A complementarity-based approach to phase in finite-dimensional quantum systems

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Abstract
We develop a comprehensive theory of phase for finite-dimensional quantum systems. The only physical requirement we impose is that phase is complementary to amplitude. To implement this complementarity we use the notion of mutually unbiased bases, which exist for dimensions that are powers of a prime. For a d-dimensional system (qudit) we explicitly construct $d + 1$ classes of maximally commuting operators, each one consisting of $d - 1$ operators. One of these classes consists of diagonal operators that represent amplitudes (or inversions). By finite Fourier transformation, it is mapped onto ladder operators that can be appropriately interpreted as phase variables. We discuss examples of qubits and qutrits, and show how these results generalize previous approaches.

Keywords: complementarity, quantum phase, finite quantum systems, finite Fourier transform

1. Introduction

The standard formalism of quantum optics is usually introduced in the context of the harmonic oscillator, where both position and momentum are represented by unbounded operators with eigenvalues in the real numbers. On the other hand, systems living in a finite-dimensional Hilbert space were studied originally by Weyl [1] and also by Schwinger [2], but except for some relevant exceptions (for a complete review see [3]), they received the attention they rightly deserve only after becoming an essential ingredient in the development of the emerging field of quantum information [4, 5]. Indeed, the promise of futuristic technologies like safe cryptography and new 'supercomputers', capable of handling otherwise intractable problems, relies on the ability to control the quantum states of a small number of qubits [6, 7].

In the modern parlance of quantum information the concept of phase for a d-dimensional system (or qudit) is ubiquitous. However, in spite of being a primitive of the theory, this notion is rather imprecise and, roughly speaking, three quite distinct conceptions can be discerned.

In the first, phase is considered as a parameter and the problem is reduced to the optimal estimation of the value of the phase shift undergone by the qudit under quantum operations [8]. Although very operational in style, it accommodates perfectly the practical requirements of typical applications handled in quantum information.

In the second, a semiclassical approach is adopted: the phase is assumed to be linked to the geometry of the state space. For example, for a qubit this space is the Poincaré sphere and the phase is identified with the angle that the state representative makes with the Z axis [9]. This pictorial understanding of phase as an angle makes intuitive contact with the classical world, but once more merely considers the phase as a state parameter instead of a full quantum variable.

The third major concept emphasizes the idea that phase is a physical property and, by any orthodox picture of quantum mechanics, must be associated with a self-adjoint operator (or at least with a family of positive operator-valued measures). In this vein, phase operators have been constructed via a polar decomposition for qubits and qutrits [10].

The main goal of this paper is to look at the fundamental problem of properly defining phase from quite a different perspective. On closer examination, one immediately discovers that the idea of complementarity is at the root of all the previous approaches: phase is complementary to some amplitude, by which we loosely mean that the precise knowledge of one implies that all possible outcomes of the other are equally probable [11]. This idea of *unbiasedness* leads directly to introducing mutually unbiased bases (MUBs) [12], which, for a variety of reasons, are becoming an important tool in quantum optics [13]. It is known that the maximum number of such bases cannot be greater than $d + 1$ and that this limit is certainly reached if d is a power of a prime [14]. It is not known whether there are nonprime-power values of d for which this bound is attained. We shall not be concerned with this problem in this paper, and assume for simplicity that we are always working in a prime dimension.

It is essential to recall that complementarity for the position–momentum pair is implemented by the Fourier transformation, which exchanges both operators. Using the ideas introduced in [15], we construct $d + 1$ disjoint classes of maximally commuting unitary matrices (each set having $d - 1$ operators). We then note that one of these classes consists solely of diagonal operators (which we can relate to *inversions*) that can be mapped, using the finite Fourier transform, to operators acting cyclically on basis states.

This perspective leads to a natural notion of *phases* as complementary to *inversions*. The advantage of this approach is that it does not rely on polar decompositions or semiclassical arguments and provides a clear understanding of the behaviour of such basic variables.

