Representations of the Weyl group and Wigner functions for SU(3)

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Bases for SU(3) irreps are constructed on a space of three-particle tensor products of two-dimensional harmonic oscillator wave functions. The Weyl group is represented as the symmetric group of permutations of the particle coordinates of these spaces. Wigner functions for SU(3) are expressed as products of SU(2) Wigner functions and matrix elements of Weyl transformations. The constructions make explicit use of dual reductive pairs which are shown to be particularly relevant to problems in optics and quantum interferometry. © *1999 American Institute of Physics.* [S0022-2488(99)01506-6]

I. INTRODUCTION

Considerable progress has been made in the development of systematic algorithms for computing matrix elements of the infinitesimal generators of Lie groups in an arbitrary representation. Much less is known about the matrices of finite group elements other than those of SU(2), and the related groups E(2), HW(1), and SU(1,1).[1]

The matrix elements of finite SU(2) transformations are the well-known Wigner $\mathcal{D}$ functions. These functions are used in many areas of physics, notably in nuclear, atomic, and molecular spectroscopy. Recently, it has been shown that the Wigner functions of SU(2) (Ref. 2) and higher unitary groups[3] are needed in the analysis of quantum interferometers. Because of the Peter–Weyl theorem, Wigner functions also play a central role in the theory of harmonic analysis.

We consider here the Wigner functions for SU(3); such functions are needed, for example, in computing SU(3) Clebsch–Gordan coefficients in an SO(3) basis.[4] Expressions for SU(3) Wigner functions were first derived, to our knowledge, by Chacón and Moshinsky,[5] in terms of SU(2) Wigner functions and matrix elements of Weyl reflection operators. Matrix elements of some Weyl reflections were derived by Macfarlane *et al.*[6] and Mukunda and Pandit.[7] The latter gave the matrix elements as products of three SU(2) Clebsch–Gordan coefficients. Chacón and Moshinsky gave expressions for matrix elements of other Weyl reflections as SU(2) Racah coefficients. These results raise the question: what does the Weyl group have to do with SU(2)? The answer appears to be that basis states for SU(3) irreps (irreducible representations) are naturally expressed in an SU(2)-coupled basis, and elements of the Weyl group for SU(3), which is isomorphic to the permutation group S_3, act on such states as SU(2) recoupling operators. More explicitly, if one constructs basis states for SU(3) by SU(2) coupling the wave functions for three particles in two-dimensional harmonic oscillator states, then the Weyl reflection operators permute the coordinates of the particles. A similar interpretation of the Weyl reflections was given by Gal and Lipkin[8] as the permutations of a coupled system of three spin-$\frac{1}{2}$ quarks.

In deriving our results, we make use of two mutually commuting subgroups, U(3) and U(2), of U(6). When acting within the space of a fully symmetric representation of U(6), these subgroups are said to form a *dual reductive pair*.[9] Such dual pairs are particularly relevant for

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describing the properties of three particles in a two-dimensional harmonic oscillator or three spin-half quarks. An overview of these and other dual pairs and their uses in optics and quantum interferometry is given in the Discussion section at the end of this paper.

II. PARAMETRIZATION OF SU(3)

Many parametrizations of SU(3) elements are possible. The most useful ones would appear to arise from factorization of SU(3) group elements into products of subgroup elements whose Wigner functions are known. Three obvious candidates for suitable subgroups are the groups $SU(2)_{12}$, $SU(2)_{13}$, and $SU(2)_{23}$, the three SU(2) subgroups whose root systems are subsystems of the SU(3) root system shown in Fig. 1. We denote an element of $SU(2)_{ij}$ by $R_{ij}(\alpha,\beta,\gamma)$, where (α,β,γ) are the standard Euler angles.

Murnaghan[10] has shown that a possible parametrization of an element $g\in SU(3)$ is given by

$$g(\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\delta_1,\delta_2)=e^{-i(h_1\delta_1+h_2\delta_2)}R_{23}(\alpha_1/2,\beta_1,-\alpha_1/2)$$
$$\times R_{13}(\alpha_2/2,\beta_2,-\alpha_2/2)R_{12}(\alpha_3/2,\beta_3,-\alpha_3/2), \tag{1}$$

where h_1 and h_2 are elements of the Cartan subalgebra.

A similar parametrization, with a different ordering, was proposed by Reck *et al.*[3] These authors showed that one can factor a general $N\times N$ unitary matrix as a product of U(2) matrices and an overall phase, with the added insight that each U(2) transformation can be realized experimentally as an optical element.

