ij}\hat{B}\right)=\hat{\mathfrak{C}}_{ij}W_{\hat{B}}(\Omega), \tag{23}$$

Table 2. Relationship between $(\nu_1\nu_2\nu_3)I$ and $(\bar{\nu}_1\bar{\nu}_2\bar{\nu}_3)I$.

$(\nu_1\nu_2\nu_3)I$	$(\bar{\nu}_1\bar{\nu}_2\bar{\nu}_3)I$	$(\nu_1\nu_2\nu_3)I$	$(\bar{\nu}_1\bar{\nu}_2\bar{\nu}_3)I$
$(210)\frac{1}{2}$	$(\tau+1,\tau,\tau-1)\frac{1}{2}$	$(111)1$	$(\tau,\tau,\tau)1$
$(201)\frac{1}{2}$	$(\tau+1,\tau-1,\tau)\frac{1}{2}$	$(102)1$	$(\tau,\tau-1,\tau+1)1$
$(021)\frac{1}{2}$	$(\tau-1,\tau+1,\tau)\frac{1}{2}$	$(120)1$	$(\tau,\tau+1,\tau-1)1$
$(012)\frac{1}{2}$	$(\tau-1,\tau,\tau+1)\frac{1}{2}$	$(111)0$	$(\tau,\tau,\tau)0$

where $\hat{\mathfrak{C}}_{ij}$ is a differential operator. It is clear that it suffices to express the right and left products of first-rank tensor operators on the Wigner kernel as an operator acting on the argument of the kernel.

Let us start with the left action. It is convenient to separate the calculations into two steps. First, we reduce the problem of arbitrary tensors multiplication to one involving a diagonal form by using equation (21):

$$\begin{aligned}\hat{T}^{\lambda}_{1;\alpha J}\hat{w}_\lambda(\Omega) &= \hat{T}^{\lambda}_{1;\alpha J}\hat{R}(\Omega)\hat{w}_\lambda(0)\hat{R}^\dagger(\Omega)\\ &= \hat{R}(\Omega)\left[\hat{R}^\dagger(\Omega)\hat{T}^{\lambda}_{1;\alpha J}\hat{R}(\Omega)\right]\hat{w}_\lambda(0)\hat{R}^\dagger(\Omega)\\ &= \sum_{\nu I} D^{(1,1)}_{\nu I;\alpha J}(\Omega^{-1})\left[\hat{R}(\Omega)\,\hat{T}^{\lambda}_{1;\nu I}\hat{w}_\lambda(0)\,\hat{R}^\dagger(\Omega)\right]. \end{aligned} \tag{24}$$

The possible values of νI are given as part of table 2. The coupling $\hat{T}^{\lambda}_{1;\nu I}\hat{w}_\lambda(0)$ has a form of a linear combination

$$\hat{T}^{\lambda}_{1;\nu I}\hat{w}_\lambda(0) = \sum_{\sigma=0}^{\lambda} F^{\lambda}_{\sigma}\,\hat{T}^{\lambda}_{1;\nu I}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma)0} \tag{25}$$

$$= \sum_{\tau} F^{\lambda}_{\tau}\, a^{L}_{\nu_1 I}(\lambda;\tau)\hat{T}^{\lambda}_{\tau;\bar{\nu}I} \tag{26}$$

where the coefficients $a^{L}_{\nu_1 I}(\lambda;\tau)$ are evaluated explicitly in appendix B and ν and $\bar{\nu}$ are related in table 2.

Thus, equation (24) is transformed to

$$\hat{T}^{\lambda}_{1;\alpha J}\hat{w}_\lambda(\Omega) = \sum_{\nu I} D^{(1,1)}_{\nu I;\alpha J}(\Omega^{-1})\sum_{\tau\nu' I'} F^{\lambda}_{\tau} a^{L}_{\nu_1 I}(\lambda;\tau) D^{(\tau,\tau)}_{\nu' I',\bar{\nu}I}(\Omega)\hat{T}^{\lambda}_{\tau;\nu' I'}. \tag{27}$$

A more useful explicit form is given by

$$\begin{aligned}&\hat{T}^{\lambda}_{1;\alpha J}\hat{w}_\lambda(\Omega)\\ &= \sum_{\nu_1=0,2}\sum_{\nu_2\nu_3} D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\sum_{\tau\nu' I'} F^{\lambda}_{\tau} a^{L}_{\nu_1\frac{1}{2}}(\lambda;\tau) D^{(\tau,\tau)}_{\nu' I';\bar{\nu}\frac{1}{2}}(\Omega)\hat{T}^{\lambda}_{\tau;\nu' I'}\\ &\quad+\sum_{\nu_2\nu_3} D^{(1,1)}_{(1\nu_2\nu_3)1;\alpha J}(\Omega^{-1})\sum_{\tau\nu' I'} F^{\lambda}_{\tau} a^{L}_{11}(\lambda;\tau) D^{(\tau,\tau)}_{\nu' I';\bar{\nu}1}(\Omega)\hat{T}^{\lambda}_{\tau;\nu' I'}\\ &\quad+ D^{(1,1)}_{(111)0;\alpha J}(\Omega^{-1})\sum_{\tau\nu' I'} F^{\lambda}_{\tau} a^{L}_{10}(\lambda;\tau) D^{(\tau,\tau)}_{\nu' I';\bar{\nu}0}(\Omega)\hat{T}^{\lambda}_{\tau;\nu' I'}. \end{aligned} \tag{28}$$

The right-hand side of equation (28) contains $SU(3)$ D-functions which are not harmonic functions of the type of equation (16), but which can be related to such functions by applying a differential operator transforming a coset function into an *adjacent* function, differing from the coset function by some $0, \pm 1$ changes in the occupation numbers. Details on obtaining these adjacency relations can be found in appendix C.2.

The adjacency relations for $\nu_1 = 0, 2$ and $I = \frac{1}{2}$ are similar to those for the $SU(2)$ D-functions [27] and can be compactly represented as

$$\hat{\mathbb{S}}_{\nu\frac{1}{2}} D^{(\tau,\tau)}_{\mu J;(\tau\tau\tau)0}(\Omega) = (-1)^{\nu_1/2}\sqrt{\frac{\tau(\tau+2)}{2}} D^{(\tau,\tau)}_{\mu J;\bar{\nu}\frac{1}{2}}(\Omega)\,, \tag{29}$$

where each $\hat{\mathbb{S}}_{\nu\frac{1}{2}}$ is a first order differential operator

$$\hat{\mathbb{S}}_{\nu\frac{1}{2}} = \sum_k d_{\nu\frac{1}{2}}(\Omega_k)\frac{\partial}{\partial\Omega_k}\,, \quad (\Omega_1,\Omega_2,\Omega_3,\Omega_4) = (\alpha_1,\beta_1,\alpha_2,\beta_2)\,. \tag{30}$$

The coefficients $d_{\nu\frac{1}{2}}(\Omega_k)$ are given in table C3.

To continue, we note that the expansion of equation (28) contains terms of the type $D^{(\tau,\tau)}_{\nu' I';\bar{\nu}1}(\Omega)$. We argue here that these cannot be expressed as a first order differential operator acting on a coset function. To see how this fundamental difference with $SU(2)$ comes about, one must recognize that the action of any first order differential operator $\hat{\mathcal{D}}$ on a coset function will drop operators linear in $\hat{C}_{ij}$ from the exponential $\hat{R}(\Omega)$; one can reorganize the resulting expression to

$$\begin{aligned}\hat{\mathcal{D}} D^{(\tau,\tau)}_{\nu' I';(\tau\tau\tau)0}(\Omega) = &\sum_{i\neq j} c_{ij}\langle(\tau,\tau)\nu' I'|\hat{R}(\Omega)\hat{C}_{ij}|(\tau,\tau)(\tau\tau\tau)0\rangle \\ &+ \langle(\tau,\tau)\nu' I'|\hat{R}(\Omega)(c_1\hat{H}_1 + c_2\hat{H}_2)|(\tau,\tau)(\tau\tau\tau)0\rangle\end{aligned} \tag{31}$$

by bringing any $\hat{C}_{ij}$ to the right of the $SU(3)$ rotation $\hat{R}(\Omega)$. The action of the $\mathfrak{su}(2)$ ladder operators $\hat{C}_{23}, \hat{C}_{32}$ kills $|(\tau,\tau)(\tau\tau\tau)0\rangle$, since by construction this state is an $\mathfrak{su}(2)$ scalar; the action of the remaining ladder operator $\hat{C}_{13}, \hat{C}_{31}, \hat{C}_{12}$ and $\hat{C}_{21}$ will lead to functions of the form $D^{(\tau,\tau)}_{\nu' I';\mu'\frac{1}{2}}(\Omega)$ by inspection, and thus not of the desired form. Finally the operators $\hat{H}_1$ and $\hat{H}_2$ both annihilate $|(\tau,\tau)(\tau\tau\tau)0\rangle$ by definition.

Instead, we find the adjacency relations for functions of the $D^{(\tau,\tau)}_{\nu' I';\bar{\nu}1}(\Omega)$ type to be of the form

$$\begin{aligned}&\hat{\mathbb{S}}^{(2)}_{1\nu_2\nu_3;1} D^{(\tau,\tau)}_{\mu J;(\tau\tau\tau)0}(\Omega) \\ &\qquad = -\tau(\tau+2)\sqrt{\frac{(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}{6}} D^{(\tau,\tau)}_{\mu J;\bar{\nu}1}(\Omega)\,,\end{aligned} \tag{32}$$

where $\hat{\mathbb{S}}^{(2)}_{1\nu_2\nu_3;1}$ are *second order differential operators*

$$\begin{aligned}\hat{\mathbb{S}}^{(2)}_{(120);1} = &\hat{\mathbb{S}}_{(021)\frac{1}{2}}\hat{\mathbb{S}}_{(210)\frac{1}{2}} + f_{(021)\frac{1}{2};(102)\frac{1}{2}}\hat{\mathbb{S}}_{(201)\frac{1}{2}} \\ &+ \frac{\sqrt{2}}{2} f_{(021)\frac{1}{2};(111)1}\hat{\mathbb{S}}_{(210)\frac{1}{2}} - \sqrt{\frac{3}{2}} f_{(021)\frac{1}{2};(111)0}\hat{\mathbb{S}}_{(210)\frac{1}{2}}\,,\end{aligned} \tag{33}$$

$$\begin{aligned}\hat{\mathbb{S}}^{(2)}_{(102);1} = &\hat{\mathbb{S}}_{(012)\frac{1}{2}}\hat{\mathbb{S}}_{(201)\frac{1}{2}} - f_{(012)\frac{1}{2};(120)1}\hat{\mathbb{S}}_{(210)\frac{1}{2}} \\ &- \left(\frac{\sqrt{2}}{2} f_{(012)\frac{1}{2};(111)1} + \sqrt{\frac{3}{2}} f_{(012)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{(201)\frac{1}{2}}\,,\end{aligned} \tag{34}$$

$$
\begin{aligned}
-\frac{1}{\sqrt{3}}\hat{\mathbb{S}}^{(2)}_{(111);1} = {} & \hat{\mathbb{S}}_{021;\frac{1}{2}}\hat{\mathbb{S}}_{201;\frac{1}{2}} + \hat{\mathbb{S}}_{012;\frac{1}{2}}\hat{\mathbb{S}}_{210;\frac{1}{2}} \\
& - f_{(021)\frac{1}{2};(120)1}\hat{\mathbb{S}}_{210;\frac{1}{2}} - f_{(012)\frac{1}{2};(102)1}\hat{\mathbb{S}}_{201;\frac{1}{2}} \\
& - \left(\sqrt{\frac{3}{2}} f_{(021)\frac{1}{2};(111)1} + \frac{\sqrt{2}}{2} f_{(021)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{201;\frac{1}{2}} \\
& - \left(\sqrt{\frac{3}{2}} f_{(012)\frac{1}{2};(111)1} - \frac{\sqrt{2}}{2} f_{(012)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{210;\frac{1}{2}},
\end{aligned} \tag{35}
$$

since they contain products of first order operators $\hat{\mathbb{S}}_{\mu\frac{1}{2}}$. The coefficients $f_{\beta a}$ are given in table C4.