2. Multicomplementary operators for finite-dimensional quantum systems

The objects we study in this paper are quantum systems described in a d-dimensional Hilbert space $\mathcal{H}_d$. We recall [12] that two different orthonormal bases $\mathcal{A}$ and $\mathcal{B}$ are said to be mutually unbiased if a system prepared in an eigenstate of any element of $\mathcal{A}$ (such as $|a\rangle$) has a uniform probability distribution of being found in any element of $\mathcal{B}$,

$$|\langle a|b\rangle|^2 = \frac{1}{d}, \qquad (2.1)$$

for all $a \in \mathcal{A}$ and all $b \in \mathcal{B}$. As anticipated in the introduction, we shall be concerned solely in cases where d is a prime number, as then we know that there are $d + 1$ MUBs. For dimensions which are powers of a prime the argument can be easily extended with some modifications [16].

If $|n\rangle$ ($n = 0, \ldots, d - 1$) is the standard (computational) basis in $\mathcal{H}_d$, we introduce the generalized Pauli matrices X and Z via the following action:

$$\begin{aligned} X|n\rangle &= |n + 1\rangle, \\ Z|n\rangle &= \omega^n|n\rangle, \end{aligned} \qquad (2.2)$$

where

$$\omega = \exp(2\pi i/d). \qquad (2.3)$$

Note that throughout this paper addition and multiplication must be understood mod d. These operators X and Z, which

are generalizations of the Pauli matrices, were studied by Patera and Zassenhaus [17] in a purely mathematical context, and have been used recently by many authors in a variety of applications [18]. Under multiplication, they generate a finite subgroup of $SU(d)$, known as the generalized Pauli group, and obey the finite-dimensional version of the Weyl form of the commutation relations:

$$ZX = \omega XZ. \qquad (2.4)$$

It is easily shown that the eigenvectors of X and those of Z satisfy (2.1).

To simplify as much as possible the following computation, we introduce the following labelling scheme: let

$$\mathfrak{X}_0 = Z, \qquad \mathfrak{X}_k = XZ^{k-1}, \qquad k = 1, \ldots, d. \quad (2.5)$$

Since we shall also need powers of these operators, we denote by $\mathfrak{C}_{\mathfrak{X}_k}$ the set

$$\mathfrak{C}_{\mathfrak{X}_k} = \{\mathfrak{X}_k, \mathfrak{X}_k^2, \ldots, \mathfrak{X}_k^{d-1}\}. \qquad (2.6)$$

The $d - 1$ operators in the class $\mathfrak{C}_{\mathfrak{X}_k}$ clearly commute one with another and therefore represent a maximal set of commuting operators.

Following the ideas of [15], consider now the following set (each containing $d + 1$ operators):

$$\begin{aligned} \mathcal{S} &= \{\mathfrak{X}_0, \mathfrak{X}_1, \ldots, \mathfrak{X}_d\} \\ &= \{Z, X, XZ, \ldots, XZ^{d-1}\}. \end{aligned} \qquad (2.7)$$

By virtue of the relation (2.4), any two operators in this set are complementary, in the sense that their eigenvectors satisfy the unbiasedness condition (2.1). Furthermore, a complete MUB is obtained by constructing every eigenvector of every element in $\mathcal{S}$, so we refer to $\mathcal{S}$ as a maximal set of multicomplementary operators.

In the case of the standard position–momentum complementary variables, their eigenvectors form bases related by the Fourier transform. The finite-dimensional Fourier transform is defined as [3]

$$F = \frac{1}{\sqrt{d}} \sum_{n,n'=0}^{d-1} \omega^{nn'} |n\rangle\langle n'|, \qquad (2.8)$$

with the properties

$$FF^\dagger = F^\dagger F = 1, \qquad F^4 = 1. \qquad (2.9)$$

Using this definition one can check that X and Z are indeed Fourier pairs:

$$X = F^\dagger ZF. \qquad (2.10)$$

There exists also an operator V that effects the transformation $X \mapsto XZ$. It has a diagonal form:

$$V = \sum_{n=0}^{d-1} \omega^{-(n^2-n)(d+1)/2} |n\rangle\langle n|, \qquad (2.11)$$

so

$$XZ^k = V^{\dagger k} X V^k, \qquad (2.12)$$

where we have assumed an odd-prime dimension (the case $d = 2$ needs minor modifications, as we shall see in the next section).

