IOP Publishing | Royal Swedish Academy of Sciences Physica Scripta

Phys. Scr. 90 (2015) 074030 (7pp) doi:10.1088/0031-8949/90/7/074030

Classical polarization multipoles: paraxial versus nonparaxial

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Received 12 November 2014, revised 21 January 2015
Accepted for publication 9 February 2015
Published 18 June 2015

Abstract

We discuss the polarization of paraxial and nonparaxial classical light fields by resorting to a multipole expansion of the corresponding polarization matrix. It turns out that only a dipolar term contributes when one considers SU(2) (paraxial) or SU(3) (nonparaxial) as fundamental symmetries. In this latter case, one can alternatively expand in SU(2) multipoles, and then both a dipolar and a quadrupolar component contribute, which explains the richer structure of this nonparaxial instance. These multipoles uniquely determine Wigner functions, in terms of which we examine some intriguing hallmarks arising in this classical scenario.

Keywords: polarisation, multipolar expansion, near fields

(Some figures may appear in colour only in the online journal)

1. Introduction

The standard theory of polarization optics deals with paraxial fields with a well-defined direction of propagation and thus a specific transverse plane. Such beamlike fields are described by two orthogonal electric field components and, consequently, their polarization is characterized by a 2×2 correlation matrix, usually called the polarization matrix (Mandel and Wolf 1995). When we expand this matrix onto the Pauli basis, the corresponding coefficients are nothing but the time honored Stokes parameters. They determine a locus on the Poincaré sphere, wherein the state of polarization is elegantly visualized; actually, the degree of paraxial polarization can be seen as the length of the Stokes vector.

The necessity of addressing new issues, such as highly nonparaxial fields (Ash and Nicholls 1972), narrow-band imaging systems (Pohl *et al* 1984), and the recognition of associated propagation questions (Petruccelli *et al* 2010), has fuelled interest in extending this two-dimensional (2D) approach to fully three-dimensional (3D) field distributions, where, in general, there is no well-defined propagation direction, and the Hermitian coherence matrix is 3×3 (Roman 1959, Brosseau 1998). However, when discussing the corresponding degree of polarization, some discrepancies occur (Samson 1973, Barakat 1977, Setälä *et al* 2002, Korotkova and Wolf 2004, Ellis *et al* 2005, Luis 2005, Dennis 2007, Moya-Cessa *et al* 2008, Sheppard 2011), mainly because notions that are equivalent in a 2D world lead to different definitions when extended to the 3D domain.

In this paper, we look at this problem from the perspective of a multipolar expansion that has been successfully employed in quantum optics (de la Hoz *et al* 2013, 2014, Sánchez-Soto *et al* 2013). For the 2D case, SU(2) is the natural symmetry group of the problem and the corresponding expansion of the polarization matrix contains only a dipolar term. The 3D polarization can be treated in quite a similar

way by taking now SU(3) as the symmetry group; again, only the SU(3) dipole contributes. Yet we can also expand the 3×3 polarization matrix in SU(2) tensors, getting an SU(2) dipole and a quadrupole. We discuss the role of these multipoles and construct Wigner functions that can be very useful in appreciating the substantial differences between paraxial and nonparaxial polarization.

2. 2D polarization

2.1. SU(2) polarization structure

We briefly review here the ingredients we shall need later for a proper understanding of the 3D polarization. We consider a monochromatic beam propagating in the z direction. The electric field can be resolved in the transverse plane in terms of horizontal (x) and vertical (y) components, which are taken to be a probabilistic ensemble given by E_x and E_y. The corresponding 2×2 (equal-time) polarization matrix (also called the coherence matrix) is defined as (Mandel and Wolf 1995)

$$\mathbf{C} = \begin{pmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{pmatrix}. \tag{2.1}$$

Here, the brackets denote ensemble averaging over different realizations. The diagonal elements of the Hermitian matrix $\mathbf{C}$ represent the energy distribution between the two components of the field: $I = \langle |E_x|^2 \rangle + \langle |E_y|^2 \rangle = \mathrm{Tr}\,(\mathbf{C})$, where Tr is the trace of the matrix. Without loss of generality, we henceforth normalize this intensity to unity. On the other hand, the off-diagonal elements describe the correlations between the field components.