In this paper, we choose a parametrization that takes advantage of the fact that, in a canonical basis, one constructs $U(N)$ irreps in a basis that reduces a particular $U(N-1)$ subgroup. Thus, an arbitrary $SU(N)$ matrix is factored

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & X_{N-1} & \\ 0 & & & \end{array}\right)\left(\begin{array}{cc|c} e^{i\alpha}\cos(\beta/2) & -\sin(\beta/2) & \\ \sin(\beta/2) & e^{-i\alpha}\cos(\beta/2) & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{N-2} \end{array}\right)\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & Y_{N-1} & \\ 0 & & & \end{array}\right) \tag{2}$$

where X_{N-1} and Y_{N-1} are $SU(N-1)$ matrices; I_{N-2} is the $(N-2)\times(N-2)$ identity matrix. For SU(2) [with the indices ordered (z,x,y)] this gives the usual factorization $R(\alpha,\beta,\gamma)=R_z(\alpha)R_y(\beta)R_z(\gamma)$. For $g\in SU(3)$, we obtain

$$g(\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\gamma_3)=R_{23}(\alpha_1,\beta_1,\gamma_1)R_{12}(\alpha_2,\beta_2,\alpha_2)R_{23}(\alpha_3,\beta_3,\gamma_3). \tag{3}$$

The parameters in this expression are derived for an arbitrary $g\in SU(3)$ in the Appendix, by a method communicated to us by J. Repka.

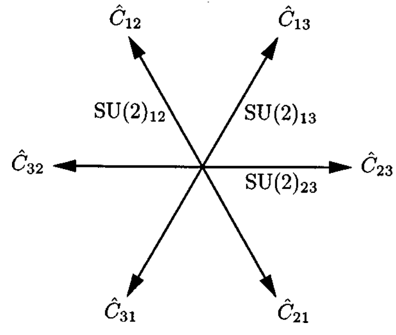

FIG. 1. Three SU(2) subsystems of the SU(3) root system.

All of the above factorizations enable one to express the SU(3) Wigner functions in terms of matrix elements of finite SU(2) transformations.

III. BASIS STATES

A. Highest weight states

An SU(3) irrep is characterized by a highest weight (λ,μ) and a corresponding highest weight state $|\phi(\lambda,\mu)\rangle$, defined as follows. The su(3) Lie algebra is spanned in the usual way by the subset of u(3) operators

$$\begin{gathered}\hat{C}_{ij}, \quad i<j \ \text{ raising operators,}\\ \hat{C}_{ij}, \quad i>j \ \text{ lowering operators,}\\ \hat{h}_1=\hat{C}_{11}-\hat{C}_{22}, \quad \hat{h}_2=\hat{C}_{22}-\hat{C}_{33}, \quad \text{Cartan operators,}\end{gathered} \tag{4}$$

where the $\{\hat{C}_{ij}\}$ operators satisfy the commutation relations

$$[\hat{C}_{ij},\hat{C}_{kl}]=\delta_{jk}\hat{C}_{il}-\delta_{il}\hat{C}_{kj}. \tag{5}$$

The highest weight state $|\phi(\lambda,\mu)\rangle$ then satisfies the equations

$$\begin{gathered}\hat{C}_{ij}|\phi(\lambda,\mu)\rangle=0, \quad i<j,\\ \hat{h}_1|\phi(\lambda,\mu)\rangle=\lambda|\phi(\lambda,\mu)\rangle, \quad \hat{h}_2|\phi(\lambda,\mu)\rangle=\mu|\phi(\lambda,\mu)\rangle.\end{gathered} \tag{6}$$

Without loss of generality, we suppose that $|\phi(\lambda,\mu)\rangle$ is also an eigenstate of the operator $\hat{C}_{33}$ with zero eigenvalue. It then satisfies the equations

$$\hat{C}_{11}|\phi(\lambda,\mu)\rangle=(\lambda+\mu)|\phi(\lambda,\mu)\rangle, \quad \hat{C}_{22}|\phi(\lambda,\mu)\rangle=\mu|\phi(\lambda,\mu)\rangle, \quad \hat{C}_{33}|\phi(\lambda,\mu)\rangle=0. \tag{7}$$

The Hilbert space, $\mathbb{H}^{(\lambda,\mu)}$, for the SU(3) irrep with highest weight (λ,μ) thereby becomes a Hilbert space for a U(3) irrep of highest weight $(\lambda+\mu,\mu,0)$.