In physical applications, only $d - 1$ populations can vary independently in a d-level system. In consequence, it is usual to work with $d - 1$ traceless operators h_j that measure population inversions between the corresponding levels, i.e.,

$$h_j = S_{jj} - S_{j+1j+1}, \qquad (2.13)$$

where

$$S_{ij} = |i\rangle\langle j|. \qquad (2.14)$$

These h_j (usually known as the Cartan–Weyl generators) constitute a maximal Abelian subalgebra. Note that the diagonal operators in the class $\mathfrak{C}_{\mathfrak{X}_0} = \{Z^k\}$ are linear combinations of h_j, so they can be used interchangeably.

On physical grounds, we expect phases to be complementary to inversions. But inversions are invariant under phase shifts: if

$$U(\varphi) = \exp\left(-i \sum_j \varphi_j h_j\right), \qquad (2.15)$$

where φ denotes $(\varphi_1, \ldots, \varphi_{d-1})$, then

$$U^\dagger(\varphi) h_k U(\varphi) = h_k. \qquad (2.16)$$

Thus, we can construct a continuous family of operators, all complementary to inversions, by conjugating any operator in the class $\mathfrak{C}_{\mathfrak{X}_k}$ by means of $U(\varphi)$. In particular, the diagonal operator V, which maps X to XZ^k as per equation (2.12) (and that is also of the form (2.15) for a definite choice of the parameters), allows us to construct any $\mathfrak{X}_k$ starting with $\mathfrak{X}_1 = X$. For this reason, and without any loss of generality, we shall henceforth focus on the elements of the class $\mathfrak{C}_{\mathfrak{X}_1} = \{X, X^2, \ldots, X^{d-1}\}$ to discuss general properties of complementary phase operators. We thus define $d - 1$ families of operators representing the exponential of the phase via

$$E^k(\varphi) = U^\dagger(\varphi) X^k U(\varphi), \qquad k = 1, \ldots, d - 1, \quad (2.17)$$

obtained by taking successive powers of $E(\varphi)$.

If $|s\rangle$ is an eigenstate of h_j with eigenvalue h_{js}, then the expectation value of $E^k(\varphi)$ on an arbitrary state $|\psi\rangle = \sum_s c_s |s\rangle$ is simply

$$\langle E^k(\varphi)\rangle = \sum_s c_{s+k}^* c_s U_{s+k}^*(\varphi) U_s(\varphi), \qquad (2.18)$$

where

$$U_s(\varphi) = \exp\left(-i \sum_j \varphi_j h_{js}\right). \qquad (2.19)$$

This allows us to introduce an operator kernel

$$\Pi(\varphi) = \frac{1}{(2\pi)^{d-1}}\left[\mathbb{I} + \sum_{k=1}^{d-1} \langle E^k(\varphi)\rangle^* X^k\right], \qquad (2.20)$$

which is properly normalized and generates all the moments through the relation

$$\langle E^\ell(\varphi)\rangle = \frac{(2\pi)^{d-1}}{d} \text{Tr}[\Pi(\varphi) X^\ell], \qquad (2.21)$$

so the phase distribution is obtained as

$$P(\varphi) = \frac{1}{(2\pi)^{d-1}}\left(1 + \sum_{k=1}^{d-1} \langle E^k(\varphi)\rangle^* \langle X^k\rangle\right). \qquad (2.22)$$

We note in passing that any positive operator-valued measure of the general form

$$\Delta(\varphi) = \frac{1}{(2\pi)^{d-1}}\left[\mathbb{I} + \sum_{k=1}^{d-1} \gamma_k E^k(\varphi)\right], \qquad (2.23)$$

which can be associated with (2.22), satisfies the conditions of real valuedness, positivity and normalization and possesses the obvious property

$$e^{i\varphi_j' h_j} \Delta(\varphi_1, \ldots, \varphi_{d-1}) e^{-i\varphi_j' h_j}$$
$$= \Delta(\varphi_1, \ldots, \varphi_j + \varphi_j', \ldots, \varphi_{d-1}), \qquad (2.24)$$

which meets the usual requirements of complementarity [19].