Sometimes, the polarization matrix is interpreted as a classical counterpart of the density operator. Nonetheless, we emphasize that while the latter carries complete information of a quantum system, the former specifies only first-order classical correlations.

Polarization transformations are generated by wave plates and represented by 2×2 unitary matrices of SU(2) (Simon and Mukunda 1989)

$$\mathbf{R}_g \equiv \mathbf{R}(\alpha, \beta, \gamma) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(\alpha+\gamma)/2} \cos(\beta/2) & -\mathrm{e}^{-\mathrm{i}(\alpha-\gamma)/2} \sin(\beta/2) \\ \mathrm{e}^{+\mathrm{i}(\alpha-\gamma)/2} \sin(\beta/2) & \mathrm{e}^{+\mathrm{i}(\alpha+\gamma)/2} \cos(\beta/2) \end{pmatrix}, \tag{2.2}$$

where (α, β, γ) denote the Euler angles. The action of these transformations on the polarization matrix is via conjugation

$$\mathbf{C}_g = \mathbf{R}_g\, \mathbf{C}\, \mathbf{R}_g^\dagger. \tag{2.3}$$

This symmetry seems to call for an SU(2)-covariant formulation of the problem. To this end, we recall that any matrix $\mathbf{O}$ acting in the $(2S+1)$-dimensional Hilbert space $\mathcal{H}_S$, which carries the irreducible representation (irrep) with spin S of SU(2), can be expanded as

$$\mathbf{O}^{(S)} = \sum_{K=0}^{2S} \sum_{q=-K}^{K} O_{Kq}^{(S)}\, \hat{T}_{Kq}^{(S)}. \tag{2.4}$$

Here, the irreducible tensor (or multipole) operators $\hat{T}_{Kq}^{(S)}$ constitute an orthonormal basis

$$\mathrm{Tr}\left[\hat{T}_{Kq}^{(S)}\, \hat{T}_{K'q'}^{(S')\,\dagger} \right] = \delta_{SS'} \delta_{KK'} \delta_{qq'}, \tag{2.5}$$

and have the right transformation properties under SU(2)

$$\mathbf{R}(\alpha, \beta, \gamma)\, \hat{T}_{Kq}^{(S)}\, \mathbf{R}^\dagger(\alpha, \beta, \gamma) = \sum_{q'} D_{q'q}^{S}(\alpha, \beta, \gamma)\, \hat{T}_{Kq'}^{(S)}, \tag{2.6}$$

where $D_{q'q}^{S}(\alpha, \beta, \gamma)$ stands for the Wigner D-function (Varshalovich *et al* 1988). The reader is referred to the abundant literature (Fano and Racah 1963, Blum 1981, Varshalovich *et al* 1988) to learn more about the amazing properties of these tensors. The point we wish to stress for our purposes here is that $\hat{T}_{Kq}^{(S)}$ is expressible as the Kth power of the SU(2) generators.

The corresponding expansion coefficients

$$O_{Kq}^{(S)} = \mathrm{Tr}\left[\mathbf{O}^{(S)}\, \hat{T}_{Kq}^{(S)\,\dagger} \right] \tag{2.7}$$

are known as state multipoles. The hermiticity imposes the symmetry condition

$$O_{K-q}^{(S)} = (-1)^q\, O_{Kq}^{(S)}. \tag{2.8}$$

For the case at hand in (2.1), we are dealing with the fundamental irrep of spin $S = 1/2$. Accordingly, the expansion (2.4) reduces to (we drop the superscript 1/2 henceforth, as there is no risk of confusion)

$$\mathbf{C} = C_{00} \hat{T}_{00} + \sum_{q=-1}^{+1} C_{1q} \hat{T}_{1q}, \tag{2.9}$$

and one can check that

$$\hat{T}_{00} = \frac{1}{\sqrt{2}}\, \mathbb{1}_2, \qquad \hat{T}_{1q} = \frac{1}{\sqrt{2}}\, \sigma_q, \tag{2.10}$$