B. The Gel'fand–Tsetlin basis

To use the factorization of Eq. (3) in computing Wigner functions, we need a basis for the Hilbert space $\mathbb{H}^{(\lambda,\mu)}$ that reduces the $\mathrm{SU}(3)\supset\mathrm{SU}(2)_{23}$ subgroup chain. Such a basis is the so-called canonical or Gel'fand–Tsetlin basis;[11]

$$\left\{\left|\begin{matrix}p & & q\\ & r & \end{matrix}\right\rangle \equiv \left|\begin{matrix}\lambda+\mu & & \mu & & 0\\ & p & & q & \\ & & r & & \end{matrix}\right\rangle; \begin{matrix}\lambda+\mu\geqslant p\geqslant\mu\geqslant q\geqslant 0\\ p\geqslant r\geqslant q\end{matrix}\right\}, \tag{8}$$

which reduces the chain

$$\begin{matrix}\mathrm{U}(3)\\ (\lambda+\mu,\mu,0)\end{matrix} \supset \begin{matrix}\mathrm{U}(2)_{23}\\ (p,q)\end{matrix} \supset \begin{matrix}\mathrm{U}(1)_3\\ r\end{matrix}, \tag{9}$$

where $\mathrm{U}(1)_3\subset\mathrm{U}(2)_{23}$ is the subgroup whose Lie algebra is spanned by $\hat{C}_{33}$.

The Gel'fand states are eigenstates of the weight operators; i.e.,

$$\hat{C}_{ii}\left|\begin{matrix}p & & q\\ & r & \end{matrix}\right\rangle=\nu_i\left|\begin{matrix}p & & q\\ & r & \end{matrix}\right\rangle, \quad i=1,2,3, \tag{10}$$

with

$$\nu_1 = \lambda + 2\mu - p - q, \quad \nu_2 = p + q - r, \quad \nu_3 = r. \tag{11}$$

One sees that the components of a weight $\nu = (\nu_1, \nu_2, \nu_3)$ add up to $\lambda + 2\mu$. They are linearly dependent and insufficient to define a state uniquely. However, the Gel'fand-Tsetlin states also reduce the subgroup chain

$$\begin{array}{c} \mathrm{U}(3) \\ (\lambda+\mu,\mu,0) \end{array} \supset \begin{array}{c} \mathrm{SU}(2)_{23} \\ I \end{array} \supset \begin{array}{c} \mathrm{U}(1)_{23} \\ M \end{array}, \tag{12}$$

and have $\mathrm{SU}(2)_{23}$ quantum numbers, I and M, related to p, q, and r by

$$I = \tfrac{1}{2}(p-q), \quad M = \tfrac{1}{2}(\nu_2 - \nu_3) = \tfrac{1}{2}(p+q) - r. \tag{13}$$

Thus, the weight ν and the $\mathrm{SU}(2)_{23}$ angular momentum I together uniquely define a basis state and, with the above relationships between ν, I and p, q, r, we can relabel a Gel'fand–Tsetlin state

$$|\nu I\rangle \equiv \left| \begin{array}{cc} p & q \\ \multicolumn{2}{c}{r} \end{array} \right\rangle. \tag{14}$$

We shall refer to the basis $\{|\nu I\rangle\}$ either as a Gel'fand–Tsetlin basis or as a weight basis.

C. An SU(2)-coupled realization

The Gel'fand–Tsetlin states can be constructed explicitly as three-particle SU(2)-coupled products of two-dimensional harmonic-oscillator states.

The construction makes use of a well-known duality relationship (discussed by Moshinsky and Chacón[5]) between U(3) and $\mathcal{U}(2)$ as commuting subgroups of U(6). Let $\{a^\dagger_{im}, a_{im};\ i=1,...,3,\ m=1,2\}$ denote (two-dimensional) harmonic oscillator raising and lowering operators for three particles. The operators $\{a^\dagger_{im}a_{jn}\}$ then span a u(6) Lie algebra. This algebra has two mutually commuting subalgebras: u(3) spanned by the operators

$$\hat{C}_{ij} = \sum_{m=1}^{2} a^\dagger_{im} a_{jm}, \tag{15}$$

and $u(2)$ spanned by

$$\hat{B}_{mn} = \sum_{i=1}^{3} a^\dagger_{im} a_{in}. \tag{16}$$

The algebras u(3) and $u(2)$ are examples of a so-called *dual pair*.[9] The use of a dual pair $(\mathrm{u}(N), u(n))$ and the corresponding direct sum subalgebra $\mathrm{u}(N) + u(n) \subset \mathrm{u}(Nn)$ are well known, for example, in the classification of states of N particles in an n-dimensional harmonic oscillator; cf., for example, the paper by Hagen and MacFarlane[12] which presents a method for deriving the $\mathrm{SU}(m) \times \mathcal{SU}(n)$ content of $\mathrm{SU}(mn)$ and provides tables for the $\mathrm{SU}(6) \to \mathrm{SU}(3) \times \mathcal{SU}(2)$ branching rules.