3. Application: quantum phase for finite-dimensional systems

3.1. The case of qubits

To fully appreciate the details of the method, we shall work out some relevant examples. First, we focus on the simple case of a two-dimensional Hilbert space $\mathcal{H}_2$, and a state space that coincides with the sphere $\mathbb{S}_2$.

The basic operators are the standard Pauli matrices

$$X = 2\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = 2\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.1)$$

such that

$$\sigma_x \sigma_z = -\sigma_z \sigma_x. \qquad (3.2)$$

The transformation $\sigma_z \mapsto \sigma_x$ is accomplished by using the finite Fourier transform

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \qquad (3.3)$$

However, it is impossible to find a unitary transformation V such that $\sigma_x \mapsto \sigma_x \sigma_z$. For this reason, instead of $\sigma_x \sigma_z$ the matrix $\sigma_y = i\sigma_x \sigma_z$ is used, in such a way that $\sigma_y = V^\dagger \sigma_x V$, where V is the unitary operator

$$V = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \qquad (3.4)$$

Let ϱ_A describe an arbitrary point on the Bloch sphere $\mathbb{S}_2$ in a direction parametrized by $\mathbf{n} = (\cos\varphi_A \sin\vartheta_A, \sin\varphi_A \sin\vartheta_A, \cos\vartheta_A)$. Define A to be

$$A = \mathbf{n}_A \cdot \boldsymbol{\sigma} = R(\vartheta_A, \varphi_A) \sigma_z R^{-1}(\vartheta_A, \varphi_A), \qquad (3.5)$$

with

$$R(\vartheta, \varphi) = \exp\left[\frac{\vartheta}{2}(\cos\varphi\, \sigma_x - \sin\varphi\, \sigma_y)\right]. \qquad (3.6)$$

The condition of complementarity between A and a generic operator B (expressed also as in (3.5)) can be written as [20]

$$\mathbf{n}_A \cdot \mathbf{n}_B = 0; \qquad (3.7)$$

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that is, the subspace spanned in $\mathbb{S}_2$ by $\mathbf{n}_A$ is orthogonal to that spanned by $\mathbf{n}_B$. We have then a one-parameter set of complementary operators of the general form

$$B = \mathbf{n}_B \cdot \boldsymbol{\sigma}, \qquad (3.8)$$

where the unit vector $\mathbf{n}_B$ satisfies (3.7), which can be recast as

$$\cot \vartheta_B = -\tan \vartheta_A \cos(\varphi_B - \varphi_A). \qquad (3.9)$$

In particular, the set complementary to the inversion σ_z consists of the one-parameter family

$$E(\varphi) = \cos \varphi \, \sigma_x - \sin \varphi \, \sigma_y = \begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}, \qquad (3.10)$$

where φ represents a reference or fiducial phase. This in fact agrees with the exponential of the phase operator obtained via a polar decomposition [10].