As a second step we use the relations equations (29) and (32) to explicitly determine the correspondence rules equation (28), obtaining

$$
\hat{T}^{\lambda}_{1;\alpha J}\hat{w}_{\lambda}(\Omega) := \hat{\mathfrak{C}}^{L}_{\alpha J}\hat{w}_{\lambda}(\Omega), \tag{36}
$$

$$
\begin{aligned}
\hat{\mathfrak{C}}^{L}_{\alpha J} = {} & \sqrt{2}\sum_{\nu_1=0,2}\sum_{\nu_2\nu_3} D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})(-1)^{\bar{\nu}_1/2}\hat{\mathbb{S}}_{\nu\frac{1}{2}}\hat{\mathcal{C}}_2^{-1/2}\hat{a}^{L}_{\nu_1\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2) \\
& - \sum_{\nu_2\nu_3} D^{(1,1)}_{(1\nu_2\nu_3)1;\alpha J}(\Omega^{-1})\sqrt{\frac{6}{(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}}\hat{\mathbb{S}}^{(2)}_{1\nu_2\nu_3;1}\hat{\mathcal{C}}_2^{-1}\hat{a}^{L}_{11}(\lambda;\hat{\mathcal{C}}_2) \\
& + D^{(1,1)}_{(111)0;\alpha J}(\Omega^{-1})\hat{a}^{L}_{10}(\lambda;\hat{\mathcal{C}}_2),
\end{aligned} \tag{37}
$$

where $\hat{\mathcal{C}}_2$ is the differential realization of the $su(3)$ Casimir invariant,

$$
\hat{\mathcal{C}}_2 = \sum_{i\neq j}\hat{C}_{ij}\hat{C}_{ji} + \hat{H}_1^2 + \hat{H}_2^2 \tag{38}
$$

$$
\hat{\mathcal{C}}_2\, D^{(\sigma,\sigma)}_{\nu I;(\sigma\sigma\sigma)0}(\Omega) = \tau(\tau+2)D^{(\sigma,\sigma)}_{\nu I;(\sigma\sigma\sigma)0}(\Omega)\,, \tag{39}
$$

on the harmonic functions equation (16). This differential realization is given explicitly in appendix C.1. The operators $\hat{a}^{L}_{\nu_1 I}(\lambda;\hat{\mathcal{C}}_2)$ are functions of the $\hat{\mathcal{C}}_2$, such that

$$
\hat{a}^{L}_{\mu_1 J}(\lambda;\hat{\mathcal{C}}_2)D^{(\sigma,\sigma)}_{\nu I;(\sigma\sigma\sigma)0}(\Omega) = a^{L}_{\mu_1 J}(\lambda;\sigma)D^{(\sigma,\sigma)}_{\nu I;(\sigma\sigma\sigma)0}(\Omega). \tag{40}
$$

The right action

$$
\hat{w}_{\lambda}(\Omega)\hat{T}^{\lambda}_{1;\alpha J} := \hat{\mathfrak{C}}^{R}_{\alpha J}\hat{w}_{\lambda}(\Omega) \tag{41}
$$

is obtained by replacing $a^{L}_{\nu_1 I}(\lambda;\tau)\to a^{R}_{\nu_1 I}(\lambda;\tau)$ and $\hat{\mathfrak{C}}^{L}_{\alpha J}\to\hat{\mathfrak{C}}^{R}_{\alpha J}$ in equation (37), with the coefficients $a^{R}_{\nu_1 I}(\lambda;\tau)$ and $a^{L}_{\nu_1 I}(\lambda;\tau)$ related quite simply by

$$
\begin{aligned}
& a^{R}_{0\frac{1}{2}}(\lambda;\tau) = a^{L}_{0\frac{1}{2}}(\lambda;\tau) - \sqrt{3\tau(\tau+2)} \\
& a^{R}_{11}(\lambda;\tau) = a^{L}_{11}(\lambda;\tau) \\
& a^{R}_{10}(\lambda;\tau) = a^{L}_{10}(\lambda;\tau) \\
& a^{R}_{2\frac{1}{2}}(\lambda;\tau) = a^{L}_{2\frac{1}{2}}(\lambda;\tau) + \sqrt{3\tau(\tau+2)}\,.
\end{aligned} \tag{42}
$$

The correspondence rules encapsulated in equations (36) and (41) are the main results of this paper. The appearance of second order operators, $\hat{\mathbb{S}}^{(2)}_{1\nu_2\nu_3;1}$, which are independent from the Casimir operator $\hat{\mathcal{C}}_2$, is a key difference with the $SU(2)$ formalism, where the correspondence rules contain only first order differential operators and functions of the $SU(2)$ Casimir operator. This new 'complication' for $SU(3)$ is ultimately a by-product of using coset function $D^{(\tau,\tau)}_{\nu' I';(\tau\tau\tau)0}(\Omega)$ which are invariant under right action by an element in $U(2)$; because the zero-weight subspace of (τ,τ) will in general contain more than just an $I=0$ state, the action of generators with $I=1$ eventually yields $D^{(\tau,\tau)}_{\nu' I';\bar{\nu}1}(\Omega)$, which is no longer $U(2)$-invariant under right action. Such a situation cannot occur in $SU(2)$ since the $U(1)$-invariant zero-weight space is one-dimensional. Moreover, it is clear that this complication will occur beyond $SU(3)$ for any $SU(n)$.

Note that the coefficients a^R and a^L for $\nu_1=1$ do not change sign so that, as an immediate application of equations (37) and (41) we obtain using the relations of equation (42) the commutator

$$[T^\lambda_{1;\alpha J},\hat{w}_\lambda(\Omega)] = \frac{\sqrt{24}}{N}\sum_\nu D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\mathbb{S}}_{\nu\frac{1}{2}}\hat{w}_\lambda(\Omega)\,, \tag{43}$$

which contains only first order differential operators, in a manner similar to the $SU(2)$ case [15, 18].

4. Examples

The explicit form of the correspondence rules equation (37) can be used to map Schrödinger equations for Hamiltonians polynomial in the generators $\hat{C}_{ij}$ to evolution equations for the Wigner function. Define

$$\hat{h} = \sum_{\alpha,J} h_{\alpha J}\hat{T}^\lambda_{1,\alpha J} \tag{44}$$

and consider first Hamiltonians of the type $\hat{H}=\hat{h}$ linear in the generators. With $\hat{\rho}$ an arbitrary operators, we rewrite the symbol of the commutator as

$$W_{[\hat{H},\hat{\rho}]}(\Omega) = -\mathrm{Tr}(\hat{\rho}[\hat{H},\hat{w}_\lambda(\Omega)]), \tag{45}$$

and use equation (17) together with the relation

$$D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1}) = (-1)^{\nu_1/2}\sqrt{\frac{(\lambda+1)(\lambda+2)}{24}}\,\hat{\mathbb{S}}^*_{\nu;\frac{1}{2}}W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega) \tag{46}$$

to obtain from equation (43) the following equation of motion for the Hamiltonian $\hat{H}=\hat{h}$:

$$\partial_t W_{\hat{\rho}}(\Omega) = -\epsilon^{-1}\{W_{\hat{H}}(\Omega),W_{\hat{\rho}}(\Omega)\}_{\mathcal{P}} \tag{47}$$

where

$$\epsilon^{-1} = 2\sqrt{\lambda(\lambda+3)} \tag{48}$$

is the so-called semi-classical parameter, and $\{\cdot,\cdot\}_{\mathcal{P}}$ is the Poisson bracket given [22] by

$$\{W_{\hat{H}}(\Omega),W_{\hat{\rho}}(\Omega)\}_{\mathcal{P}} = -\mathrm{i}\sum_\nu(-1)^{\frac{1}{2}\nu_1}\left(\hat{\mathbb{S}}^*_{\nu;\frac{1}{2}}W_{\hat{H}}(\Omega)\right)\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}W_{\hat{\rho}}(\Omega)\right)\,. \tag{49}$$

The sum in equation (49) is limited to $\nu_1 = 0, 2$ as these are the only ones for which $I = \frac{1}{2}$.

Suppose next that $\hat{H} = \hat{h}^2$: the Hamiltonian is the square of a linear combination of $\mathfrak{su}(3)$ generators. Equation (45) can be reduced to

$$W_{[\hat{h}^2,\hat{\rho}]} = -\mathrm{Tr}(\hat{\rho}([\hat{h},\hat{w}_\lambda(\Omega)]\hat{h} + \hat{h}[\hat{h},\hat{w}_\lambda(\Omega)])) \,. \tag{50}$$

In the specific case where $\hat{H} = (\hat{T}^{\lambda}_{1;\alpha J})^2$, we obtain from equations (37) and (43)

$$\mathrm{i}\partial_t W_{\hat{\rho}} = -\frac{\sqrt{24}}{N}\sum_{\nu} D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\mathbb{S}}_{\nu\frac{1}{2}}(\hat{\mathfrak{C}}^{L}_{\alpha J} + \hat{\mathfrak{C}}^{R}_{\alpha J})W_{\hat{\rho}} \,. \tag{51}$$

Here again, the sum over ν is restricted to $\nu_1 = 0, 2$ as only for those values can we have $I = \frac{1}{2}$.

We emphasize that equation (51) contains third order derivatives (in addition to functions of the Casimir operator). This is drastically different from the $SU(2)$ case, where third order differential operators appear only due to double derivatives in the differential expressions for Casimir functions. This third-order structure in the $SU(3)$ problem will lead to a very different long-time behaviour for Hamiltonians quadratic in $\mathfrak{su}(3)$ generators.

In particular, for Hamiltonians $\hat{H} \propto (\hat{T}^{\lambda}_{1;(\alpha)111)0})^2$ and $\hat{H} \propto (\hat{T}^{\lambda}_{1;(\alpha)111)1})^2$, the evolution describes '$\mathfrak{su}(3)$' squeezing effect [26]. The full expressions of equation (51) for these two cases can be found in appendix D. The evolution equation for other cases is quite complicated even for Hamiltonians $\hat{H} = \hat{h}^2$ quadratic in arbitrary combinations of generators.

5. Semiclassical limit

The semiclassical limit in quantum systems is associated with a large value of some physical parameter, *v.g.* the number of photons (in systems with $HW(1)$ symmetry), the size of an effective spin ($SU(2)$ symmetry), large value of a projection of angular momentum ($E(2)$ symmetry). Then, the semiclassical expansion is performed in the inverse of this 'large' parameter. In physical realizations of quantum systems with $SU(3)$ symmetry, *v.g.* the Bose–Einstein condensate in a three-well configuration [29], the semiclassical limit would correspond to the large number of total excitations. From the mathematical perspective this corresponds to the large dimension of the (symmetric) representation. Then, ϵ defined in equation (48) (or its approximate value $\epsilon \approx (2\lambda + 3)^{-1}$) can be considered as appropriate semiclassical expansion parameter whenever $\epsilon \ll 1$.

In order to obtain the asymptotic form of the correspondence rules we expand the operators $\hat{a}^{R,L}_{\nu I}(\lambda;\bar{\nu})$ appearing in equation (37) in powers of ϵ, keeping two non-vanishing orders, as given explicitly in appendix B. Then, the operators $\hat{\mathfrak{C}}^{L,R}$ take the following forms:

$$\begin{aligned}\hat{\mathfrak{C}}^{L,R}_{\alpha J} = N^{-1}\Bigg[&\sqrt{6}\sum_{\nu_1=0,2}\left(\pm 1 + \frac{3}{2}(-1)^{\nu_1/2}\epsilon\right)\left(D^{(1,1)}_{\alpha J;\nu\frac{1}{2}}(\Omega)\right)^{*}\hat{\mathbb{S}}_{\nu;\frac{1}{2}} \\ &- \epsilon\sum_{\nu_2\nu_3}\sqrt{\frac{6}{(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}}\left(D^{(1,1)}_{\alpha J;(1\nu_2\nu_3)1}(\Omega)\right)^{*}\hat{\mathbb{S}}^{(2)}_{\nu;1} \\ &+ \left(\frac{2}{\epsilon} - 3\epsilon(\hat{\mathcal{C}}_2 + 3)\right)\left(D^{(1,1)}_{\alpha J;(111)0}(\Omega)\right)^{*}\Bigg]\end{aligned} \tag{52}$$

with the $+$ and $-$ signs in the first term on the right for $\hat{\mathfrak{C}}^L_{\alpha J}$ and $\hat{\mathfrak{C}}^R_{\alpha J}$, respectively. Then, we obtain from equation (51) the following approximate equation of motion for a square of any $\mathfrak{su}(3)$ generator

$$\partial_t W_{\hat{\rho}} = -\epsilon^{-1}\{W_{\hat{h}^2}, W_{\hat{\rho}}\}_{\mathcal{P}} + O(\epsilon), \tag{53}$$

where no correction to the classical evolution of order ϵ^0 in the semiclassical parameter appears, as it is expected for the Wigner function semiclassical dynamics. It is clear that equation (53) corresponds to the so-called Truncated Wigner approximation [15], widely used in quantum systems with low-rank symmetries for the description of the semiclassical dynamic effects.

One should stress that there are two types of the second order differential operators, appearing in the first-order correction terms ($\sim\epsilon$): the first type is proportional to the Casimir operator (this is similar to the $SU(2)$ situation) and the second type is proportional to $\hat{\mathbb{S}}^{(2)}_{\nu;1}$: the appearance of this latter type qualitatively distinguishes the evolutions of systems with low or higher rank symmetries.

6. Conclusions

The correspondence rules of equation (37) can be immediately rewritten for the Wigner symbols and recast in terms of the star-product operation:

$$W_{\hat{T}^{\lambda}_{1;\alpha J}\hat{\rho}} = \hat{\mathfrak{C}}^R_{\alpha J} W_{\hat{\rho}} = W_{\hat{T}^{\lambda}_{1;\alpha J}} \star W_{\hat{\rho}}\,, \tag{54}$$

$$W_{\hat{\rho}\hat{T}^{\lambda}_{1;\alpha J}} = \hat{\mathfrak{C}}^L_{\alpha,J} W_{\hat{\rho}} = W_{\hat{\rho}} \star W_{\hat{T}^{\lambda}_{1;\alpha J}}. \tag{55}$$

The principal difference with other known correspondence rules consists in the appearance of second order derivatives in the operators $\hat{\mathfrak{C}}^{L,R}_{\alpha J}$; this is in addition to the derivatives contained in the differential form of the Casimir operators.

Apart from derivatives arising in the Casimir operator, the *exact* equation of motion describing the non-linear evolution of the Wigner function contains at least third-order differential operators; this significantly complicates the analysis of non-linear dynamics in comparison, for instance, with spin evolution.

In spite of this novel feature, the leading order term of the semiclassical expansion of the evolution equation is still reduced to the Poisson brackets on the $\mathbb{CP}^2$-manifold, and the short-time dynamics can still be well described in terms of classical trajectories. The appearance of third order derivatives in the exact equations of motion significantly affects the qualitative character of non-linear evolution beyond the semiclassical times. The explicit forms of terms of order ϵ and $1/\epsilon$ in equation (52)—terms that cancel to leading order in the semi-classical expansion of the quantum Hamiltonian evolution—are non-trivial and quite important *v.g.* for the phase-space description of $SU(3)$ dissipative channels.

As an interesting and important by-product we have obtained adjacency relations connecting some $SU(3)$ D-functions to the harmonic functions on $SU(3)/U(2)$, thereby generalizing results on spherical harmonics (see chapter 4 of [27]). We expect to use these novel relations in application of the $SU(3)$ group in quantum mechanics.

We have also formally shown that a single parameter ϵ, given in equation (48), related to the inverse eigenvalue of the Casimir, and which scales like the inverse of the label λ of the irrep $(\lambda, 0)$, naturally enters in the semiclassical limit of the nonlinear $SU(3)$ dynamics.