Now observe that, if $|0\rangle$ is the state in which all particles are in their respective harmonic oscillator ground states, the state

$$|\phi(\lambda,\mu)\rangle = (a^\dagger_{11})^\lambda (a^\dagger_{11}a^\dagger_{22} - a^\dagger_{12}a^\dagger_{21})^\mu |0\rangle \tag{17}$$

satisfies all the conditions of Eq. (6). Thus, $|\phi(\lambda,\mu)\rangle$ is an (unnormalized) SU(3) highest weight state. But it also satisfies

$$\hat{B}_{12}|\phi(\lambda,\mu)\rangle=0,$$
$$\hat{B}_{11}|\phi(\lambda,\mu)\rangle=(\lambda+\mu)|\phi(\lambda,\mu)\rangle, \quad \hat{B}_{22}|\phi(\lambda,\mu)\rangle=\mu|\phi(\lambda,\mu)\rangle, \tag{18}$$

which means that $|\phi(\lambda,\mu)\rangle$ is simultaneously a highest weight state for $u(2)$ with highest weight $(\lambda+\mu,\mu)$ and a highest weight state for u(3) with highest weight $(\lambda+\mu,\mu,0)$, cf. Eq. (7). Moreover, since the u(3) and $u(2)$ operators commute with one another, we can identify all the desired SU(3) basis states with those of the subset of $\mathrm{U}(3)\times\mathcal{U}(2)$ states that are of $\mathcal{U}(2)$ highest weight. This result is a special case of a general result for dual pairs;[9] for any N and n, the commuting algebras u(N) and $u(n)$ have a complete set of highest weight states in common within the carrier space of a fully symmetric irrep of the Lie algebra u(Nn) [i.e., an irrep of highest weight (σ,0,...), where σ, equal to $\lambda+2\mu$ in the present case, is the total number of harmonic oscillator quanta].

It is well known that basis states for an $su(2)$ irrep of spin s_i are given by

$$|s_i,m_i\rangle=\frac{(a_{i1}^\dagger)^{s_i+m_i}(a_{i2}^\dagger)^{s_i-m_i}}{\sqrt{(s_i+m_i)!(s_i-m_i)!}}|0\rangle. \tag{19}$$

These states are also a basis for a $u(2)$ irrep of highest weight $(2s_i,0)$. They are tensor products of pairs of $u(1)$ irreps of $u(1)$ spin (s_i+m_i) and $-(s_i-m_i)$, respectively. A Gel'fand basis for SU(3) can likewise be constructed from triple tensor products of $su(2)$ irreps.

Theorem: The weight basis, defined by Eqs. (8)–(14), can be expressed, to within arbitrary phase factors:

$$|\nu I\rangle=[|\tfrac{1}{2}\nu_1\rangle\otimes[|\tfrac{1}{2}\nu_2\rangle\otimes|\tfrac{1}{2}\nu_3\rangle]^I]^{\lambda/2}_{\lambda/2},$$

$$=\sum_{m_1m_2m_3(N)}(\tfrac{1}{2}\nu_3,m_3;\tfrac{1}{2}\nu_2,m_2|I,N)(I,N;\tfrac{1}{2}\nu_1,m_1|\tfrac{1}{2}\lambda,\tfrac{1}{2}\lambda)|\tfrac{1}{2}\nu_1,m_1\rangle|\tfrac{1}{2}\nu_2,m_2\rangle|\tfrac{1}{2}\nu_3,m_3\rangle, \tag{20}$$

with $\nu=(\nu_1,\nu_2,\nu_3)$.

Proof: It follows, from Eq. (15), that

$$\hat{C}_{ii}|\nu I\rangle=\nu_i|\nu I\rangle. \tag{21}$$

Thus, the states $|\nu I\rangle$ have the same weights as their Gel'fand–Tsetlin counterparts. It remains to show that a state $|\nu I\rangle$, defined by Eq. (20), has $\mathrm{SU}(2)_{23}$ angular momentum I.

Consider a set of states for particles 2 and 3 which span an irrep of $\mathrm{u}(2)\times u(2)\subset\mathrm{u}(3)\times u(2)$, where the u(2)⊂u(3) subalgebra is spanned by the operators $\{\hat{C}_{23},\hat{C}_{32},\hat{C}_{22},\hat{C}_{33}\}$. If the two-particle states transform according to a u(2) irrep (p,q) then, by duality, they also belong to $u(2)$ irreps of the same highest weight, (p,q). Thus, if a state has su(2) angular momentum $I=(p-q)/2$, it also has $su(2)$ angular momentum I. It follows that the $su(2)$-coupled two-particle state,

$$[|\tfrac{1}{2}\nu_2\rangle\otimes\tfrac{1}{2}\nu_3]^I_N, \tag{22}$$

belongs to a $u(2)$ irrep (p,q) with

$$p+q=\nu_2+\nu_3, \quad p-q=2I, \tag{23}$$

and therefore to the u(2) irrep with the same labels (p,q) and to the irrep with angular momentum $I=\frac{1}{2}(p-q)$ of the subalgebra su(2)⊂u(2). This completes the proof.