According to the approach developed in this paper, we have now one family of phase operators that can be constructed as

$$E(\varphi) = \exp(i\varphi\sigma_z/2)\sigma_x \exp(-i\varphi\sigma_z/2). \qquad (3.11)$$

This coincides also with (3.10), and can be expressed in the suggestive form

$$E(\varphi) = \underline{e}(\varphi)^{\text{t}} F \underline{\sigma}, \qquad (3.12)$$

where F is given in (3.3), t denotes the transpose and

$$\underline{e}(\varphi) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi} \\ e^{i\varphi} \end{pmatrix}, \qquad \underline{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_x \sigma_z \end{pmatrix}. \qquad (3.13)$$

This result confirms the complementary character of $E(\varphi)$ obtained via Fourier transformation. For a pure state like

$$|\psi\rangle = \begin{pmatrix} \cos(\vartheta/2) \\ \sin(\vartheta/2)e^{i\chi} \end{pmatrix}, \qquad (3.14)$$

with $0 \leqslant \vartheta \leqslant \pi, 0 \leqslant \chi \leqslant 2\pi$, the average value of $E(\varphi)$ is

$$\langle E(\varphi) \rangle = \sin \vartheta \cos(\chi + \varphi). \qquad (3.15)$$

The main features of this description are obviously independent of the reference phase φ, which is usually fixed to the value 0.

3.2. The case of qutrits

For a three-dimensional Hilbert space $\mathcal{H}_3$ the basic operators are

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad (3.16)$$

with $\omega = \exp(2\pi i/3)$. We have four classes of disjoint traceless operators, each containing two commuting operators:

$$\begin{array}{ll} \mathfrak{C}_{\mathfrak{X}_0} = \{Z, Z^2\}, & \mathfrak{C}_{\mathfrak{X}_1} = \{X, X^2\}, \\ \mathfrak{C}_{\mathfrak{X}_2} = \{XZ, (XZ)^2\}, & \mathfrak{C}_{\mathfrak{X}_3} = \{XZ^2, (XZ^2)^2\}. \end{array} \qquad (3.17)$$

The discrete Fourier transform is

$$F = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \qquad (3.18)$$

and V is the diagonal unitary matrix

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \qquad (3.19)$$

We have now two Cartan operators, each associated with the independent inversions

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad (3.20)$$

which can be easily expressed as linear combinations of Z and Z^2. Note that X and X^2 correspond to two different physical situations: in the computational basis X acts as $X|n\rangle = |n+1\rangle$, while $X^2|n\rangle = |n+2\rangle$.

Thus, we have two families of commuting phase operators:

$$\begin{aligned} E(\varphi_1, \varphi_2) &= e^{i(\varphi_1 h_1 + \varphi_2 h_2)} X e^{-i(\varphi_1 h_1 + \varphi_2 h_2)} \\ E^2(\varphi_1, \varphi_2) &= e^{i(\varphi_1 h_1 + \varphi_2 h_2)} X^2 e^{-i(\varphi_1 h_1 + \varphi_2 h_2)}. \end{aligned} \qquad (3.21)$$

In this particular case, E and E^2 essentially coincide because $E^2(\varphi_1, \varphi_2) = E^\dagger(\varphi_1, \varphi_2)$. The phase operator $E(\varphi_1, \varphi_2)$ can be represented in a form similar to (3.12), namely as

$$E(\varphi_1, \varphi_2) = \underline{e}(\varphi_1, \varphi_2)^{\text{t}} F \underline{X}, \qquad (3.22)$$

with

$$\underline{e}(\varphi_1, \varphi_2) = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i(\varphi_2 - 2\varphi_1)} \\ e^{i(\varphi_1 + \varphi_2)} \\ e^{i(\varphi_1 - 2\varphi_2)} \end{pmatrix}, \qquad \underline{X} = \begin{pmatrix} X \\ XZ \\ XZ^2 \end{pmatrix}. \qquad (3.23)$$