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Appendix A. Technical results on Clebsch, tensors and Racah

The $SU(3)$ Clebsch–Gordan coefficients factor into a reduced (or double-barred) coefficient multiplied by a $SU(2)$ coefficient:

$$\left\langle \begin{matrix}(\lambda_1,\mu_1)\\(a_1a_2a_3)I_1\end{matrix};\begin{matrix}(\lambda_2,\mu_2)\\(b_1b_2b_3)I_2\end{matrix}\middle|\begin{matrix}(\lambda,\mu)\\(c_1c_2c_3)J\end{matrix}\right\rangle_\kappa = \left\langle \begin{matrix}(\lambda_1,\mu_1)\\a_1I_1\end{matrix};\begin{matrix}(\lambda_2,\mu_2)\\b_1I_2\end{matrix}\middle\|\begin{matrix}(\lambda,\mu)\\\mu_1J\end{matrix}\right\rangle_\kappa \times\left\langle \begin{matrix}I_1\\\frac{1}{2}(a_2-a_3)\end{matrix};\begin{matrix}I_2\\\frac{1}{2}(b_2-b_3)\end{matrix}\middle|\begin{matrix}J\\\frac{1}{2}(c_2-c_3)\end{matrix}\right\rangle \tag{A.1}$$

where κ labels (where appropriate) the multiple copies of the irrep (λ,μ) in the decomposition of the tensor product $(\lambda_1,\mu_1)\otimes(\lambda_2,\mu_2)$. This index κ is omitted when (λ,μ) occurs once in $(\lambda_1,\mu_1)\otimes(\lambda_2,\mu_2)$.

A.1. Tensor operators

One can show [11] that

$$\begin{aligned}&\left\langle \begin{matrix}(\lambda,0)\\\lambda-a;\frac{1}{2}a\end{matrix};\begin{matrix}(0,\lambda)\\\sigma+a;\frac{1}{2}(\sigma+a)\end{matrix}\middle\|\begin{matrix}(\sigma,\sigma)\\2\sigma;\frac{1}{2}\sigma\end{matrix}\right\rangle\\&\quad=(-1)^a\sqrt{\frac{(\sigma+a+1)!(\lambda-\sigma)!(\lambda-a)!(2\sigma+2)!}{(\sigma+1)!(\lambda-\sigma-a)!a!\sigma!(\lambda+\sigma+2)!}}\end{aligned} \tag{A.2}$$

for the highest weight state, and more generally, for any state in (σ,σ)

$$\begin{aligned}&\left\langle \begin{matrix}(\lambda,0)\\\nu_1';\frac{1}{2}(\lambda-\nu_1')\end{matrix};\begin{matrix}(0,\lambda)\\\lambda+\sigma-\nu_1'-p;\frac{1}{2}(\lambda+\sigma-\nu_1'-p)\end{matrix}\middle\|\begin{matrix}(\sigma,\sigma)\\2\sigma-p;I\end{matrix}\right\rangle=(-1)^\lambda\frac{(\lambda-\nu_1')!}{\sigma!}\\&\times\sqrt{\frac{(\sigma-I+\frac{1}{2}(\sigma-p))!(\sigma+I+\frac{1}{2}(\sigma-p)+1)!(\lambda-\sigma)!(2\sigma+2)(\nu_1'+p-\sigma)!(1+I+\frac{1}{2}(p+\sigma))!}{(2+\lambda+\sigma)!(\nu_1')!(\lambda-\nu_1'-I+\frac{1}{2}(\sigma-p))!(1+I+\lambda-\nu_1'+\frac{1}{2}(\sigma-p))!(\frac{1}{2}(\sigma+p)-I)!}}\\&\times\sum_{\nu_1=\nu_{1,\min}}^{\nu_{1,\max}}(-1)^{\nu_1}\left(\frac{(\lambda-I-\nu_1+\frac{1}{2}(\sigma+p))!\nu_1!}{(\nu_1-\nu_1')!(p-\nu_1+\nu_1')!(\nu_1-\sigma)!(\lambda-\nu_1)!}\right)\\&\times{}_3F_2\left[\begin{matrix}\nu_1-\lambda & I+\nu_1'-\lambda+\frac{1}{2}(p-\sigma) & 1+I+\frac{1}{2}(\sigma-p)\\\nu_1'-\lambda & I+\nu_1-\lambda-\frac{1}{2}(p+\sigma) & \end{matrix};1\right]\end{aligned} \tag{A.3}$$

with ${}_3F_2$ the generalized hypergeometric function, and

$$\nu_{1,\min}=\max[\sigma,\nu_1'],\nu_{1,\max}=\min[\lambda,\nu_1'+p,\lambda-I+\frac{1}{2}(\sigma+p)]. \tag{A.4}$$

The bounds of the sum are determined by the factorials involved with ν_1.

The combinations

$$\hat{T}^\lambda_{\sigma;\nu I_\nu}=\sum_{\alpha I_\alpha\beta I_\beta}\tilde{C}^{\sigma\nu I_\nu}_{\lambda\alpha I_\alpha;\lambda^*\beta I_\beta}|(\lambda,0)\alpha I_\alpha\rangle\langle(\lambda,0)\beta I_\beta|, \tag{A.5}$$

$$\tilde{C}^{\sigma\nu I_\nu}_{\lambda\alpha I_\alpha;\lambda^*\beta^* I_\beta} = \left\langle \begin{matrix}(\lambda,0)\\ \alpha I_\alpha\end{matrix}; \begin{matrix}(0,\lambda)\\ \beta^* I_\beta\end{matrix} \middle| \begin{matrix}(\sigma,\sigma)\\ \nu I_\nu\end{matrix}\right\rangle (-1)^{\lambda-\beta_2} \tag{A.6}$$

with $\beta^* = (\lambda-\beta_1, \lambda-\beta_2, \lambda-\beta_3)$, are $\mathfrak{su}(3)$ irreducible tensor operators.

The irreducible tensor operators of equation (14) satisfy the following trace-orthogonality condition

$$\mathrm{Tr}((\hat{T}^\lambda_{\sigma;\alpha I_\alpha})^\dagger \hat{T}^\lambda_{\sigma';\alpha' I_{\alpha'}}) = \delta_{\sigma\sigma'}\delta_{\alpha\alpha'}\delta_{I_\alpha I_{\alpha'}}, \tag{A.7}$$

where

$$\hat{T}^\lambda_{\sigma;\nu J} = (-1)^{\sigma+\nu_2}(\hat{T}^\lambda_{\sigma;\nu^* J})^\dagger \tag{A.8}$$

with $\nu^* = (2\sigma-\nu_1, 2\sigma-\nu_2, 2\sigma-\nu_3)$.

One can decompose products like $T^\lambda_{1;\nu I} T^\lambda_{\sigma;(\sigma\sigma\sigma)0}$ using the definition of the Racah U-coefficients [28] and the orthogonality property of tensors:

$$\hat{T}^{(1,1)}_{\nu;J} T^{(\sigma,\sigma)}_{(\sigma\sigma\sigma)0} = \sum_{\tau=\sigma-1}^{\sigma+1} c^{(\tau,\tau)}_{\nu_1 J;\sigma} \hat{T}^{(\tau,\tau)}_{\nu';J},$$
$$c^{(\tau,\tau)}_{\nu_1 J;\sigma} = \sum_\rho \left\langle \begin{matrix}(1,1)\\ \nu_1;J\end{matrix}; \begin{matrix}(\sigma,\sigma)\\ \sigma;0\end{matrix} \middle\| \begin{matrix}(\tau,\tau)\\ \nu_1';J\end{matrix}\right\rangle_\rho U_\rho[(1,1)(\lambda,0)(\tau,\tau)(0,\lambda);(\lambda,0)(\sigma,\sigma)]. \tag{A.9}$$

Here we provide tables of reduced CG coefficients, Racah U-coefficients and others coefficients needed to obtains some intermediate results or otherwise useful in calculations.

A.2. Some CG for couplings of the type $(1,1)\otimes(\sigma,\sigma)\to(\tau,\tau)$

The highest weight for the irrep $(\sigma+1,\sigma+1)$ in the reduction of the product $(1,1)\otimes(\sigma,\sigma)$ is the product of the $(1,1)$ and (σ,σ) highest weights so that

$$\left\langle \begin{matrix}(1,1)\\ 2;\frac{1}{2}\end{matrix}; \begin{matrix}(\sigma,\sigma)\\ (2\sigma;\frac{1}{2}\sigma)\end{matrix} \middle\| \begin{matrix}(\sigma+1,\sigma+1)\\ 2(\sigma+1);\frac{1}{2}(\sigma+1)\end{matrix}\right\rangle = 1\,. \tag{A.10}$$

Some calculations also require the following expressions:

$$\left\langle \begin{matrix}(1,1)\\ 1;0\end{matrix}; \begin{matrix}(\sigma,\sigma)\\ 2\sigma-1-p;I\end{matrix} \middle\| \begin{matrix}(\sigma-1,\sigma-1)\\ 2(\sigma-1)-p;I\end{matrix}\right\rangle$$
$$= \frac{(-1)^{p-2I+\sigma+1}}{8(\sigma+1)}\sqrt{\frac{3(3+2I+p+\sigma)(1-2I+p+\sigma)}{(\sigma+1)(2\sigma+1)}}$$
$$\times\sqrt{(1+2I-p+3\sigma)(-1-2I-p+3\sigma)} \tag{A.11}$$

$$\left\langle \begin{matrix}(1,1)\\ 1;0\end{matrix}; \begin{matrix}(\sigma,\sigma)\\ 2(\sigma+1)-p-1;I\end{matrix} \middle\| \begin{matrix}(\sigma+1,\sigma+1)\\ 2(\sigma+1)-p;I\end{matrix}\right\rangle$$
$$= \frac{(-1)^{p-2I+\sigma+1}}{2(\sigma+1)}\sqrt{\frac{3(3+2I+p+\sigma)(1-2I+p+\sigma)}{(\sigma+1)(2\sigma+3)}}$$
$$\times\sqrt{(3-2I-p+3\sigma)(5+2I-p+3\sigma)}. \tag{A.12}$$

Table A1. $\left\langle \begin{matrix}(1,1)\\ \nu_1;I\end{matrix}; \begin{matrix}(\sigma,\sigma)\\ \sigma;0\end{matrix} \middle\| \begin{matrix}(\sigma+1,\sigma+1)\\ \mu;I\end{matrix}\right\rangle$.

$(\nu_1;I)$	$(\mu;I)$	$\left\langle \begin{matrix}(1,1)\\ \nu_1;I\end{matrix}; \begin{matrix}(\sigma,\sigma)\\ \sigma;0\end{matrix} \middle\| \begin{matrix}(\sigma+1,\sigma+1)\\ \mu;I\end{matrix}\right\rangle$
$(2;\frac{1}{2})$	$(\sigma+2;\frac{1}{2})$	$\frac{(\sigma+2)}{2(\sigma+1)}\sqrt{\frac{(\sigma+3)}{(2\sigma+3)}}$
$(1;1)$	$(\sigma+1;1)$	$\frac{(\sigma+2)(\sigma+3)}{2(\sigma+1)}\sqrt{\frac{1}{3(\sigma+1)(2\sigma+3)}}$
$(1;0)$	$(\sigma+1;0)$	$\frac{(\sigma+2)}{2}\sqrt{\frac{3}{(2\sigma+3)(\sigma+1)}}$
$(0;\frac{1}{2})$	$(\sigma;\frac{1}{2})$	$\frac{(\sigma+2)}{2(\sigma+1)}\sqrt{\frac{\sigma+3}{(2\sigma+3)}}$

Some final analytical expressions for the decomposition $(1,1)\otimes(\sigma,\sigma)\to(\sigma,\sigma)$ are also needed:

$$\left\langle \begin{matrix}(1,1)\\ 1;0\end{matrix}; \begin{matrix}(\sigma,\sigma)\\ 2\sigma-p;I\end{matrix} \middle\| \begin{matrix}(\sigma,\sigma)\\ 2\sigma-p;I\end{matrix}\right\rangle_{\rho=1}
= \frac{\sqrt{3}(-1)^{p+\sigma-2I}}{2(\sigma+1)\sqrt{\sigma(\sigma+2)}}\Big(\sigma(\sigma+1)-\frac{1}{4}(p-\sigma+2I)
\times(\sigma-p+2I+2)-\frac{1}{4}(p+\sigma-2I)(p+\sigma+2I+2)\Big) \tag{A.13}$$

$$\left\langle \begin{matrix}(1,1)\\ 1;0\end{matrix}; \begin{matrix}(\sigma,\sigma)\\ 2\sigma-p;I\end{matrix} \middle\| \begin{matrix}(\sigma,\sigma)\\ 2\sigma-p;I\end{matrix}\right\rangle_{\rho=2}
= \frac{(-1)^{p+\sigma-2I}\sqrt{2\sigma+1}}{2(\sigma+1)\sqrt{\sigma(\sigma+2)(2\sigma+3)}}\Big(\sigma(\sigma+1)
-\frac{3}{4}(p-\sigma+2I)(\sigma-p+2I+2)+\frac{3}{4}\frac{(p+\sigma-2I)(p+\sigma+2I+2)}{(2\sigma+1)}\Big). \tag{A.14}$$

A.3. Basic equation to evaluate $U_{su(3)}$

The basic equation to evaluate $U_{su(3)}$ coefficients is [28]:

$$\sum_{\rho}\left\langle \begin{matrix}(1,1)\\ \nu_1;I\end{matrix}; \begin{matrix}(\sigma,\sigma)\\ 2\sigma;\frac{1}{2}\sigma\end{matrix} \middle\| \begin{matrix}(\tau,\tau)\\ 2\tau;\frac{1}{2}\tau\end{matrix}\right\rangle_{\rho} U_{su(3)}\left((1,1),(\lambda,0),(\tau,\tau),(0,\lambda);(\lambda,0),(\sigma,\sigma)\right)_{\rho}
= \sum_{a_1}\left\langle \begin{matrix}(\lambda,0)\\ a_1\end{matrix}; \begin{matrix}(0,\lambda)\\ \lambda+\sigma-a_1\end{matrix} \middle\| \begin{matrix}(\sigma,\sigma)\\ 2\sigma;\frac{1}{2}\sigma\end{matrix}\right\rangle \left\langle \begin{matrix}(1,1)\\ \nu_1;I\end{matrix}; \begin{matrix}(\lambda,0)\\ a_1\end{matrix} \middle\| \begin{matrix}(\lambda,0)\\ \nu_1-1+a_1\end{matrix}\right\rangle
\times\left\langle \begin{matrix}(\lambda,0)\\ \nu_1-1+a_1\end{matrix}; \begin{matrix}(0,\lambda)\\ \lambda+\sigma-a_1\end{matrix} \middle\| \begin{matrix}(\tau,\tau)\\ 2\tau;\frac{1}{2}\tau\end{matrix}\right\rangle U_{su(2)}\left(I,I_a,\frac{1}{2}\tau,I_b;I_c,\frac{1}{2}\sigma\right) \tag{A.15}$$

where ρ labels the copies of the irrep (τ,τ) in the decomposition of the product $(1,1)\otimes(\sigma,\sigma)$. Specifically, the irreps $(\sigma+1,\sigma+1)$ and $(\sigma-1,\sigma-1)$ occur once so ρ is redundant, but the irrep (σ,σ) occurs twice so when $(\tau,\tau)=(\sigma,\sigma)$ there is a sum of the two copies of this irrep. Expressions for the relevant coupling coefficients can be found in tables A2 and A4.