IV. MATRIX ELEMENTS OF WEYL OPERATORS

The Weyl group is generated by reflections of the roots in the hyperplanes perpendicular to each of the roots. Let α_{ij} denote the SU(3) root whose root vector is $\hat{C}_{ij}$ and let P_{ij} denote the reflection in the line perpendicular to α_{ij}. Then, for example,

$$P_{12}:\alpha_{12}\rightarrow\alpha_{21},\quad \alpha_{13}\rightarrow\alpha_{23},\quad \alpha_{32}\rightarrow\alpha_{31}, \tag{24}$$

and $P_{12}^2=1$. Thus, one obtains the known result that the Weyl group for SU(3) is isomorphic to the symmetric group S_3 of permutations of three objects and that the subset of reflections correspond to transpositions.

By writing Eq. (20) in the form

$$\Psi_{\nu I}(123)\equiv\langle 123|\nu I\rangle=[\psi_{\nu_1}(1)\otimes[\psi_{\nu_2}(2)\otimes\psi_{\nu_3}(3)]^I]^{\lambda/2}_{\lambda/2}, \tag{25}$$

we obtain representations of the Weyl group for SU(3) in which, for example,

$$\begin{gathered}
[P_{12}\Psi_{\nu I}](123)=\langle 123|P_{12}|\nu I\rangle=\Psi_{\nu I}(213),\\
[P_{13}\Psi_{\nu I}](123)=\Psi_{\nu I}(321),\\
[P_{132}\Psi_{\nu I}](123)=[P_{12}P_{13}\Psi_{\nu I}](123)=\Psi_{\nu I}(312).
\end{gathered} \tag{26}$$

It follows that

$$\begin{aligned}
[P_{12}\Psi_{\nu I}](123)&=[\psi_{\nu_1}(2)\otimes[\psi_{\nu_2}(1)\otimes\psi_{\nu_3}(3)]^I]^{\lambda/2}_{\lambda/2}\\
&=\sum_{I'}(-1)^{(\nu_3-2I-2I'+2\mu-\lambda)/2}\sqrt{(2I+1)(2I'+1)}\\
&\quad\times\begin{Bmatrix}\nu_1/2 & \nu_3/2 & I'\\ \nu_2/2 & \lambda/2 & I\end{Bmatrix}[\psi_{\nu_2}(1)\otimes[\psi_{\nu_1}(2)\otimes\psi_{\nu_3}(3)]^{I'}]^{\lambda/2}_{\lambda/2},
\end{aligned} \tag{27}$$

where $\{{}^{abc}_{def}\}$ is a Wigner 6-j symbol. Thus, we obtain the matrix elements

$$\langle\nu' I'|P_{12}|\nu I\rangle=\delta_{\nu_1',\nu_2}\delta_{\nu_2',\nu_1}\delta_{\nu_3',\nu_3}(-1)^{(\nu_3-2I-2I'+2\mu-\lambda)/2}\sqrt{(2I+1)(2I'+1)}\begin{Bmatrix}\nu_1/2 & \nu_3/2 & I'\\ \nu_2/2 & \lambda/2 & I\end{Bmatrix}. \tag{28}$$

In a similar way one determines that

$$\langle\nu' I'|P_{123}|\nu I\rangle=\delta_{\nu_1',\nu_3}\delta_{\nu_2',\nu_1}\delta_{\nu_3',\nu_2}(-1)^{(\nu_1+\nu_2-2I'+2\lambda)/2}\sqrt{(2I+1)(2I'+1)}\begin{Bmatrix}\nu_1/2 & \nu_2/2 & I'\\ \nu_3/2 & \lambda/2 & I\end{Bmatrix} \tag{29}$$

and

$$\langle\nu' I'|P_{132}|\nu I\rangle=\delta_{\nu_1',\nu_2}\delta_{\nu_2',\nu_3}\delta_{\nu_3',\nu_1}(-1)^{(\nu_1+2I+2\mu+\lambda)/2}\sqrt{(2I+1)(2I'+1)}\begin{Bmatrix}\nu_1/2 & \nu_3/2 & I'\\ \nu_2/2 & \lambda/2 & I\end{Bmatrix}. \tag{30}$$

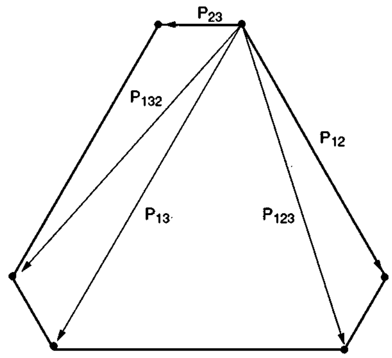

FIG. 2. The action of Weyl group elements on the highest weight of an SU(3) irrep.