where $\mathbb{1}_2$ is the identity matrix and $q = 0, \pm 1$ runs over the spherical basis. The unit vectors in that basis $(\mathfrak{e}_{-1}, \mathfrak{e}_0, \mathfrak{e}_{+1},)$ are related to the Cartesian ones $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by

$$\mathfrak{e}_{\pm 1} = \mp \frac{1}{\sqrt{2}} \left(\mathbf{e}_1 \pm \mathrm{i}\, \mathbf{e}_2 \right), \qquad \mathfrak{e}_0 = \mathbf{e}_3, \tag{2.11}$$

and $(\sigma_1, \sigma_2, \sigma_3)$ is the standard Pauli basis.

Obviously, $C_{00} = 1/\sqrt{2}$ and the physical relevant information comes from the dipole C_{1q}. In fact,

$$n_q \equiv \sqrt{2}\, C_{1q} = \mathrm{Tr}\left(\mathbf{C} \sigma_q \right), \qquad q = 0, \pm 1, \tag{2.12}$$

are nothing but the components of the Stokes vector, which in Cartesian coordinates $\mathbf{n} = (n_1, n_2, n_3)$ provides geometric information about the polarization ellipse: n_1 and n_3 carry information about the alignment of the ellipse axes, while πn_2 gives the ellipse area, signed according to polarization handedness.

With this notation, (2.9) can be recast as

$$\mathbf{C} = \frac{1}{2}\begin{pmatrix} 1 + n_3 & n_1 - \mathrm{i}n_2 \\ n_1 + \mathrm{i}n_2 & 1 - n_3 \end{pmatrix}. \tag{2.13}$$

so that for each polarization matrix $\mathbf{C}$ we have a natural map onto a dipole $\mathbf{C} \mapsto \mathbf{n} = (n_1, n_2, n_3)$. This is consistent with the fact that at the classical level only first-order moments of the Stokes variables are taken into consideration.

The length of $\mathbf{n}$ will be denoted as

$$\mathbb{P}^{(2)} = |\mathbf{n}| = \sqrt{n_1^2 + n_2^2 + n_3^2}, \tag{2.14}$$

and is the conventional degree of polarization for classical 2D fields. If the relation between the E_x and E_y is completely deterministic, the field is fully polarized. For such a pure state (borrowing the terminology from quantum optics), the polarization matrix is idempotent, i.e.,

$$\mathbf{C}_{\mathrm{pol}}^2 = \mathbf{C}_{\mathrm{pol}}, \tag{2.15}$$

and we get $\mathbb{P}^{(2)}_{\mathrm{pol}} = 1$. On the other hand, if the components of the field are fully uncorrelated, the off-diagonal elements are zero. If, in addition, the energy is distributed evenly between the x and y components,

$$\mathbf{C}_{\mathrm{unpol}} = \frac{1}{2}\,\mathbb{1}_2, \tag{2.16}$$

and we have $\mathbb{P}^{(2)}_{\mathrm{unpol}} = 0$.

Note that the SU(2) polarization transformations (2.3) induce rotations on the Stokes vector $\mathbf{n}$, as confirmed by the well-known relation between SU(2) and the group of rotations SO(3) (Cornwell 1997). Therefore, $\mathbb{P}^{(2)}$ is clearly unchanged by these transformations.

Finally, we stress that for quantum fields with N photons, the spin of the associated irrep is $S = N/2$, so classical polarization is formally identical with single-photon quantum polarization, which, in turn, is the prototype of a qubit.

2.2. Wigner function on the Poincaré sphere

In signal processing, the Wigner representation yields a description displaying both the time and frequency features, which are related via the Fourier transform. This arises naturally in music, for instance, where a signal is usually described not by a time function nor by the Fourier transform of that function, but by its musical score (i.e., a prescription of the frequencies of the tones that should be present at a certain moment). It arises also in mechanics, where the position and the momentum of a particle are given simultaneously, leading to a simple interpretation in phase space.

Although originally introduced to represent quantum mechanical phenomena in phase space (Wigner 1932), the Wigner distribution function was established in optics (Walther 1968) to relate partial coherence with radiometry. Since then, a great number of applications of this function have been reported (Dragoman 1997, Mecklenbraüker and Hlawatsch 1997, Galleani and Cohen 2002, Bastiaans 2009, Alonso 2011).