In addition, we have:

$$I_a=\frac{1}{2}(\lambda-a_1)\,,\quad I_b=\frac{1}{2}(\lambda+\sigma-a_1)\,,\quad I_c=\frac{1}{2}(\lambda-\nu_1-a_1+1)\,. \tag{A.16}$$

Table A2. $\left\langle \begin{smallmatrix}(1,1)\\ \nu_1;J\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ n_1;I_n\end{smallmatrix} \,\Big\|\, \begin{smallmatrix}(\sigma-1,\sigma-1)\\ 2(\sigma-1);\frac{1}{2}(\sigma-1)\end{smallmatrix}\right\rangle$.

$(\nu_1;J)$	$(n_1;I_n)$	$\left\langle \begin{smallmatrix}(1,1)\\ \nu_1;J\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ n_1;I_n\end{smallmatrix} \,\Big\|\, \begin{smallmatrix}(\sigma-1,\sigma-1)\\ 2(\sigma-1);\frac{1}{2}(\sigma-1)\end{smallmatrix}\right\rangle$
$(2;\frac{1}{2})$	$(2(\sigma-1);\frac{1}{2}\sigma-1)$	$-\frac{1}{\sigma+1}\sqrt{\frac{(\sigma-1)}{(2\sigma+1)}}$
$(2;\frac{1}{2})$	$(2(\sigma-1);\frac{1}{2}\sigma)$	$\frac{1}{\sigma+1}\sqrt{\frac{\sigma+2}{2\sigma+1}}$
$(1;1)$	$(2\sigma-1;\frac{1}{2}(\sigma-1))$	$\frac{\sigma}{\sigma+1}\sqrt{\frac{(\sigma-1)}{2(\sigma+1)(2\sigma+1)}}$
$(1;1)$	$(2\sigma-1;\frac{1}{2}(\sigma+1))$	$-\frac{1}{\sigma+1}\sqrt{\frac{\sigma(\sigma+2)}{(\sigma+1)}}$
$(1;0)$	$(2\sigma-1;\frac{1}{2}(\sigma-1))$	$\frac{\sigma}{\sigma+1}\sqrt{\frac{3}{2(2\sigma+1)}}$

Table A3. $\left\langle \begin{smallmatrix}(1,1)\\ \nu_1;J\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix} \,\Big\|\, \begin{smallmatrix}(\sigma-1,\sigma-1)\\ \mu;J\end{smallmatrix}\right\rangle$.

$(\nu_1;J)$	$(\mu;J)$	$\left\langle \begin{smallmatrix}(1,1)\\ \nu_1;J\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix} \,\Big\|\, \begin{smallmatrix}(\sigma-1,\sigma-1)\\ \mu;J\end{smallmatrix}\right\rangle$
$(2;\frac{1}{2})$	$(\sigma;\frac{1}{2})$	$-\frac{\sigma}{2(\sigma+1)}\sqrt{\frac{(\sigma-1)}{(2\sigma+1)}}$
$(1;1)$	$(\sigma-1,1)$	$\frac{\sigma(\sigma-1)}{2(\sigma+1)\sqrt{3(\sigma+1)(2\sigma+1)}}$
$(1;0)$	$(\sigma-1;0)$	$\frac{\sigma}{\sigma+1}\sqrt{\frac{3}{2(2\sigma+1)}}$
$(0;\frac{1}{2})$	$(\sigma-2;\frac{1}{2})$	$\frac{\sigma}{2}\sqrt{\frac{3}{(2\sigma+1)(\sigma+1)}}$

Table A4. The $SU(3)$ reduced CG $\left\langle \begin{smallmatrix}(1,1)\\ \nu_1 I_1\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ n_1 I_2\end{smallmatrix} \,\Big\|\, \begin{smallmatrix}(\sigma,\sigma)\\ 2\sigma;\frac{1}{2}\sigma\end{smallmatrix}\right\rangle_\rho$. The $\rho=1$ copy is chosen using the usual convention that the $SU(3)$ CGs agree with the Wigner–Eckart theorem when the generators are considered as $SU(3)$ tensors transforming by the $(1,1)$ representation. The $\rho=2$ copy is chosen to be orthogonal to the $\rho=1$ copy.

$\nu_1;I_1$	$n_1;I_2$	$\left\langle \begin{smallmatrix}(1,1)\\ \nu_1 I_1\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ n_1 I_2\end{smallmatrix} \,\Big\|\, \begin{smallmatrix}(\sigma,\sigma)\\ 2\sigma;\frac{1}{2}\sigma\end{smallmatrix}\right\rangle_1$	$\left\langle \begin{smallmatrix}(1,1)\\ \nu_1 I_1\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ n_1 I_2\end{smallmatrix} \,\Big\|\, \begin{smallmatrix}(\sigma,\sigma)\\ 2\sigma;\frac{1}{2}\sigma\end{smallmatrix}\right\rangle_2$
1;1	$2\sigma;\frac{1}{2}\sigma$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}\sqrt{\frac{2\sigma+1}{2\sigma+3}}$
1;0	$2\sigma;\frac{1}{2}\sigma$	$\frac{1}{2}\sqrt{\frac{3\sigma}{\sigma+2}}$	$\frac{1}{2}\sqrt{\frac{\sigma(2\sigma+1)}{(\sigma+2)(2\sigma+3)}}$
$2;\frac{1}{2}$	$2\sigma-1,\frac{1}{2}(\sigma+1)$	$\sqrt{\frac{\sigma+2}{2(\sigma+1)(\sigma+2)}}$	$\sqrt{\frac{3(2\sigma+1)}{2(\sigma+1)(2\sigma+3)}}$
$2;\frac{1}{2}$	$2\sigma-1;\frac{1}{2}(\sigma-1)$	$-\sqrt{\frac{(2\sigma+1)}{2(\sigma+1)(\sigma+2)}}$	$\sqrt{\frac{3}{2(\sigma+1)(\sigma+2)(2\sigma+3)}}$

In equation (A.15), the indices b^* and c are implicit and related to a and ν through

$$\begin{aligned} b_1^* &= \lambda+\sigma-a_1\,, \qquad & b_2^* &= \lambda-a_2\,, \qquad & b_3 &= \lambda-\sigma-a_3 \\ c_1 &= \nu_1-1+a_1\,, \qquad & c_2 &= \nu_2-1+a_2 \qquad & c_3 &= \nu_3-1+a_3. \end{aligned} \tag{A.17}$$

On the right hand side, we need CGs of the type given in table A1. The Racah $U_{su(2)}$ coefficient is related to the Wigner 6j-symbol by

$$U(abcd;ef) = (-1)^{a+b+c+d}\sqrt{(2e+1)(2f+1)}\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}. \tag{A.18}$$

Table A5. The $SU(3)$ reduced CG $\left\langle \begin{smallmatrix}(1,1)\\ \nu_1 I_1\end{smallmatrix};\begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix}\middle\|\begin{smallmatrix}(\sigma,\sigma)\\ N_1 J\end{smallmatrix}\right\rangle_\rho$.

$\nu_1;I_1$	$N_1;J$	$\left\langle \begin{smallmatrix}(1,1)\\ \nu_1 I_1\end{smallmatrix};\begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix}\middle\|\begin{smallmatrix}(\sigma,\sigma)\\ N_1 J\end{smallmatrix}\right\rangle_1$	$\left\langle \begin{smallmatrix}(1,1)\\ \nu_1 I_1\end{smallmatrix};\begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix}\middle\|\begin{smallmatrix}(\sigma,\sigma)\\ N_1 J\end{smallmatrix}\right\rangle_2$
$2;\frac{1}{2}$	$\sigma+1;\frac{1}{2}$	$-\frac{1}{2}$	$\sqrt{\frac{3}{4(2\sigma+1)(2\sigma+3)}}$
1;0	$\sigma;0$	0	$\sqrt{\frac{\sigma(\sigma+2)}{(2\sigma+1)(2\sigma+3)}}$
1;1	$\sigma;1$	0	$-\sqrt{\frac{\sigma(\sigma+2)}{(2\sigma+1)(2\sigma+3)}}$
$0;\frac{1}{2}$	$\sigma-1;\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}\frac{1}{\sqrt{(2\sigma+1)(2\sigma+3)}}$

Table A6. The Racah U-coefficients.

τ	ρ	$U_\rho\left[(1,1)(\lambda,0)(\tau,\tau)(0,\lambda);(\lambda,0)(\sigma,\sigma)\right]_\rho$
$\sigma+1$		$\frac{(\sigma+1)}{2}\sqrt{\frac{3(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+3)(\sigma+2)(2\sigma+3)}}$
σ	1	$\frac{\sqrt{3}}{2}\sqrt{\frac{\sigma(\sigma+2)}{\lambda(\lambda+3)}}$
σ	2	$\frac{(2\lambda+3)}{2}\sqrt{\frac{\sigma(\sigma+2)}{\lambda(\lambda+3)(2\sigma+1)(2\sigma+3)}}$
$\sigma-1$		$-\frac{(\sigma+1)}{2}\sqrt{\frac{3(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+3)\sigma(2\sigma+1)}}$

Reduced CG's of the type $\left\langle \begin{smallmatrix}(1,1)\\ \nu_1;I\end{smallmatrix};\begin{smallmatrix}(\lambda,0)\\ a_1\end{smallmatrix}\middle\|\begin{smallmatrix}(\lambda,0)\\ \nu_1-1+a_1\end{smallmatrix}\right\rangle$ can be obtained from the algorithm of [11] and depend on three cases, which are also tied to the relation between τ and σ. It is also useful to note the symmetry relation

$$\left\langle \begin{matrix}(\lambda,0)\\ \alpha_1\alpha_2\alpha_3\end{matrix};\begin{matrix}(0,\lambda)\\ \lambda-\beta_1,\lambda-\beta_2,\lambda-\beta_3\end{matrix}\middle|\begin{matrix}(1,1)\\ \gamma_1\gamma_2\gamma_3;I\end{matrix}\right\rangle = (-1)^{\beta_2}\sqrt{\frac{16}{(\lambda+1)(\lambda+2)}}\left\langle \begin{matrix}(1,1)\\ \gamma_1\gamma_2\gamma_3;I\end{matrix};\begin{matrix}(\lambda,0)\\ \beta_1\beta_2,\beta_3\end{matrix}\middle|\begin{matrix}(\lambda,0)\\ \alpha_1\alpha_2\alpha_3\end{matrix}\right\rangle \tag{A.19}$$

which can be obtained from [11].