To check these results, it is useful to apply them to the highest weight state. We find that

$$P_{12}\left|(\lambda+\mu,\mu,0)\,\frac{\mu}{2}\right\rangle=(-1)^{\mu}\left|(\mu,\lambda+\mu,0)\,\frac{\lambda+\mu}{2}\right\rangle,$$

$$P_{123}\left|(\lambda+\mu,\mu,0)\,\frac{\mu}{2}\right\rangle=\left|(0,\lambda+\mu,\mu)\,\frac{\lambda}{2}\right\rangle, \tag{31}$$

$$P_{132}\left|(\lambda+\mu,\mu,0)\,\frac{\mu}{2}\right\rangle=(-1)^{\mu}\left|(\mu,0,\lambda+\mu)\,\frac{\lambda+\mu}{2}\right\rangle,$$

consistent with the known action on the highest weight shown in Fig. 2. As expected, Weyl group elements map extremal states into other extremal states.

V. WIGNER FUNCTIONS

Matrix elements of $\mathrm{SU}(2)_{23}$ group elements are given immediately in the $\{|\nu I\rangle\}$ basis as SU(2) Wigner functions; viz.,

$$\langle \nu' I'|R_{23}(\alpha,\beta,\gamma)|\nu I\rangle=\delta_{\nu_1',\nu_1}\delta_{I'I}\mathcal{D}^{I}_{(\nu_2'-\nu_3')/2,(\nu_2-\nu_3)/2}(\alpha,\beta,\gamma), \tag{32}$$

where $\mathcal{D}^{I}_{M,N}$ is a standard SU(2) Wigner function.

To evaluate matrix elements of the other $\mathrm{SU}(2)_{ij}$ subgroups, we make use of the fact (noted by Chacón and Moshinsky[5]) that the different $\mathrm{SU}(2)_{ij}$ subgroups are Weyl transforms of one another. Thus, for example, the infinitesimal generators of $\mathrm{SU}(2)_{12}$

$$\hat{C}_{12},\quad \hat{C}_{21},\quad \tfrac{1}{2}(\hat{C}_{11}-\hat{C}_{22}), \tag{33}$$

are related to those of $\mathrm{SU}(2)_{23}$ by

$$\hat{C}_{12}=P_{132}\hat{C}_{23}P_{132}^{-1}=P_{132}\hat{C}_{23}P_{123}. \tag{34}$$

It follows that

$$R_{12}(\alpha,\beta,\gamma)=P_{132}R_{23}(\alpha,\beta,\gamma)P_{123}. \tag{35}$$

Similarly, one finds that

$$R_{13}(\alpha,\beta,\gamma)=P_{12}R_{23}(\alpha,\beta,\gamma)P_{12}. \tag{36}$$

Thus, with the parametrization given by Eq. (3), we obtain the SU(3) Wigner functions

$$\begin{aligned}
&D^{(\lambda\mu)}_{\nu' I',\nu I}(\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\gamma_3)\\
&\quad=\sum \mathcal{D}^{I'}_{(\nu_2'-\nu_3')/2,(\tau_2-\sigma_3)/2}(\alpha_1,\beta_1,\gamma_1)\langle(\nu_1',\tau_2,\sigma_3)I'|P_{132}|(\sigma_3,\nu_1',\tau_2)J\rangle\\
&\qquad\times\mathcal{D}^{J}_{(\nu_1'-\tau_2)/2,(\nu_1-\sigma_2)/2}(\alpha_2,\beta_2,\alpha_2)\langle(\sigma_3,\nu_1,\sigma_2)J|P_{123}|(\nu_1,\sigma_2,\sigma_3)I\rangle\\
&\qquad\times\mathcal{D}^{I}_{(\sigma_2-\sigma_3)/2,(\nu_2-\nu_3)/2}(\alpha_3,\beta_3,\gamma_3),
\end{aligned} \tag{37}$$

where the sum is over all σ, τ, and J values allowed by Eqs. (11), (13), and the betweenness conditions (8).

VI. MATRIX ELEMENTS OF SO(3)

If SO(3)$\subset$SU(3) is the subgroup whose infinitesimal generators are the angular momentum operators

$$\hat{L}_z=-i(\hat{C}_{23}-\hat{C}_{32}),\quad \hat{L}_x=-i(\hat{C}_{31}-\hat{C}_{13}),\quad \hat{L}_y=-i(\hat{C}_{12}-\hat{C}_{21}), \tag{38}$$

then we have the identities

$$\hat{L}_z=2\hat{I}_y,\quad \hat{L}_x=-2\hat{F}_y,\quad \hat{L}_y=2\hat{T}_y, \tag{39}$$