With regard to polarization, the SU(2) symmetry discussed earlier allows us to take advantage of the pioneering papers by Stratonovich (1956) and Berezin (1975), who worked out *bona fide* Wigner distributions on the sphere. This construction was later generalized by others (Agarwal 1981, Brif and Mann 1998, Heiss and Weigert 2000, Klimov and Chumakov 2000, Klimov and Romero 2008) and has proved to be very useful in visualizing properties of spinlike systems (Dowling *et al* 1994, Chumakov *et al* 1999, Klimov 2002).

The Wigner function associated to the operator $\mathbf{O}^{(S)}$ is uniquely defined as (Dowling *et al* 1994)

$$W_{\mathbf{O}}^{(S)}(\theta, \phi) = \sum_{K=0}^{2S} \sum_{q=-K}^{K} O_{Kq}^{(S)}\, Y_{Kq}(\theta, \phi), \tag{2.17}$$

where $Y_{Kq}(\theta, \phi)$ are the standard spherical harmonics. In quantum optics, $\mathbf{O}$ is most often taken as the density operator. One can check that, in this way, the function $W_{\mathbf{O}}^{(S)}(\theta, \phi)$ satisfies all the pertinent requirements. We have the normalization

$$\mathrm{Tr}\left[\mathbf{O}^{(S)}\right] = \sqrt{\frac{2S+1}{4\pi}} \int_{S_2} W_{\mathbf{O}}^{(S)}(\theta, \phi)\, \mathrm{d}\Omega, \tag{2.18}$$

where $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta \mathrm{d}\phi$ is the invariant measure on the unit sphere $\mathcal{S}_2$.

Besides, $W_{\mathbf{O}}^{(S)}(\theta, \phi)$ is covariant under rotations, which means that for a rotated operator $\mathbf{O}_g^{(S)} = \mathbf{R}_g \mathbf{O}^{(S)}\, \mathbf{R}_g^{\dagger}$ one has

$$W_{\mathbf{O}_g}^{(S)}(\Omega) = W_{\mathbf{O}}^{(S)}\left(\mathbf{R}_g^{-1}\Omega\right), \tag{2.19}$$

where $\Omega = (\theta, \phi)$ parametrizes the points in $\mathcal{S}_2$. The Wigner function of the rotated state follows thus the rotation rigidly without deformation, reflecting the fact that physics should not depend on the orientation of the reference frame.

For the classical polarization matrix $\mathbf{C}$, the associated Wigner function reads (we also omit the superscript 1/2)

$$W_{\mathbf{n}}(\theta, \phi) = \frac{1}{\sqrt{2}}\left[Y_{00} + \sum_{q=-1}^{+1} n_q\, Y_{1q}(\theta, \phi) \right]. \tag{2.20}$$

In figure 1 we plot this Wigner function for a fully polarized field with $\mathbf{n} = (0, 0, 1)$. The covariance under rotations guarantees that this is indeed the form (apart from trivial rotations) for any other polarized state.

In the opposite instance of unpolarized states only the monopole term contribute: the resulting Wigner function is just an isotropic sphere. Partially polarized states smoothly interpolate In between these two limiting cases.

As we can appreciate, the Wigner distribution is not a delta-like function, as one would expect from the accepted practice of picturing the state by the point determined by $\mathbf{n}$. This would be compatible with the assumption that classically, since there are no fluctuations, the polarization direction can be measured in a single-shot measurement. Instead, the Wigner function has some width because if one measures a classically polarized state in a rotated Stokes basis, then one will only detect part of the intensity. This means that the detected fraction of the polarization correspond to the overlap