Its explicit form is

$$E(\varphi_1, \varphi_2) = \begin{pmatrix} 0 & 0 & e^{i(\varphi_1 + \varphi_2)} \\ e^{-i(2\varphi_1 - \varphi_2)} & 0 & 0 \\ 0 & e^{i(\varphi_1 - 2\varphi_2)} & 0 \end{pmatrix}, \qquad (3.24)$$

where again φ_1 and φ_2 are reference phases. The average value of $E(\varphi_1, \varphi_2)$ in an arbitrary pure state such as

$$|\psi\rangle = \begin{pmatrix} \cos(\vartheta/2) \\ \sin(\vartheta/2) \cos(\xi/2)e^{i\chi_1} \\ \sin(\vartheta/2) \sin(\xi/2)e^{i\chi_2} \end{pmatrix}, \qquad (3.25)$$

is simply

$$\begin{aligned} \langle E(\varphi_1, \varphi_2) \rangle &= \tfrac{1}{2} \sin^2(\vartheta/2) \sin \xi \, e^{i[(\chi_1 - \chi_2) + (\varphi_1 - 2\varphi_2)]} \\ &+ \tfrac{1}{2} \sin \vartheta \{ \sin(\xi/2)e^{i[\chi_2 + (\varphi_1 + \varphi_2)]} \\ &+ \cos(\xi/2)e^{-i[\chi_1 + (2\varphi_1 - \varphi_2)]} \}. \end{aligned} \qquad (3.26)$$

We observe that when only two levels are involved, $\xi = 0$,

$$\langle E(\varphi_1, \varphi_2) \rangle = \tfrac{1}{2} \sin \vartheta \, e^{-i[\chi_1 + (2\varphi_1 - \varphi_2)]}, \qquad (3.27)$$

which measures the relative phase of the basis states $|1\rangle$ and $|2\rangle$, and depends only on a single effective phase $2\varphi_1 - \varphi_2$. Since $E^2(\varphi_1, \varphi_2) = E^\dagger(\varphi_1, \varphi_2)$, which includes all the possible moments, the phase distribution (2.22) remains constant in the direction $2\varphi_1 - \varphi_2 = \text{constant}$.

It is instructive to compare our construction of phase operators with the more common algorithm, based on the polar decomposition of S_{12}, S_{23}, and S_{13}, defined in equation (2.14).

The polar decomposition of S_{ji} implicitly focuses of the $i \to j$ transition, without analysing the role of the third 'spectator' state of the system. This observation can be used to explain the lack of uniqueness in the polar decomposition of singular operators like S_{12}, for instance. This approach produces a phase operator E_{12} of the form

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ x & 0 & y \\ y^* & 0 & -x^* \end{pmatrix}, \qquad |x|^2 + |y|^2 = 1, \quad (3.28)$$

where $S_{12} = E_{12}R_{12}$ and $R_{12} = \sqrt{S_{12}S_{21}}$ is the 'modulus'. The only constraints on E_{12} are imposed by the requirement of unitarity.

A particular solution to (3.28) is obtained by 'isolating' the third state from the first two by choosing $y = 0$ and thus $x = e^{i\varphi_{12}}$. Since (S_{12}, S_{21}, h_1) span an $su(2)$ subalgebra, this choice amounts to limiting the action of the phase operator E_{12} to a specified $su(2)$ subspace. A similar argument holds for E_{23}, with (S_{23}, S_{32}, h_2) spanning another $su(2)$ subalgebra.

This perspective in terms of polar decomposition and transitions leads to phase operators that fulfil the requirements of complementarity only for that between pairs of states involved in each transition. The corresponding positive operator-valued measure obtained in this construction is still of the general form found in (2.23). However, this restricted point of view is to be contrasted with the approach of this paper, where complementarity is imposed for the three-level system as a whole. It is obvious from the explicit forms (3.24) and (3.28) that they give quite different results.

4. Concluding remarks

Mutually unbiased bases are a primitive of quantum theory, as they embody the importance of the superposition principle. In this paper we have used them to develop a comprehensive quantum theory of the phases as complementary to inversions in finite-dimensional systems.

The construction presented in this paper is devoid of any ambiguity associated with the non-uniqueness of polar decomposition of ladder operators. In prime dimensions, phase operators and inversions are elegantly related by a finite Fourier transform, much like positions and momenta are related by an ordinary Fourier transform in infinite-dimensional systems, and this provides an appealing way of treating a concept as central as the phases of a system.

Acknowledgments

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