A.3.1. $\nu_1=2$. If $\nu_1=2$, then $I'=\frac{1}{2}$ and $\tau=\sigma+1$. We can combine equation (A.10), use equation (A.19) and the tables already provided to obtain

$$\left\langle \begin{matrix}(1,1)\\ 2;\frac{1}{2}\end{matrix};\begin{matrix}(\lambda,0)\\ a_1\end{matrix}\middle\|\begin{matrix}(\lambda,0)\\ a_1+1\end{matrix}\right\rangle = \sqrt{\frac{3(\lambda-a_1+1)(a_1+1)}{2\lambda(\lambda+3)}}, \tag{A.20}$$

$$U_{su(2)}\left(\frac{1}{2},I_a,\frac{1}{2}\sigma+\frac{1}{2},\frac{1}{2}(\lambda+\sigma-a_1);I_a-\frac{1}{2},\frac{1}{2}\sigma\right) = -\sqrt{\frac{(\lambda-a_1)(\sigma+1)}{(\lambda-a_1+1)(\sigma+2)}} \tag{A.21}$$

and evaluate the sum on the right of equation (A.15) as

$$U_{su(3)}\left((1,1),(\lambda,0),(\sigma+1,\sigma+1),(0,\lambda);(\lambda,0),(\sigma,\sigma)\right) = \frac{(\sigma+1)}{2}\sqrt{\frac{3(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+3)(\sigma+2)(2\sigma+3)}}. \tag{A.22}$$

A.3.2. $\nu_1 = 1$. When $\nu_1 = 1$, we have $I' = 0$ or $I' = 1$, and $\tau = \sigma$. For $I' = 0$ we have:

$$\left\langle \begin{matrix}(1,1)\\1;0\end{matrix};\begin{matrix}(\lambda,0)\\a_1\end{matrix}\middle\|\begin{matrix}(\lambda,0)\\a_1\end{matrix}\right\rangle = \frac{3a_1-\lambda}{2\sqrt{\lambda(\lambda+3)}}$$
$$U_{su(2)}\left(0,I_a,\frac{1}{2}\sigma,\frac{1}{2}(\lambda+\sigma-a_1);I_a,\frac{1}{2}\sigma\right) = 1\,,$$
$$\left\langle \begin{matrix}(1,1)\\1;0\end{matrix};\begin{matrix}(\sigma,\sigma)\\2\sigma;\frac{1}{2}\sigma\end{matrix}\middle\|\begin{matrix}(\sigma,\sigma)\\2\sigma;\frac{1}{2}\sigma\end{matrix}\right\rangle_{\rho=1} = \frac{\sqrt{3}}{2}\sqrt{\frac{\sigma}{(\sigma+2)}}\,,$$
$$\left\langle \begin{matrix}(1,1)\\1;0\end{matrix};\begin{matrix}(\sigma,\sigma)\\2\sigma;\frac{1}{2}\sigma\end{matrix}\middle\|\begin{matrix}(\sigma,\sigma)\\2\sigma;\frac{1}{2}\sigma\end{matrix}\right\rangle_{\rho=2} = \frac{1}{2}\sqrt{\frac{\sigma(2\sigma+1)}{(\sigma+2)(2\sigma+3)}}\,. \tag{A.23}$$

The sum on RHS of equation (A.15) can here again be evaluated in closed form to produce

$$\begin{aligned}&\frac{\sqrt{3}}{2}\sqrt{\frac{\sigma}{\sigma+2}}\,U_{su(3)}\left((1,1),(\lambda,0),(\sigma,\sigma),(0,\lambda);(\lambda,0),(\sigma,\sigma)\right)_{\rho=1}\\&+\sqrt{\frac{\sigma(2\sigma+1)}{4(\sigma+2)(2\sigma+3)}}U_{su(3)}\left((1,1),(\lambda,0),(\sigma,\sigma),(0,\lambda);(\lambda,0),(\sigma,\sigma)\right)_{\rho=2}\\&=\frac{\sigma(\lambda+3\sigma+6)}{2(2\sigma+3)\sqrt{\lambda(\lambda+3)}}\,.\end{aligned} \tag{A.24}$$

For $I' = 1$, we find

$$\left\langle \begin{matrix}(1,1)\\1;1\end{matrix};\begin{matrix}(\lambda,0)\\a_1\end{matrix}\middle\|\begin{matrix}(\lambda,0)\\a_1\end{matrix}\right\rangle = \frac{1}{2}\sqrt{\frac{3(\lambda-a_1)(\lambda-a_1+2)}{\lambda(\lambda+3)}}\,,$$
$$U_{su(2)}\left(1,I_a,\frac{1}{2}\sigma,\frac{1}{2}(\lambda+\sigma-a_1);I_a,\frac{1}{2}\sigma\right) = -\sqrt{\frac{(\lambda-a_1)\sigma}{(\sigma+2)(\lambda-a_1+2)}}\,,$$
$$\left\langle \begin{matrix}(1,1)\\1;1\end{matrix};\begin{matrix}(\sigma,\sigma)\\2\sigma;\frac{1}{2}\sigma\end{matrix}\middle\|\begin{matrix}(\sigma,\sigma)\\2\sigma;\frac{1}{2}\sigma\end{matrix}\right\rangle = \frac{1}{2} \tag{A.25}$$

and this time

$$\begin{aligned}&\frac{1}{2}U_{su(3)}\left((1,1),(\lambda,0),(\sigma,\sigma),(0,\lambda);(\lambda,0),(\sigma,\sigma)\right)_{\rho=1}\\&-\frac{\sqrt{3}}{2}\sqrt{\frac{2\sigma+1}{2\sigma+3}}U_{su(3)}\left((1,1),(\lambda,0),(\sigma,\sigma),(0,\lambda);(\lambda,0),(\sigma,\sigma)\right)_{\rho=2}\\&\quad=-\frac{(\lambda-\sigma)}{2(2\sigma+3)}\sqrt{\frac{3\sigma(\sigma+2)}{\lambda(\lambda+3)}}.\end{aligned} \tag{A.26}$$

This system is inverted to obtain the final expressions

$$U_{su(3)}\left((1,1),(\lambda,0),(\sigma,\sigma),(0,\lambda);(\lambda,0),(\sigma,\sigma)\right)_{\rho=1} = \frac{\sqrt{3}}{2}\sqrt{\frac{\sigma(\sigma+2)}{\lambda(\lambda+3)}}, \tag{A.27}$$

$$U_{su(3)}\left((1,1),(\lambda,0),(\sigma,\sigma),(0,\lambda);(\lambda,0),(\sigma,\sigma)\right)_{\rho=2} = \frac{(2\lambda+3)}{2}\sqrt{\frac{\sigma(\sigma+2)}{\lambda(\lambda+3)(2\sigma+1)(2\sigma+3)}}. \tag{A.28}$$

A.3.3. $\nu_1=0$. When $\nu_1=0$ then $\tau=\sigma-1$ and $I'=\frac{1}{2}$. We now have

$$\left\langle \begin{matrix}(1,1)\\ 0;\frac{1}{2}\end{matrix};\begin{matrix}(\lambda,0)\\ a_1\end{matrix}\,\middle\|\,\begin{matrix}(\lambda,0)\\ a_1-1\end{matrix}\right\rangle = \sqrt{\frac{3a_1(\lambda-a_1+1)}{2\lambda(\lambda+3)}}$$

and a few straightforward steps yield

$$U_{su(3)}\left((1,1),(\lambda,0),(\sigma-1,\sigma-1),(0,\lambda);(\lambda,0),(\sigma,\sigma)\right) = \frac{(\sigma+1)}{2}\sqrt{\frac{3(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+3)\sigma(2\sigma+1)}}. \tag{A.29}$$

Appendix B. Calculation of a^L and a^R coefficients and their asymptotics

The coefficients $a^L_{\nu_1 I}(\lambda;\tau)$ have the following general form:

$$a^L_{\nu I}(\lambda;\tau) = \sqrt{\frac{16}{(\lambda+1)(\lambda+2)}}\left[\sum_{\sigma=\tau-1}^{\tau+1}\frac{F^\sigma_\lambda}{F^\tau_\lambda} \times \sum_\rho \left\langle \begin{matrix}(1,1)\\ \nu_1 I\end{matrix};\begin{matrix}(\sigma,\sigma)\\ \sigma;0\end{matrix}\,\middle\|\,\begin{matrix}(\tau,\tau)\\ \bar{\nu}_1;I\end{matrix}\right\rangle_\rho U_\rho\left[(1,1)(\lambda,0)(\tau,\tau)(0,\lambda);(\lambda,0)(\sigma,\sigma)\right]\right] \tag{B.1}$$

where the required four Racah coefficient are given in table A6. Using the $SU(3)$ CG coefficients of tables A3 and A5, one obtains the explicit expressions

$$Na^L_{2\frac{1}{2}}(\lambda;\tau) = \sqrt{3\tau(\tau+2)}\left(\frac{\tau\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)}}{(\tau+1)(2\tau+1)} - \frac{(\tau+2)\sqrt{\lambda-\tau}\sqrt{\lambda+\tau+3}}{(\tau+1)(2\tau+3)} + \frac{2(\lambda-2\tau(\tau+2))}{4\tau(\tau+2)+3}\right), \tag{B.2}$$

$$Na^L_{11}(\lambda;\tau) = \frac{\tau(\tau+2)}{(\tau+1)(2\tau+1)(2\tau+3)}\left(-2(2\lambda+3)(\tau+1) + \sqrt{(\lambda-\tau)(\lambda+\tau+3)} + (2\tau+3)\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)}\right), \tag{B.3}$$

$$Na^L_{10}(\lambda;\tau) = \frac{3\tau^2\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)}}{(\tau+1)(2\tau+1)} + \frac{2(2\lambda+3)(\tau+2)\tau}{4\tau(\tau+2)+3} + \frac{3(\tau+2)^2\sqrt{\lambda-\tau}\sqrt{\lambda+\tau+3}}{(\tau+1)(2\tau+3)}, \tag{B.4}$$

$$Na^L_{0\frac{1}{2}}(\lambda;\tau-1) = \sqrt{3\tau(\tau+2)}\left(\frac{\tau\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)}}{(\tau+1)(2\tau+1)} - \frac{(\tau+2)\sqrt{\lambda-\tau}\sqrt{\lambda+\tau+3}}{(\tau+1)(2\tau+3)} + \frac{2(\lambda+2\tau(\tau+2)+3)}{4\tau(\tau+2)+3}\right), \tag{B.5}$$

where the normalization factor N is given in (18). The coefficients $a^R_{\nu_1 I}(\lambda;\tau)$ are evaluated in the same manner, yielding equation (42).

In the limit of large dimension of a representation the coefficients $a^{L,R}_{\tau;\nu_1}(\lambda;\bar{\nu}_1, I)$ can be expanded in inverse powers of semiclassical parameter ϵ:

$$Na^L_{11}(\lambda;\tau) = Na^R_{11};(\lambda;\tau) \sim -\epsilon\tau(\tau+2), \tag{B.6}$$

$$Na^L_{10}(\lambda;\tau) = Na^R_{10}(\lambda;\tau,0) \sim \frac{2}{\epsilon} - 3\epsilon(\tau(\tau+2)+3), \tag{B.7}$$

$$Na^L_{0\frac{1}{2}}(\lambda;\tau) = Na^R_{0\frac{1}{2}}(\lambda;\tau) \sim \sqrt{3\tau(\tau+2)}\left(1+\frac{3\epsilon}{2}\right), \tag{B.8}$$

$$Na^L_{2\frac{1}{2}}(\lambda;\tau) = Na^R_{0\frac{1}{2}}(\lambda;\tau) \sim \sqrt{3\tau(\tau+2)}\left(-1+\frac{3\epsilon}{2}\right). \tag{B.9}$$

Appendix C. Differential operators

C.1. The Casimir operator

$$\begin{aligned}\hat{\mathcal{C}}_2 = &-2\frac{\partial^2}{\partial\beta_2^2} - \frac{4}{1-\cos(\beta_2)}\frac{\partial^2}{\partial\beta_1^2} \\ &+ \frac{1}{2}\left(\cos(\beta_1)\left(\cos(\beta_2)-1\right) - \cos(\beta_2) - 3\right)\csc^2(\beta_2)\sec^2\left(\frac{\beta_1}{2}\right)\frac{\partial^2}{\partial\alpha_2^2} \\ &- 2\left(2\cot(\beta_2) + \csc(\beta_2)\right)\frac{\partial}{\partial\beta_2} + \csc^2\left(\frac{\beta_2}{2}\right)\sec^2\left(\frac{\beta_1}{2}\right)\frac{\partial^2}{\partial\alpha_1\partial\alpha_2} \\ &+ 2\csc^2(\beta_1)\csc^2\left(\frac{\beta_2}{2}\right)\frac{\partial^2}{\partial\alpha_1^2} - 2\cot(\beta_1)\csc^2\left(\frac{\beta_2}{2}\right)\frac{\partial}{\partial\beta_1}.\end{aligned} \tag{C.1}$$

C.2. Anaturdjacency relations

We want to replace the functions $D^{(\tau,\tau)}_{\mu j;\bar{\nu}I}(\Omega)$ with differential operators $\hat{\mathbb{S}}_{\nu I}$ acting on functions of the type $D^{(\tau,\tau)}_{\mu j;(\tau\tau\tau)0}(\Omega)$, i.e. we need to find differential operators $\hat{\mathbb{S}}_{\nu I}$ acting on these functions so that

Table C1. The relation between $\hat{C}_{\nu I}$ and generators.

$\hat{C}_{210;\frac{1}{2}}$	$\hat{C}_{13}$	$\hat{C}_{201;\frac{1}{2}}$	$-\hat{C}_{12}$
$\hat{C}_{120;1}$	$\hat{C}_{23}$	$\hat{C}_{111;1}$	$-\frac{1}{\sqrt{2}}\left(\hat{C}_{22}-\hat{C}_{33}\right)$
$\hat{C}_{102;1}$	$-\hat{C}_{32}$	$\hat{C}_{111;0}$	$\frac{1}{\sqrt{6}}\left(2\hat{C}_{11}-\hat{C}_{22}-C_{33}\right)$
$\hat{C}_{021;\frac{1}{2}}$	$\hat{C}_{21}$	$\hat{C}_{012;\frac{1}{2}}$	$\hat{C}_{31}$

$$\hat{\mathbb{S}}_{\nu I} D^{(\tau,\tau)}_{\mu J;(\tau\tau\tau)0}(\Omega) \propto D^{(\tau,\tau)}_{\mu J;\bar{\nu} I}(\Omega)\,. \tag{C.2}$$

First we can recast this as follows. Let $\Omega_k \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ and start with

$$\begin{aligned}\frac{\partial}{\partial\Omega_k} D^{(\tau,\tau)}_{\mu J;(\tau,\tau\tau)0}(\Omega) &= \frac{\partial}{\partial\Omega_k}\langle(\tau,\tau)\mu J|\hat{R}(\Omega)|(\tau,\tau)\tau\tau\tau;0\rangle \\ &= \langle(\tau,\tau)\mu J|\frac{\partial}{\partial\Omega_k}\hat{R}(\Omega)|(\tau,\tau)\tau\tau\tau;0\rangle\end{aligned} \tag{C.3}$$

$$= \langle(\tau,\tau)\mu J|\hat{R}(\Omega)\left(\sum_{\nu I} c_{\nu I}(\Omega_k)\hat{C}_{\nu I}\right)|(\tau,\tau)\tau\tau\tau;0\rangle\,, \tag{C.4}$$

where table C1 gives the $\hat{C}_{\nu I}$ in terms of the $\hat{C}_{ij}$.

Defined in this way, the operators $\hat{C}_{\nu I}$ differ from the generators $\hat{C}_{ij}$ by at most a sign and from the tensor operators $\hat{T}^{\lambda}_{1;\nu I}$ by a normalization that is a function of the $\mathfrak{su}(3)$ quadratic Casimir invariant and the dimension of the irrep on which the tensors act.