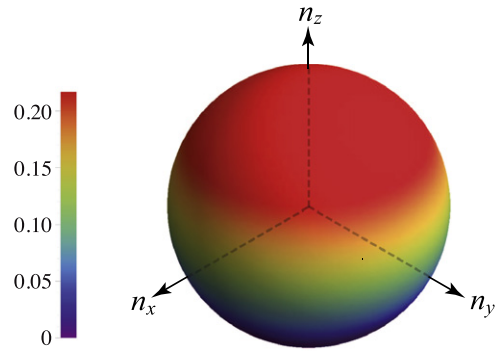

Figure 1. Density plot over the unit sphere (the scale is indicated on the left) of the Wigner distribution for a 2D polarization state with $n = (0, 0, 1)$.

of the Wigner function for the state and the corresponding one for the detection direction on the Poincaré sphere. In fact, if we denote by $W_{\mathbf{n}}$ and $W_{\mathbf{n}'}$ these two Wigner functions, then

$$\int W_{\mathbf{n}}(\theta, \phi) W_{\mathbf{n}'}(\theta, \phi)\, \mathrm{d}\Omega = \cos^2\left(\frac{\chi}{2}\right), \tag{2.21}$$

where $\mathbf{n} \cdot \mathbf{n}' = \cos\chi$ is the angle between the two vectors. This is just another way to look at the Malus law from a phase-space perspective (Wódkiewicz 1995).

3. 3D polarization

3.1. SU(3) polarization structure

Next, we loosen the restriction of planar geometry and examine the behavior of electric fields having three non-vanishing components, in directions we denote as x, y, and z, respectively. Now, the vibrations of the field are no longer constrained to a plane and the polarization must be described by a 3×3 matrix

$$C = \begin{pmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle & \langle E_x E_z^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle & \langle E_y E_z^* \rangle \\ \langle E_z E_x^* \rangle & \langle E_z E_y^* \rangle & \langle E_z E_z^* \rangle \end{pmatrix}. \tag{3.1}$$

Once more, we normalize the total intensity $I = \langle |E_x|^2 \rangle + \langle |E_y|^2 \rangle + \langle |E_z|^2 \rangle = \mathrm{Tr}\,(C)$, to unity. If all of the components are completely uncorrelated (and their energies are equal) the field is unpolarized and its direction is random. If one of the components has less energy than the other two, the vibrations are less random and, consequently, the field is more polarized than in the equal-energy case. Any field having only two non-vanishing components is thus never unpolarized in the 3D sense, regardless of the correlations between the components. Hence, a planar field, which is commonly called unpolarized in 2D, is not fully unpolarized but partially polarized in a 3D description.

As in 2D, the field is called fully polarized if all of the field components are completely correlated. Hence, in contrast to an unpolarized field, a planar field that is fully polarized is always fully polarized also in the 3D sense. One of the most remarkable differences between 2D and 3D is that the 3×3 polarization matrix cannot be generally expressed as a sum of unpolarized and fully polarized parts (Ellis *et al* 2005).

The 3D polarization transformations are represented by 3×3 matrices of SU(3), which we write as (Rowe *et al* 1999)

$$\begin{aligned} \mathbf{R}_g = \mathbf{R}_g(\omega) \equiv \mathbf{T}_{23}\left(\alpha_1, \beta_1, -\alpha_1\right)\mathbf{T}_{12}\left(\alpha_2, \beta_2, -\alpha_2\right) \\ \times \mathbf{T}_{23}\left(\alpha_3, \beta_3, -\alpha_3\right)\boldsymbol{\Phi}(\gamma_1, \gamma_2), \end{aligned} \tag{3.2}$$

where ω is an octuple of Euler-like angles $\omega = (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma_1, \gamma_2)$ and the set $\{\mathbf{T}_{ij}\}$ comprises SU(2) subgroup matrices

$$\mathbf{T}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}(\alpha+\gamma)/2}\cos(\beta/2) & -\mathrm{e}^{-\mathrm{i}(\alpha-\gamma)/2}\sin(\beta/2) \\ 0 & \mathrm{e}^{+\mathrm{i}(\alpha-\gamma)/2}\sin(\beta/2) & \mathrm{e}^{+\mathrm{i}(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix}, \tag{3.3}$$