where $\hat{I}_y$, $\hat{T}_y$ and $\hat{F}_y$ belong to the Lie algebras of $\mathrm{SU}(2)_{23}$, $\mathrm{SU}(2)_{13}$, and $\mathrm{SU}(2)_{12}$, respectively. Thus, with the standard parametrization of an SO(3) element

$$\Omega(\alpha,\beta,\gamma)=e^{-i\alpha\hat{L}_z}e^{-i\beta\hat{L}_y}e^{-i\gamma\hat{L}_x}, \tag{40}$$

we have the identity

$$\Omega(\alpha,\beta,\gamma)=R_{23}(0,2\alpha,0)R_{12}(0,2\beta,0)R_{23}(0,2\gamma,0)=R_{23}(0,2\alpha,0)P_{132}R_{23}(0,2\beta,0)P_{123}R_{23}(0,2\gamma,0), \tag{41}$$

and the matrix elements

$$\begin{aligned}
\langle\nu' I'|\Omega(\alpha,\beta,\gamma)|\nu I\rangle=&\sum_{\sigma\tau J} d^{I'}_{(\nu_2'-\nu_3')/2,(\tau_3-\sigma_3)/2}(2\alpha)\\
&\times\langle(\nu_1',\tau_3,\sigma_3)I'|P_{132}|(\sigma_3,\nu_1',\tau_3)J\rangle d^{J}_{(\nu_1'-\tau_3)/2,(\nu_1-\sigma_2)/2}(2\beta)\\
&\times\langle(\sigma_3,\nu_1,\sigma_2)J|P_{123}|(\nu_1,\sigma_2,\sigma_3)I\rangle d^{L}_{(\sigma_2-\sigma_3)/2,(\nu_2-\nu_3)/2}(2\gamma),
\end{aligned} \tag{42}$$

where d^I_{MN} is a reduced SU(2) Wigner function.

VII. DISCUSSION

We have derived matrix elements of Weyl group elements and expressions for SU(3) Wigner functions by making use of the dual actions of U(3) and U(2) on the carrier spaces of symmetric representations of U(6).

The groups U(3) and U(2) are special cases of U(N) and U(n) groups that form dual pairs on the carrier spaces of fully symmetric irreps [i.e., irreps of highest weight (σ,0,...)] of U($N\times n$); they are also dual on a direct sum of such spaces.

The essential property of a dual pair[9,13] is that the constituent groups are the centralizers of each other's actions on a specified vector space. The classic example is the Schur–Weyl pair[14] of unitary, $\mathrm{U}(n)$, and symmetric, $\mathbf{S}_N$, groups which have commuting actions on the N-fold tensor product, $\mathbb{C}^{N\times n}$, of a complex n-dimensional vector space, $\mathbb{C}^n$. The Schur–Weyl duality has been used effectively to relate the characters of unitary groups, which are infinite Lie groups, to those of the finite symmetric groups. It also underlies the famous Littlewood–Richardson rules[15] for tensor products and the methods of King, Wybourne, and others[16] for inferring branching rules.

Another famous dual pair comprises the orthogonal, $\mathrm{O}(N)$, and symplectic, $\mathrm{Sp}(n,\mathbb{R})$, groups acting on the N-fold tensor product $\mathbb{H}^{N\times n}$ of the n-dimensional harmonic oscillator Hilbert space $\mathbb{H}^n$.[17] Whereas the Schur–Weyl duality relates the properties of a finite-dimensional irrep of a Lie group to those of a discrete group, the symplectic-orthogonal duality relates the properties of an infinite-dimensional irrep of a noncompact Lie group to those of a compact Lie group. This duality was used, for example, to infer the $\mathrm{Sp}(n,\mathbb{R})\rightarrow\mathrm{U}(n)$ branching rules from known properties of $\mathrm{O}(N)$.[18]

It is interesting to note that $\mathrm{U}(n)\times\mathrm{U}(N)$ and $\mathrm{Sp}(n)\times\mathrm{O}(N)$ are both direct products of dual pairs on a common harmonic oscillator Hilbert space $\mathbb{H}^{N\times n}$. Thus, one has the useful concept of dual subgroup chains

$$\mathrm{Sp}(n,\mathbb{R})\supset\mathrm{U}(n)\leftrightarrow\mathrm{O}(N)\supset\mathrm{U}(N), \tag{43}$$

involving the two dual pairs $\mathrm{Sp}(n,\mathbb{R})\times\mathrm{O}(N)$ and $\mathrm{U}(n)\times\mathrm{U}(N)$. These duality relations have been used[19] to relate the representations and tensor products of $\mathrm{U}(N)$ in an $\mathrm{O}(N)$ basis to those of $\mathrm{Sp}(n,\mathbb{R})$ in a $\mathrm{U}(n)$ basis. They also play an essential role in the microscopic theory of nuclear collective motion[20] with $\mathbb{H}^{N\times n}$ regarded as the Hilbert space for N-particles in an n-dimensional space.