From this we now have the general relation

$$\frac{\partial}{\partial\Omega_k}\hat{R}(\Omega) = \sum_{\nu I} c_{\nu I}(\Omega_k)\hat{R}(\Omega)\hat{C}_{\nu I}. \tag{C.5}$$

It is important to notice that this relation does not depend on the $\mathfrak{su}(3)$ irrep so the coefficients $c_{\nu I}(\Omega_k)$ can be found using any irrep. The most expeditious choice is the 3×3 irrep $(1,0)$. For this representation the operators $\hat{C}_{\nu I}$ are orthonormal under trace:

$$\mathrm{Tr}\left((\hat{C}_{\nu' I'})^\dagger \hat{C}_{\nu I}\right) = \delta_{\nu'\nu}\delta_{I'I} \tag{C.6}$$

so we can easily write

$$c_{\nu' I'}(\Omega_k) = \mathrm{Tr}\left((\hat{C}_{\nu' I'})^\dagger \hat{R}^\dagger(\Omega)\frac{\partial}{\partial\Omega_k}\hat{R}(\Omega)\right)\,. \tag{C.7}$$

The coefficients $c_{\nu I}(\Omega_k)$ are given in table C2.

To continue, it is convenient to divide the generators in two sets. The first contains elements in the $\mathfrak{u}(2)$ subalgebra: $\{\hat{C}_{120;1}, \hat{C}_{111;1}, \hat{C}_{102;1}, \hat{C}_{111;0}\}$ and will be labeled by roman letters $a, b, c \ldots$. The second contains the remaining operators $\{\hat{C}_{210;\frac{1}{2}}, \hat{C}_{201;\frac{1}{2}}\hat{C}_{021;\frac{1}{2}}, \hat{C}_{012;\frac{1}{2}}\}$ and will be labeled using Greek letters $\alpha, \beta \ldots$.

Table C2. The $c_{\nu I}(\Omega_k)$ coefficients of equation (C.5).

νI	$c_{\nu I}(\alpha_1)$	$c_{\nu I}(\beta_1)$
$210;\frac{1}{2}$	$\mathrm{i}\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\left(\cos^2\left(\frac{\beta_2}{4}\right)+\cos(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\right)\sin\left(\frac{\beta_2}{2}\right)$	$-\frac{1}{2}\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\cos\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)$
$201;\frac{1}{2}$	$\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\sin(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\sin\left(\frac{\beta_2}{2}\right)$	$-\frac{1}{2}\mathrm{e}^{-\mathrm{i}\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)$
$120;1$	$-\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha_1}\sin(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\left(\cos^2\left(\frac{\beta_2}{4}\right)+\cos(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\right)$	$\mathrm{e}^{-\mathrm{i}\alpha_1}\sin^2\left(\frac{\beta_2}{4}\right)$
$111;1$	$-2\mathrm{i}\sqrt{2}\sin^2\left(\frac{\beta_1}{2}\right)\sin^2\left(\frac{\beta_2}{4}\right)\left(\cos(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)+1\right)$	0
$102;1$	$\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha_1}\sin(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\left(\cos^2\left(\frac{\beta_2}{4}\right)+\cos(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\right)$	$\mathrm{e}^{\mathrm{i}\alpha_1}\sin^2\left(\frac{\beta_2}{4}\right)$
$111;0$	$\mathrm{i}\sqrt{\frac{3}{2}}\sin^2\left(\frac{\beta_1}{2}\right)\sin^2\left(\frac{\beta_2}{2}\right)$	0
$021;\frac{1}{2}$	$-\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\sin(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\sin\left(\frac{\beta_2}{2}\right)$	$-\frac{1}{2}\mathrm{e}^{\mathrm{i}\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)$
$012;\frac{1}{2}$	$\mathrm{i}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\left(\cos^2\left(\frac{\beta_2}{4}\right)+\cos(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\right)\sin\left(\frac{\beta_2}{2}\right)$	$\frac{1}{2}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\cos\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)$
νI	$c_{\nu I}(\alpha_2)$	$c_{\nu I}(\beta_2)$
$210;\frac{1}{2}$	$\frac{1}{2}\mathrm{i}\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\sin(\beta_2)$	$-\frac{1}{2}\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)$
$201\frac{1}{2}$	$-\frac{1}{2}\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\sin(\beta_2)$	$\frac{1}{2}\mathrm{e}^{-\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)$
$120;1$	$-\frac{1}{2}\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha_1}\sin(\beta_1)\sin^2\left(\frac{\beta_2}{2}\right)$	0
$111;1$	$\frac{\mathrm{i}\cos(\beta_1)\sin^2\left(\frac{\beta_2}{2}\right)}{\sqrt{2}}$	0
$102;1$	$\frac{1}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha_1}\sin(\beta_1)\sin^2\left(\frac{\beta_2}{2}\right)$	0
$111;0$	$\mathrm{i}\sqrt{\frac{3}{2}}\sin^2\left(\frac{\beta_2}{2}\right)$	0
$021;\frac{1}{2}$	$\frac{1}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\sin(\beta_2)$	$\frac{1}{2}\mathrm{e}^{\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)$
$012;\frac{1}{2}$	$\frac{1}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\sin(\beta_2)$	$\frac{1}{2}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)$

Table C3. The $d_{\nu I}(\Omega_k)$ coefficients.

	$\nu I=210;\frac{1}{2}$	$\nu I=201;\frac{1}{2}$
$d_{\nu I}(\alpha_1)$	$-\frac{1}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\csc\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$	$-\frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}\alpha_2}\csc\left(\frac{\beta_2}{2}\right)\sec\left(\frac{\beta_1}{2}\right)$
$d_{\nu I}(\beta_1)$	$-\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\cos\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$	$-\mathrm{e}^{\mathrm{i}\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$
$d_{\nu I}(\alpha_2)$	$-2\mathrm{i}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\sin^2\left(\frac{\beta_2}{4}\right)\csc(\beta_2)$	$-\frac{\mathrm{i}}{2}\sin\left(\frac{\beta_1}{2}\right)\mathrm{e}^{\mathrm{i}\alpha_2}\left(\cot(\beta_1)\csc\left(\frac{\beta_2}{2}\right) -2\left(\cos(\beta_1)+\cos\left(\frac{\beta_2}{2}\right)+1\right)\csc(\beta_1)\csc(\beta_2)\right)$
$d_{\nu I}(\beta_2)$	$-\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)$	$\mathrm{e}^{\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)$
	$\nu I=012;\frac{1}{2}$	$\nu I=021;\frac{1}{2}$
$d_{\nu I}(\alpha_1)$	$-\frac{1}{2}\mathrm{i}\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\csc\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$	$\frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}\alpha_2}\csc\left(\frac{\beta_2}{2}\right)\sec\left(\frac{\beta_1}{2}\right)$
$d_{\nu I}(\beta_1)$	$\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\cos\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$	$-\mathrm{e}^{-\mathrm{i}\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$
$d_{\nu I}(\alpha_2)$	$-2\mathrm{i}\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\sin^2\left(\frac{\beta_2}{4}\right)\csc(\beta_2)$	$\frac{1}{2}\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\left(\cot(\beta_1)\csc\left(\frac{\beta_2}{2}\right) -2\left(\cos(\beta_1)+\cos\left(\frac{\beta_2}{2}\right)+1\right)\csc(\beta_1)\csc(\beta_2)\right)$
$d_{\nu I}(\beta_2)$	$\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)$	$\mathrm{e}^{-\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)$

Table C4. The coefficients $f_{\beta a}$.

β	$a=120;1$	$111;1$
$012;\frac{1}{2}$	$-2\mathrm{e}^{-\mathrm{i}(2\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)$ $\times\sin(\beta_1)\sin^4\left(\frac{\beta_2}{4}\right)\csc(\beta_2)$	$\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\frac{1}{\sqrt{2}}\left(\sin^2\left(\frac{\beta_2}{4}\right)\cos(\beta_1)\sec\left(\frac{\beta_2}{2}\right)-1\right)$
$021;\frac{1}{2}$	$-\frac{1}{4}\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\left(\sec\left(\frac{\beta_2}{2}\right)\left(2\sin^2\left(\frac{\beta_2}{4}\right)\cos(\beta_1)+1\right)+3\right)$	$\frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\left(\sin^2\left(\frac{\beta_2}{4}\right)\cos(\beta_1)\sec\left(\frac{\beta_2}{2}\right)+1\right)$
$201;\frac{1}{2}$	$\mathrm{e}^{-\mathrm{i}\alpha_1}\sin\alpha_2(\cot\alpha_2+\mathrm{i})$ $\times\sin^2\beta_1\csc\left(\frac{\beta_1}{2}\right)\sin^4\left(\frac{\beta_2}{4}\right)\csc\beta_2$	$-\frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\left(\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1\sec\left(\frac{\beta_2}{2}\right)+1\right)$
$210;\frac{1}{2}$	$\mathrm{e}^{\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\frac{1}{4}\left(\sec\left(\frac{\beta_2}{2}\right)\left(2\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1-1\right)-3\right)$	$\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\frac{1}{\sqrt{2}}\left(\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1\sec\left(\frac{\beta_2}{2}\right)-1\right)$
β	$a=102;1$	$111;0$
$012;\frac{1}{2}$	$\mathrm{e}^{-\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\frac{1}{4}\left(\sec\left(\frac{\beta_2}{2}\right)\left(1-2\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1\right)+3\right)$	$\frac{1}{2}\sqrt{\frac{3}{2}}\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{2}\right)$
$021;\frac{1}{2}$	$-\mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\frac{1}{4}\left(\sec\left(\frac{\beta_2}{2}\right)\left(2\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1+1\right)+3\right)$	$\frac{1}{2}\sqrt{\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{2}\right)$
$201;\frac{1}{2}$	$\mathrm{e}^{-\mathrm{i}\alpha_1}\sin\alpha_2\csc\left(\frac{\beta_1}{2}\right)\csc\beta_2$ $\times\sin^2(\beta_1)\sin^4\left(\frac{\beta_2}{4}\right)(\cot(\alpha_2)+\mathrm{i})$	$-\frac{1}{2}\sqrt{\frac{3}{2}}\mathrm{e}^{\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{2}\right)$
$210;\frac{1}{2}$	$\frac{1}{4}\mathrm{e}^{\mathrm{i}\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\left(\sec\left(\frac{\beta_2}{2}\right)\left(2\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1-1\right)-3\right)$	$\frac{1}{2}\sqrt{\frac{3}{2}}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{2}\right)$

Table D1. The non-zero function coefficients $c_{(\Omega_i\Omega_j)}$ in the operator $\mathfrak{S}_{(111)0;11}=\sum_{ij}c_{(\Omega_i\Omega_j)}\partial^2/\partial\Omega_i\partial\Omega_j$ of equation (D.14).

Ω_i or (Ω_i,Ω_j)	c_{Ω_i} or $c_{\Omega_i\Omega_j}$	Ω_i or (Ω_i,Ω_j)	c_{Ω_i} or $c_{\Omega_i\Omega_j}$
β_1	$\frac{3}{4}\tan\left(\frac{1}{2}\beta_2\right)$	β_2	$\frac{3}{2}\cot\beta_1$
(α_2,α_2)	$\frac{3}{8}\left(\tan^2(\frac{1}{2}\beta_1)-\frac{2}{1+\cos\beta_2}\right)$	(α_1,α_2)	$-\frac{3}{2(1+\cos\beta_1)}$
(β_2,β_2)	$\frac{3}{4}(\cos\beta_2-1)$	(β_1,β_1)	$\frac{3}{2}$
(α_1,α_1)	$\frac{3}{2}\csc^2(\beta_1)$		

Consider now

$$\sum_k d_\beta(\Omega_k)\frac{\partial}{\partial\Omega_k}\hat{R}(\Omega) = \sum_{k\alpha} d_\beta(\Omega_k)c_\alpha(\Omega_k)\hat{R}(\Omega)\hat{\mathcal{C}}_\alpha + \sum_{ak} d_\beta(\Omega_k)c_a(\Omega_k)\hat{R}(\Omega)\hat{\mathcal{C}}_a \tag{C.8}$$

and choose $d_\beta(\Omega_k)$ so that

$$\sum_k d_\beta(\Omega_k)c_\alpha(\Omega_k) = \delta_{\beta\alpha}\,, \tag{C.9}$$

so yielding

$$\sum_k d_\beta(\Omega_k)\frac{\partial}{\partial\Omega_k}\hat{R}(\Omega) = \hat{R}(\Omega)\hat{\mathcal{C}}_\beta + \sum_{ak} d_\beta(\Omega_k)c_a(\Omega_k)\hat{R}(\Omega)\hat{\mathcal{C}}_a\,. \tag{C.10}$$

If we recall from equation (C.3) that this sum will act on $|(\tau,\tau)\tau\tau\tau;0\rangle$, and that $|(\tau,\tau)\tau\tau\tau;0\rangle$ is by construction annihilated by $\hat{\mathcal{C}}_a$, we see that equation (C.9) is simply a linear system for d_β which can be easily solved. The solution coefficients $d_\beta(\Omega_k)$ are found in table C3.

With this:

$$\hat{\mathbb{S}}_{\nu\frac{1}{2}} D^{(\tau,\tau)}_{\mu J,(\tau\tau\tau)0}(\Omega) = \sum_k d_{\nu\frac{1}{2}}(\Omega_k)\frac{\partial}{\partial\Omega_k} D^{(\tau,\tau)}_{\mu J,(\tau\tau\tau)0}(\Omega) \tag{C.11}$$

where $\hat{\mathbb{S}}_{\nu\frac{1}{2}}$ is a differential operator that shifts the function $D^{(\tau,\tau)}_{\mu J,(\tau\tau\tau)0}(\Omega)$ to $D^{(\tau,\tau)}_{\mu J;\bar{\nu}\frac{1}{2}}(\Omega)$ up to a proportionality term.