or

$$\mathbf{T}_{12} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(\alpha+\gamma)/2}\cos(\beta/2) & -\mathrm{e}^{-\mathrm{i}(\alpha-\gamma)/2}\sin(\beta/2) & 0 \\ \mathrm{e}^{+\mathrm{i}(\alpha-\gamma)/2}\sin(\beta/2) & \mathrm{e}^{+\mathrm{i}(\alpha+\gamma)/2}\cos(\beta/2) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.4}$$

depending on the values of (ij). Also,

$$\boldsymbol{\Phi}(\gamma_1, \gamma_2) = \mathrm{diag}\left(\mathrm{e}^{-2\mathrm{i}\gamma_1}, \mathrm{e}^{\mathrm{i}(\gamma_1-\gamma_2/2)}, \mathrm{e}^{\mathrm{i}(\gamma_1+\gamma_2/2)}\right). \tag{3.5}$$

Equation (3.2) factorizes then into SU(2) submatrices, with parameters defined by the corresponding Euler angles.

The action of these transformations on C is via conjugation, as in (2.3), and induces rotations of the vector $\mathbf{n}$. However, one word of caution seems pertinent here: there is no obvious physical interpretation via optical elements of SU(3) transformations, as now the plane waves averaging to the 3×3 polarization matrix do not share a common propagation direction, in general. Any physical device represented by a SU(3) transformation should be insensitive to the propagation directions of the separate members of the ensemble (Dennis 2004).

After our discussion in section 2, one might be tempted to look for an expansion of C in SU(3) irreducible tensors. The corresponding formalism is now much more involved than for SU(2), and we refer the reader to (Banyai *et al* 1966) for a full account. Without going into inessential details, such an expansion reads

$$C = C_{00}\hat{\mathcal{T}}_{00} + \sum_{\nu=1}^{8} C_{1\nu}\hat{\mathcal{T}}_{1\nu}, \tag{3.6}$$

where the relevant SU(3) tensors are

$$\hat{\mathcal{T}}_{00} = \frac{1}{\sqrt{3}}\mathbb{1}_3, \qquad \hat{\mathcal{T}}_{1\nu} = \frac{1}{\sqrt{2}}\lambda_\nu, \tag{3.7}$$

and $\{\lambda_\nu\}$ (the ν index running from 1 to 8) are the Gell-Mann matrices, whose properties are discussed in detail e.g., in (Cornwell 1997).

Therefore, $C_{00} = 1/\sqrt{3}$ and the SU(3) dipole is $C_{1\nu}$. Actually, if we define

$$n_\nu \equiv \sqrt{3}\, C_{1\nu} = \frac{\sqrt{3}}{2}\,\mathrm{Tr}\left(C\lambda_\nu\right), \tag{3.8}$$

then we have

$$C = \frac{1}{3}\left(\mathbb{1}_3 + \sqrt{3}\,\mathbf{n}\cdot\boldsymbol{\lambda}\right) = \frac{1}{3} \times \begin{pmatrix} 1+\sqrt{3}n_3+n_8 & \sqrt{3}\left(n_1 - \mathrm{i}n_2\right) & \sqrt{3}\left(n_4 - \mathrm{i}n_5\right) \\ \sqrt{3}\left(n_1 + \mathrm{i}n_2\right) & 1-\sqrt{3}n_3+n_8 & \sqrt{3}\left(n_6 - \mathrm{i}n_7\right) \\ \sqrt{3}\left(n_4 + \mathrm{i}n_5\right) & \sqrt{3}\left(n_6 + \mathrm{i}n_7\right) & 1-2n_8 \end{pmatrix}. \tag{3.9}$$

This is the SU(3)-equivalent version of (2.13). Consequently, the SU(3) symmetry gives a natural degree of polarization as

$$\mathbb{P}^{(3)} = |\mathbf{n}| = \sqrt{\sum_{\nu=1}^{8} n_\nu^2}, \tag{3.10}$$

i.e., again the length of the Stokes vector, which is readily shown to verify $0 \leqslant \mathbb{P}^{(3)} \leqslant 1$.

3.2. Picturing multipoles on the Poincaré sphere

Although the previous SU(3) approach is mathematically correct, it is not clear physically what $\mathbb{P}^{(3)}$ represents. Unlike in 2D, where the Stokes vector represents the complete state of polarization and can be easily visualized, the generalized Stokes vector is eight-dimensional and the geometrical space supporting this vector is not intuitive at all.