If we substitute equation (C.8) into equation (C.11) we find

$$\begin{aligned}
\hat{\mathbb{S}}_{\nu\frac{1}{2}} & D^{(\tau,\tau)}_{\mu J,(\tau\tau\tau)0}(\Omega) \\
&= \langle(\tau,\tau)\mu J|\hat{R}(\Omega)\left(\hat{\mathcal{C}}_{\nu\frac{1}{2}} + \sum_{ak} d_{\nu\frac{1}{2}}(\Omega_k)c_a(\Omega_k)\hat{\mathcal{C}}_a\right)|(\tau,\tau)\tau\tau\tau;0\rangle \\
&= \langle(\tau,\tau)\mu J|\hat{R}(\Omega)\hat{\mathcal{C}}_{\nu\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle \\
&= \langle(\tau,\tau)\mu J|\hat{R}(\Omega)|(\tau,\tau)\bar{\nu};\frac{1}{2}\rangle\langle(\tau,\tau)\bar{\nu};\frac{1}{2}|\hat{\mathcal{C}}_{\nu\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle \\
&= D^{(\tau,\tau)}_{\mu J;\bar{\nu}\frac{1}{2}}(\Omega)\langle(\tau,\tau)\bar{\nu};\frac{1}{2}|\hat{\mathcal{C}}_{\nu\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle
\end{aligned} \tag{C.12}$$

since $\hat{\mathcal{C}}_a|(\tau,\tau)\tau\tau\tau;0\rangle = 0$.

Finally, we can evaluate $\langle(\tau,\tau)\bar{\nu};\frac{1}{2}|\hat{\mathcal{C}}_{\nu\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle$. It turns out that this expression is quite simply expressed in terms of ν:

$$\langle(\tau,\tau)\bar{\nu};\frac{1}{2}|\hat{\mathcal{C}}_{\nu\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle = (-1)^{\bar{\nu}_1/2}\sqrt{\frac{\tau(\tau+2)}{2}}\,, \tag{C.13}$$

where ν and $\bar{\nu}$ are related in table 2. Combining this with equation (C.12), we now have

$$\hat{\mathbb{S}}_{\nu\frac{1}{2}} D^{(\tau,\tau)}_{\mu J;(\tau\tau\tau)0}(\Omega) = (-1)^{\bar{\nu}_1/2}\sqrt{\frac{\tau(\tau+2)}{2}} D^{(\tau,\tau)}_{\mu J;\bar{\nu}\frac{1}{2}}(\Omega) \tag{C.14}$$

$$= \langle(\tau,\tau)\mu J|R(\Omega)\hat{\mathcal{C}}_{\nu;\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle \quad , \tag{C.15}$$

where the $\hat{\mathbb{S}}_{\nu_{\frac{1}{2}}}$ operators are of first order only.

Next we investigate adjacency relations of the form

$$\hat{\mathbb{S}}^{(2)}_{(1\nu_2\nu_3)I} D^{(\sigma,\sigma)}_{\mu J;\bar{\nu}1}(\Omega)\,, \tag{C.16}$$

and show, in agreement with the argument presented in section 3, that this operator is of second order in the derivatives.

We consider

$$\begin{aligned}
&\hat{\mathbb{S}}_\alpha \hat{\mathbb{S}}_\beta D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega) \\
&= \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\left(\hat{\mathcal{C}}_\alpha + \sum_{ak} d_\alpha(\Omega_k) c_a(\Omega_k)\hat{\mathcal{C}}_a\right)\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\,, \\
&= \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{\mathcal{C}}_\alpha\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle \\
&\quad + \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\sum_{ak} d_\beta(\Omega_k) c_a(\Omega_k)\hat{\mathcal{C}}_a\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\,.
\end{aligned} \tag{C.17}$$

Now, since $\hat{\mathcal{C}}_\beta$ is an element of the $\mathfrak{u}(2)$ subalgebra, we have

$$\begin{aligned}
\hat{\mathcal{C}}_a\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle &= [\hat{\mathcal{C}}_a,\hat{\mathcal{C}}_\beta]|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle + \hat{\mathcal{C}}_\beta\hat{\mathcal{C}}_a|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle \\
&= [\hat{\mathcal{C}}_a,\hat{\mathcal{C}}_\beta]|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle \\
&= \sum_\gamma g^\gamma_{a\beta}\hat{\mathcal{C}}_\gamma|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle.
\end{aligned} \tag{C.18}$$

As we have for $\hat{\mathcal{C}}_\gamma$

$$\hat{\mathbb{S}}_\gamma D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega) = \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{\mathcal{C}}_\gamma|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\,, \tag{C.19}$$

we find that

$$\begin{aligned}
&\hat{\mathbb{S}}_\alpha \hat{\mathbb{S}}_\beta D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega) \\
&= \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\left(\hat{\mathcal{C}}_\alpha + \sum_{ak} d_\alpha(\Omega_k) c_a(\Omega_k)\hat{\mathcal{C}}_a\right)\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle \\
&= \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{\mathcal{C}}_\alpha\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle \\
&\qquad + \sum_{ak\gamma} d_\alpha(\Omega_k) c_a(\Omega_k) g^\gamma_{a\beta}\hat{\mathbb{S}}_\gamma D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)
\end{aligned} \tag{C.20}$$

or

$$\begin{aligned}
&\left(\hat{\mathbb{S}}_\alpha\hat{\mathbb{S}}_\beta - \sum_{a\gamma} f_{\alpha a} g^\gamma_{a\beta}\hat{\mathbb{S}}_\gamma\right) D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega) \\
&= \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{\mathcal{C}}_\alpha\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\,,
\end{aligned} \tag{C.21}$$

where, for economy, we denote

$$f_{\alpha a} := \sum_k d_\alpha(\Omega_k) c_a(\Omega_k)\,. \tag{C.22}$$

These coefficients are given in table C4.

C.3. The second order operator $\hat{\mathbb{S}}^{(2)}_{(1\nu_2\nu_3)I}$

In order to obtain operators which shifts the I label by 1, we work with products of the operators $\hat{\mathbb{S}}_{\nu\frac{1}{2}}$.

If we consider

$$\begin{aligned}&\hat{\mathbb{S}}_{(021)\frac{1}{2}}\hat{\mathbb{S}}_{(210)\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu,J;(\sigma,\sigma,\sigma)0}(\Omega)\\&=\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\hat{C}_{(021)\frac{1}{2}}\hat{C}_{(210)\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\&+\sum_a f_{(021)\frac{1}{2};a}\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\left[\hat{C}_a,\hat{C}_{(210)\frac{1}{2}}\right]|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\quad,\end{aligned}\tag{C.23}$$

where the $f_{\alpha a}$ coefficients that appear in equation (C.23) are given in table C4. Using

$$\begin{aligned}&\sum_a f_{(021)\frac{1}{2};a}\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\left[\hat{C}_a,\hat{C}_{(210)\frac{1}{2}}\right]|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\&=-f_{(021)\frac{1}{2};(102)\frac{1}{2}}\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\hat{C}_{(201)\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\&+\left(-\frac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)1}+\sqrt{\frac{3}{2}}f_{(021)\frac{1}{2};(111)0}\right)\end{aligned}\tag{C.24}$$

$$\begin{aligned}&\times\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\hat{C}_{(210)\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\Big)\\&=-f_{(021)\frac{1}{2};(102)\frac{1}{2}}\hat{\mathbb{S}}_{(201)\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\\&+\left(\sqrt{\frac{3}{2}}f_{(021)\frac{1}{2};(111)0}-\frac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)1}\right)\hat{\mathbb{S}}_{(210)\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\end{aligned}\tag{C.25}$$

and

$$\begin{aligned}&\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\hat{C}_{(021)\frac{1}{2}}\hat{C}_{(210)\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\&=-\frac{\sigma(\sigma+2)}{\sqrt{6}}D^{(\sigma,\sigma)}_{\mu J;(\sigma,\sigma+1,\sigma-1)1}(\Omega)\end{aligned}\tag{C.26}$$

we obtain the expression

$$\begin{aligned}&-\frac{\sigma(\sigma+2)}{\sqrt{6}}D^{(\sigma,\sigma)}_{\mu J;(\sigma,\sigma+1,\sigma-1)1}(\Omega)\\&=\hat{\mathbb{S}}_{(021)\frac{1}{2}}\hat{\mathbb{S}}_{(210)\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}\\&+\left(\frac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)1}\hat{\mathbb{S}}_{(210)\frac{1}{2}}-\sqrt{\frac{3}{2}}f_{(021)\frac{1}{2};(111)0}\hat{\mathbb{S}}_{(210)\frac{1}{2}}\right)D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\\&+f_{(021)\frac{1}{2};(102)\frac{1}{2}}\hat{\mathbb{S}}_{(201)\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\,,\end{aligned}\tag{C.27}$$

$$:=\hat{\mathbb{S}}^{(2)}_{(120);1}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\,.\tag{C.28}$$

Similarly, starting with

$$\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\hat{C}_{(012)\frac{1}{2}}\hat{C}_{(201)\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle$$
$$= -\frac{\sigma(\sigma+2)}{\sqrt{6}}D^{(\sigma,\sigma)}_{\mu,J;(\sigma,\sigma-1,\sigma+1)1}(\Omega)\,, \tag{C.29}$$

we easily reach

$$\begin{aligned}
&-\frac{\sigma(\sigma+2)}{\sqrt{6}}D^{(\sigma,\sigma)}_{\mu J;(\sigma,\sigma-1,\sigma+1)1}(\Omega)\\
&\quad=\hat{\mathbb{S}}_{(012)\frac{1}{2}}\hat{\mathbb{S}}_{(201)\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}\\
&+\left(\frac{\sqrt{2}}{2}f_{(012)\frac{1}{2};(111)1}+\sqrt{\frac{3}{2}}f_{(012)\frac{1}{2};(111)0}\hat{\mathbb{S}}_{(201)\frac{1}{2}}\right)D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\\
&\quad-f_{(012)\frac{1}{2};(120)1}\hat{\mathbb{S}}_{(210)\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)
\end{aligned} \tag{C.30}$$

$$:=\hat{\mathbb{S}}^{(2)}_{(102);1}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\quad . \tag{C.31}$$

Finally, we consider the action

$$\begin{aligned}
&\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\left(\hat{C}_{021;\frac{1}{2}}\hat{C}_{201;\frac{1}{2}}+\hat{C}_{012;\frac{1}{2}}\hat{C}_{210;\frac{1}{2}}\right)|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&=-\frac{\sigma(\sigma+2)}{\sqrt{3}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)1}(\Omega)\\
&:=-\frac{1}{\sqrt{3}}\hat{\mathbb{S}}^{(2)}_{(111);1}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\,.
\end{aligned} \tag{C.32}$$

We can then verify that

$$\begin{aligned}
&\hat{\mathbb{S}}_{021;\frac{1}{2}}\hat{\mathbb{S}}_{201;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)}(\Omega)\\
&\quad=\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\hat{C}_{021;\frac{1}{2}}\hat{C}_{201;\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&+f_{(021)\frac{1}{2};(120)1}\langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{C}_{210;\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\quad+\left(\sqrt{\frac{3}{2}}f_{(021)\frac{1}{2};(111)1}+\frac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)0}\right)\\
&\qquad\times\langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{C}_{201;\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\,,
\end{aligned} \tag{C.33}$$

$$\begin{aligned}
&=\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\hat{C}_{021;\frac{1}{2}}\hat{C}_{201;\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\quad+\left(\sqrt{\frac{3}{2}}f_{(021)\frac{1}{2};(111)1}+\frac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{201;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)}(\Omega)\\
&\quad+f_{(021)\frac{1}{2};(120)1}\hat{\mathbb{S}}_{210;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)}(\Omega)\,,
\end{aligned} \tag{C.34}$$

where equation (C.15) has been used.

Duplicating the same steps, this time for $\hat{\mathbb{S}}_{012;\frac{1}{2}}\hat{\mathbb{S}}_{210;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)}(\Omega)$, yields

$$
\begin{aligned}
&\hat{\mathbb{S}}_{012;\frac{1}{2}}\hat{\mathbb{S}}_{210;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)}(\Omega)\\
&\quad=\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\hat{C}_{012;\frac{1}{2}}\hat{C}_{210;\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\quad-f_{(012)\frac{1}{2};(102)1}\langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{C}_{201;\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\qquad+\left(\sqrt{\frac{3}{2}}f_{(012)\frac{1}{2};(111)1}-\frac{\sqrt{2}}{2}f_{(012)\frac{1}{2};(111)0}\right)\\
&\qquad\quad\times\langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{C}_{210;\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\,,
\end{aligned}
\tag{C.35}
$$

$$
\begin{aligned}
&=\langle(\sigma,\sigma)\mu;J|\hat{R}(\Omega)\hat{C}_{021;\frac{1}{2}}\hat{C}_{201;\frac{1}{2}}|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\quad+\left(\sqrt{\frac{3}{2}}f_{(012)\frac{1}{2};(111)1}-\frac{\sqrt{2}}{2}f_{(012)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{210;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)}(\Omega)\\
&\quad+f_{(012)\frac{1}{2};(102)1}\hat{\mathbb{S}}_{201;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)}(\Omega)\,.
\end{aligned}
\tag{C.36}
$$

Hence:

$$
\begin{aligned}
&-\frac{1}{\sqrt{3}}\hat{\mathbb{S}}^{(2)}_{(111);1}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\\
&=\left(\hat{\mathbb{S}}_{021;\frac{1}{2}}\hat{\mathbb{S}}_{201;\frac{1}{2}}-f_{(021)\frac{1}{2};(120)1}\hat{\mathbb{S}}_{210;\frac{1}{2}}\right)D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\\
&-\left(\sqrt{\frac{3}{2}}f_{(021)\frac{1}{2};(111)1}+\frac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{201;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\\
&+\left(\hat{\mathbb{S}}_{012;\frac{1}{2}}\hat{\mathbb{S}}_{210;\frac{1}{2}}-f_{(012)\frac{1}{2};(102)1}\hat{\mathbb{S}}_{201;\frac{1}{2}}\right)D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\\
&-\left(\sqrt{\frac{3}{2}}f_{(012)\frac{1}{2};(111)1}-\frac{\sqrt{2}}{2}f_{(012)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{210;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\,.
\end{aligned}
\tag{C.37}
$$