To proceed further, we note that the 3×3 matrix C acts also in the carrier space of the irrep $S = 1$ of SU(2). In consequence, we can alternatively expand C following the general prescription (2.4). Omitting for simplicity the superscript 1, we get

$$C = C_{00}\hat{T}_{00} + \sum_{q=-1}^{+1} C_{1q}\hat{T}_{1q} + \sum_{q=-2}^{+2} C_{2q}\hat{T}_{2q}, \tag{3.11}$$

where now the SU(2) tensors are (Varshalovich *et al* 1988)

$$\begin{aligned} \hat{T}_{00} &= \frac{1}{\sqrt{3}}\mathbb{1}_3, \\ \hat{T}_{1q} &= \frac{1}{\sqrt{2}}S_q, \qquad q = 0, \pm 1, \\ \hat{T}_{2q} &= \sum_{r,r'=-1}^{+1} C^{2q}_{1r,1r'}\, S_r S_{r'}, \qquad q = 0, \pm 1, \pm 2. \end{aligned} \tag{3.12}$$

Here, $C^{2q}_{1r,1r'}$ is the corresponding Clebsch–Gordan coefficient and S_q stands for the spherical components of the su(2) sub algebra generated by $\mathbf{S} = (\lambda_2, -\lambda_5, \lambda_7)$.

The resulting multipoles can be expressed as

$$\begin{aligned} C_{00} &= 1/\sqrt{3}\,, \\ D_q &\equiv \sqrt{3}\, C_{1q} = \sqrt{3}\ \mathrm{Tr}\left(C\hat{T}_{1q}\right), \\ Q_q &\equiv C_{2q} = \sqrt{3}\ \mathrm{Tr}\left(C\hat{T}_{2q}\right). \end{aligned} \tag{3.13}$$

As a consequence, this leads naturally to an SU(2) Wigner function:

$$W(\theta, \phi) = \frac{1}{\sqrt{3}}\left[Y_{00} + \sum_{q=-1}^{+1} D_q\, Y_{1q}(\theta, \phi) + \sum_{q=-2}^{+2} Q_q\, Y_{2q}(\theta, \phi) \right]. \tag{3.14}$$

What it is remarkable is that, in this way, we can picture the 3D polarization state in the sphere S_2 associated to SU(2). In figure 2 we have plotted this Wigner function for the state with eight-dimensional Stokes vector $\mathbf{n} = (0, 0, \sqrt{3}/2, 0, 0, 0, 0, 1/2)$, which is 3D fully polarized, as $\mathbb{P}^{(3)} = 1$. However, this state has $\mathbf{D} = (0, 0, 1/2)$, so is not SU(2) polarized and presents a quadrupole contribution along the z axis. Note that the different multipolar contributions can be plotted separately, as in the figure. Unpolarized states have only monopole contribution, so their Wigner function is isotropic.

To investigate this point, we recall that it is conventional to use a quadrupolar tensor, with Cartesian components

$$\hat{Q}_{ik} = \frac{1}{2}\left(S_i S_k + S_k S_i - \frac{4}{3}\delta_{jk}\right), \qquad i, k \in \{1, 2, 3\}, \tag{3.15}$$

which are related with $\hat{T}_{2q}$ in the form

$$\begin{aligned} \hat{T}_{2\pm 2} &= \frac{1}{2}\left(\hat{Q}_{11} - \hat{Q}_{22} \pm 2\mathrm{i}\hat{Q}_{12}\right), \\ \hat{T}_{2\pm 1} &= \mp\left(\hat{Q}_{13} \pm \mathrm{i}\hat{Q}_{23}\right), \\ \hat{T}_{20} &= \sqrt{\frac{3}{2}}\,\hat{Q}_{33}. \end{aligned} \tag{3.16}$$

In this case, the nondiagonal elements of this symmetric tensor are $\hat{Q}_{12} = \lambda_6$, $\hat{Q}_{13} = \lambda_1$ and $\hat{Q}_{23} = \lambda_4$, whereas the diagonal ones can be expressed as

$$\hat{Q}_{11} - \hat{Q}_{22} = 2\lambda_3, \qquad 2\hat{Q}_{23} - \hat{Q}_{11} - \hat{Q}_{33} = 2\sqrt{3}\,\lambda_8, \tag{3.17}$$

where (λ_3, λ_8) is just the Cartan subalgebra (Rowe *et al* 1989).

The decomposition in terms of SU(2) multipoles is known as the irreducible embedding of SU(2) in SU(3) and has been widely employed in nuclear physics (Dalitz 1952, Ward 1982). In optics, this was first noticed by (Carozzi *et al* 2000) and later on its physical meaning was elucidated from different perspectives (Dennis 2004, Petrov 2008, Sheppard 2014). The dipole terms are precisely Im $\langle E_i E_j^* \rangle$, so they measure the strength of two of the oscillating fields in antiphase, as two distinguishable sources in antiphase constitute a dipole. The three components (n_6, n_4, n_1) of the quadrupole correspond to Re $\langle E_i E_j^* \rangle$ and measure the strength of two of the oscillating fields in phase: now two distinguishable sources oscillating in phase generate a field with a quadropole component.

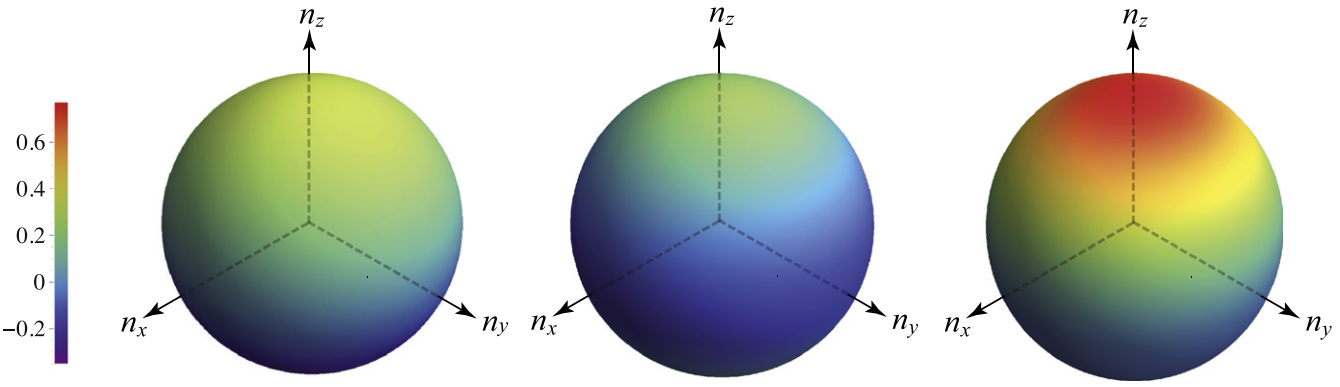

Figure 2. From left to right, density plots (the scale is included) for the Wigner function associated with the dipolar component, the quadrupolar component, and total one for the state $\mathbf{n} = (0, 0, \sqrt{3}/2, 0, 0, 0, 0, 1/2)$.

4. Concluding remarks

We have explored the use of Wigner functions to depict the behavior of 2D and 3D polarizations. Although this has been mostly considered as a quantum tool, it has proven to be also quite an efficient approach to deal with classical fields. Especially, in the 3D case, we have tailored an efficient procedure to represent state on the sphere. We hope that this analysis clarifies the discussion on 3D polarization in the literature.

Acknowledgments

Financial support from the Swedish Foundation for International Cooperation in Research and Higher Education (STINT), the Swedish Research Council (VR) through its Linnaeus Center of Excellence ADOPT and Contract No. 621-2011-4575, the CONACyT (Grant 106525), Canadian NSERC, the European Union FP7 (Grant Q-ESSENCE), the Spanish MINECO (Grant FIS2011-26786) and the Program UCM-Banco Santander (Grant GR3/14) is gratefully acknowledged. GB thanks the Max-Planck-Institut für die Physik des Lichts for hosting him during the fall 2014 and the Wenner-Gren Foundation for economic support.

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