We can summarize equations (C.28), (C.31) and (C.37) as

$$
\begin{aligned}
&\hat{\mathbb{S}}^{(2)}_{1\nu_2\nu_3;1}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\\
&\quad=-\sigma(\sigma+2)\sqrt{\frac{(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}{6}}D^{(\sigma,\sigma)}_{\mu J;(1\nu_2\nu_3)1}(\Omega)\,.
\end{aligned}
\tag{C.38}
$$

Appendix D. Explicit form of equation (51) for $\alpha J=(111)0$ and $(111)1$

D.1. General expressions

For $\nu'_1=0,2$, define

$$
\hat{\mathcal{S}}_{\alpha J;\nu'_1\frac{1}{2}}:=\sum_{\nu'_2\nu'_3}D^{(1,1)}_{(\nu'_1\nu'_2\nu'_3)\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\mathbb{S}}_{(\nu'_1\nu'_2\nu'_3)\frac{1}{2}}\,,
\tag{D.1}
$$

$$
\hat{\mathcal{S}}_{\alpha J}=\hat{\mathcal{S}}_{\alpha J;0\frac{1}{2}}+\hat{\mathcal{S}}_{\alpha J;2\frac{1}{2}}
\tag{D.2}
$$

and denote by $g(\Omega)\equiv g(\alpha_1,\beta_1,\alpha_2,\beta_2)$. In this way equation (51) becomes

$$\mathrm{i}\partial_t W_{\hat{\rho}}(\Omega) = -\frac{\sqrt{24}}{N}\hat{\mathcal{S}}_{\alpha J}\hat{\mathfrak{C}}_{\alpha J} W_{\rho}(\Omega) \qquad \hat{H} = (\hat{T}^{\lambda}_{1;\alpha J})^2\,, \tag{D.3}$$

where

$$\hat{\mathfrak{C}}_{\alpha J} W_{\hat{\rho}}(\Omega) := (\hat{\mathfrak{C}}^{L}_{\alpha J} + \hat{\mathfrak{C}}^{R}_{\alpha J}) W_{\hat{\rho}}(\Omega) \tag{D.4}$$

as per equation (51). $\hat{\mathfrak{C}}_{\alpha J}$ itself is a sum, so for notational convenience define

$$\begin{aligned}\hat{\mathfrak{C}}_{\alpha J;0\frac{1}{2}} &= \sqrt{2}\sum_{\nu_2\nu_3} D^{(1,1)}_{(0\nu_2\nu_3)\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\mathbb{S}}_{(0\nu_2\nu_3)\frac{1}{2}} \\ &\qquad \times \left(\hat{a}^{R}_{0\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}^{L}_{0\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2)\right)\hat{\mathcal{C}}_2^{-1/2}\,,\end{aligned} \tag{D.5}$$

$$\begin{aligned}\hat{\mathfrak{C}}_{\alpha J;2\frac{1}{2}} &= -\sqrt{2}\sum_{\nu_2\nu_3} D^{(1,1)}_{(2\nu_2\nu_3)\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\mathbb{S}}_{(2\nu_2\nu_3)\frac{1}{2}} \\ &\qquad \left(\hat{a}^{L}_{2\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}^{R}_{2\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2)\right)\hat{\mathcal{C}}_2^{-1/2}\,,\end{aligned} \tag{D.6}$$

$$\begin{aligned}\hat{\mathfrak{C}}_{\alpha J;11} &= \sum_{\nu_2\nu_3} D^{(1,1)}_{(1\nu_2\nu_3)1;\alpha J}(\Omega^{-1})\sqrt{\frac{6}{(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}}\,\hat{\mathbb{S}}^{(2)}_{1\nu_2\nu_3;1} \\ &\qquad \times \left(\hat{a}^{L}_{11}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}^{R}_{11}(\lambda;\hat{\mathcal{C}}_2)\right)\hat{\mathcal{C}}_2^{-1}\,,\end{aligned} \tag{D.7}$$

$$\hat{\mathfrak{C}}_{\alpha J;10} = D^{(1,1)}_{(111)0;\alpha J}(\Omega^{-1})\left(\hat{a}^{L}_{10}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}^{R}_{10}(\lambda;\hat{\mathcal{C}}_2)\right)\,, \tag{D.8}$$

$$\hat{\mathfrak{C}}_{\alpha J} = \hat{\mathfrak{C}}_{\alpha J;0\frac{1}{2}} + \hat{\mathfrak{C}}_{\alpha J;2\frac{1}{2}} + \hat{\mathfrak{C}}_{\alpha J;11} + \hat{\mathfrak{C}}_{\alpha J;10}. \tag{D.9}$$

Note that

$$\begin{aligned}&\left(\hat{a}^{L}_{0\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}^{R}_{0\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2)\right)\hat{\mathcal{C}}_2^{-1/2}\,g(\Omega) \\ &\quad = \left(\hat{a}^{L}_{2\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}^{R}_{2\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2)\right)\hat{\mathcal{C}}_2^{-1/2}\,g(\Omega)\end{aligned} \tag{D.10}$$

acting on any $g(\Omega)$, so that

$$\begin{aligned}&\left(\hat{\mathfrak{C}}_{\alpha J;0\frac{1}{2}} + \hat{\mathfrak{C}}_{\alpha J;2\frac{1}{2}}\right)g(\Omega) \\ &= \sqrt{2}\sum_{\nu_2\nu_3}\left(D^{(1,1)}_{(0\nu_2\nu_3)\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\mathbb{S}}_{(0\nu_2\nu_3)\frac{1}{2}} - D^{(1,1)}_{(2\nu_2\nu_3)\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\mathbb{S}}_{(2\nu_2\nu_3)\frac{1}{2}}\right) \\ &\quad \times \left(\hat{a}^{L}_{0\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}^{R}_{0\frac{1}{2}}(\lambda;\hat{\mathcal{C}}_2)\right)\hat{\mathcal{C}}_2^{-1/2}\,g(\Omega).\end{aligned} \tag{D.11}$$

D.2. $(\alpha J) = (111)0$.

This is the case where $\hat{H} = (T^{\lambda}_{(111)0})^2$ in equation (51). Let $g(\Omega) = g(\alpha_1, \beta_1, \alpha_2, \beta_2)$ be otherwise arbitrary; then we have

$$\hat{\mathcal{S}}_{(111)0}\,\hat{\mathfrak{C}}_{(111)0;\nu_1 I}\,g(\Omega) = \mathrm{i}\sqrt{\frac{3}{2}}\frac{\partial}{\partial\alpha_2}\,\hat{\mathfrak{C}}_{(111)0;\nu_1 I}\,g(\Omega). \tag{D.12}$$

Thus, we concentrate on the action of $\hat{\mathfrak{C}}_{(111)0;\nu_1 I}$ on $g(\Omega)$. For the $(\nu', I) = (0, \frac{1}{2})$ and $(2, \frac{1}{2})$ we obtain

$$\left(\sum_{\nu_1'=0,2} \hat{\mathfrak{C}}_{(111)0;\nu_1'\frac{1}{2}}\right) g(\Omega) = -\sqrt{3}\sin\beta_2 \frac{\partial}{\partial\beta_2}\left(\hat{\mathcal{C}}_2^{-1/2}\, g(\Omega)\right). \tag{D.13}$$

As to the terms with $(\nu_1', I) = (1,1)$ and $(1,0)$, we find

$$\begin{aligned}
&\hat{\mathfrak{C}}_{(111)0;11} g(\Omega) \\
&= \left(c_{\beta_2}^{(111)0}\frac{\partial}{\partial\beta_2} + c_{\beta_1}^{(111)0}\frac{\partial}{\partial\beta_1} + c_{(\alpha_2,\alpha_2)}^{(111)0}\frac{\partial^2}{\partial\alpha_2^2} + c_{(\alpha_1,\alpha_2)}^{(111)0}\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}\right. \\
&\qquad \left. + c_{(\beta_2,\beta_2)}^{(111)0}\frac{\partial^2}{\partial\beta_2^2} + c_{(\beta_1,\beta_1)}^{(111)0}\frac{\partial^2}{\partial\beta_1^2} + c_{(\alpha_1,\alpha_1)}^{(111)0}\frac{\partial^2}{\partial\alpha_1^2}\right) \\
&\qquad \times \hat{\mathcal{C}}_2^{-1}\left(\hat{a}_{11}^{L}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}_{11}^{R}(\lambda;\hat{\mathcal{C}}_2)\right) g(\Omega),
\end{aligned} \tag{D.14}$$

$$\hat{\mathfrak{C}}_{(111)0;10} g(\Omega) = \frac{1}{4}(1 + 3\cos\beta_2)\left(\hat{a}_{10}^{L}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}_{10}^{R}(\lambda;\hat{\mathcal{C}}_2)\right) g(\Omega), \tag{D.15}$$

where the coefficients $c_{\Omega_k}^{(111)0}$ and $c_{(\Omega_j,\Omega_k)}^{(111)0}$ are given in table D1.

D.3. $(\alpha J) = (111)1$

This is the case where $\hat{H} = (\hat{T}^{\lambda}_{(111)1})^2$ in equation (51). Here, we have

$$\left(\hat{\mathcal{S}}_{(111)1}\right) g(\Omega) = \frac{\mathrm{i}}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha_2} - 2\frac{\partial}{\partial\alpha_1}\right) g(\Omega). \tag{D.16}$$

For terms with $\nu_1' = 0$ and $\nu' = 2$:

$$\begin{aligned}
&\hat{\mathfrak{C}}_{(111)1;0\frac{1}{2}}\, g(\Omega) \\
&\quad = \left(-\frac{1}{2}\cos\beta_1\sin\beta_2\frac{\partial}{\partial\beta_2} + \sin\beta_1\frac{\partial}{\partial\beta_1} + \frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial\alpha_2} - 2\frac{\partial}{\partial\alpha_1}\right)\right) \\
&\qquad \times \left(\hat{a}_{0\frac{1}{2}}^{L}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}_{0\frac{1}{2}}^{R}(\lambda;\hat{\mathcal{C}}_2)\right)\hat{\mathcal{C}}_2^{-1/2} g(\Omega),
\end{aligned} \tag{D.17}$$

$$\begin{aligned}
&\hat{\mathfrak{C}}_{(111)1;2\frac{1}{2}}\, g(\Omega) \\
&= \left(-\frac{1}{2}\cos\beta_1\sin\beta_2\frac{\partial}{\partial\beta_2} + \sin\beta_1\frac{\partial}{\partial\beta_1} - \frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial\alpha_2} - 2\frac{\partial}{\partial\alpha_1}\right)\right) \\
&\qquad \times \left(\hat{a}_{0\frac{1}{2}}^{L}(\lambda;\hat{\mathcal{C}}_2) + \hat{a}_{0\frac{1}{2}}^{R}(\lambda;\hat{\mathcal{C}}_2)\right)\hat{\mathcal{C}}_2^{-1/2} g(\Omega),
\end{aligned} \tag{D.18}$$

the sum simplifies this time to

$$\begin{aligned}
&\left(\sum_{\nu_1'=0,2} \hat{\mathfrak{C}}_{(111)0;\nu_1'\frac{1}{2}}\right) g(\Omega) \\
&= \left(-\cos\beta_1\sin\beta_2\frac{\partial}{\partial\beta_2} + 2\sin\beta_1\frac{\partial}{\partial\beta_1}\right)\hat{\mathcal{C}}_2^{-1/2}\, g(\Omega).
\end{aligned} \tag{D.19}$$

As to the terms with $\nu_1' = 1$, we find

$$\hat{\mathfrak{C}}_{(111)1;11}g(\Omega)$$
$$= \left[-\frac{\sqrt{3}\cos(\beta_1)(3+\cos(\beta_2))}{4\sin(\beta_2)} \left(\frac{\partial}{\partial\beta_2} + 2\cot(\beta_1)\cot\left(\frac{1}{2}\beta_2\right)\frac{\partial}{\partial\beta_1} \right) \right.$$
$$+ \frac{\sqrt{3}}{8}\left[2\csc^2\left(\frac{1}{2}\beta_2\right)\sec^2\left(\frac{1}{2}\beta_1\right) + \cos(\beta_1)\left(\tan^2\left(\frac{1}{2}\beta_1\right) + \sec^2\left(\frac{1}{2}\beta_2\right)\right)\right]\frac{\partial^2}{\partial\alpha_2^2}$$
$$+ \frac{\sqrt{3}}{8}(\cos(\beta_1)(\cos(\beta_2)-1)-4)\csc^2\left(\frac{\beta_2}{2}\right)\sec^2\left(\frac{\beta_1}{2}\right)\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}$$
$$+\frac{3}{4}(\cos\beta_2 - 1)\frac{\partial^2}{\partial\beta_2^2} - \frac{\sqrt{3}}{4}\cos(\beta_1)(\cos(\beta_2)+3)\csc^2\left(\frac{\beta_2}{2}\right)\frac{\partial^2}{\partial\beta_1^2}$$
$$\left. +\frac{3}{2}\csc^2(\beta_1)\frac{\partial^2}{\partial\alpha_1^2}\right]\hat{\mathcal{C}}_2^{-1}\left(\hat{a}_{11}^L(\lambda;\hat{\mathcal{C}}_2) + \hat{a}_{11}^R(\lambda;\hat{\mathcal{C}}_2)\right)g(\Omega)\,, \tag{D.20}$$

$$\hat{\mathfrak{C}}_{(111)0;10}g(\Omega)$$
$$= \frac{1}{4}(1+3\cos\beta_2)\left(\hat{a}_{10}^L(\lambda;\hat{\mathcal{C}}_2) + \hat{a}_{10}^R(\lambda;\hat{\mathcal{C}}_2)\right)g(\Omega). \tag{D.21}$$

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