It should be mentioned that dual subgroup chains were discovered long ago by Brauer[21] who extended the Schur–Weyl duality by observing that the centralizer of the orthogonal subgroup $\mathrm{O}(n)\subset\mathrm{U}(n)$ on $\mathbb{C}^{N\times n}$ is a group (also an algebra) that contains the symmetric group S_N as a subgroup [cf. Ref. 13 for a discussion of the $\mathrm{O}(n)$–Brauer duality].

The physical significance of several of the above dual pairs is illustrated effectively by applications to optics and quantum interferometry, applications which motivated the present investigation.

It has long been known that geometrical optics is an application of Hamiltonian mechanics. Moreover, in the linear approximation, the transformation of a light beam by an optical element, such as a lens, is an $\mathrm{Sp}(2,\mathbb{R})$ transformation. This observation is important because it means that the combined effects of many optical elements can be inferred by matrix multiplication. More importantly, one can go beyond the linear approximation to compute the aberrations of an optical system and how to correct them. The techniques for doing this have been developed into a fine art by Dragt and his students[22] and have revolutionized the design of charged-particle and optical beam systems; an introduction to the subject has been given by Guillemin and Sternberg.[23]

We note that there also exists a dual group action on optical systems. If a beam of light or charged particles is polarizable or has intrinsic spin degrees of freedom, then, in addition to the symplectic group action on its spatial phase-space coordinates, there is a dual orthogonal group action on its polarization state. Thus, for example, for light, with two linearly independent polarizations, or for spin-half particle beams, one has a dual $\mathrm{Sp}(2,\mathbb{R})\times\mathrm{O}(2)$ action on the combined space-spin degrees of freedom. [Note that we mean by $\mathrm{Sp}(2,\mathbb{R})$ the rank-2 group of real canonical transformation of a four-dimensional phase space; some authors denote the same group by $\mathrm{Sp}(4,\mathbb{R})$.] Thus, one can extend the dynamical group for an optical system from $\mathrm{Sp}(2,\mathbb{R})$ to the direct product group $\mathrm{Sp}(2,\mathbb{R})\times\mathrm{O}(2)$ and thereby admit polarizing (spin rotation) as well as focusing elements. One can further extend the dynamical group to $\mathrm{Sp}(4,\mathbb{R})\supset\mathrm{Sp}(2,\mathbb{R})\times\mathrm{O}(2)$ to include combinations of the two. [It is of interest to note that a general polarizing element is not restricted to $\mathrm{O}(2)$ and may induce a $\mathrm{U}(2)$ transformation that lies inside $\mathrm{Sp}(4,\mathbb{R})$ but which does not commute with the group $\mathrm{Sp}(2,\mathbb{R})$ of spatial transformations.]

Such extensions are relevant for describing the quantum interference of light or particle beams. In this case, one is interested in the detailed quantum states of many-photon (many-particle) system. Thus, one is interested in the unitary representations of the dynamical group and, as we have shown explicitly for $U(3)\times U(2)$ in Sec. III C, the irreps of a dynamical group are determined by those of its dual and vice versa.

It has recently been proposed that quantum interferometers should be analyzed in terms of unitary groups.[2,3] A typical quantum interferometer comprises a sequence of elements in which two input modes of the electromagnetic field (beams) are transformed linearly into two output modes. It has been shown that the transformation of the two modes by such an optical element is a $U(2)$ transformation [an $SU(2)$ transformation together with a phase shift].[2] It has also been shown[2] that a so-called *active* interferometer can similarly be represented by an $SU(1,1)$ transformation [note that $SU(1,1)$ is isomorphic to $Sp(1,\mathbb{R})$] and that a linear optical system, comprising n input modes, is represented by an $SU(n)$ transformation.[3]

The use of dual pairs provides a natural framework for the extension of these methods to include polarization and optical elements whose parameters depend on the polarization state of the input fields. To include polarization, one simply extends the $U(n)$ group to $U(n)\times U(2)$ and to include combinations of polarizers and beam splitters, for example, one extends to $U(2n)\supset U(n)\times U(2)$. This is particularly relevant in the quantal context because the input states available to σ photons, when there are n input modes and two linearly independent polarizations for each photon, span an irrep of highest weight $(\sigma,0,...)$ of the group $U(2n)$. The duality properties imply that the subrepresentations available to the subgroup $U(n)\times U(2)$, on restriction of the $U(2n)$ representation $(\sigma,0,...)$, are the so-called two-rowed irreps of type $(\lambda_1,\lambda_2,0,...)\times(\lambda_1,\lambda_2)$ (i.e., irreps whose highest weights have no more than two nonzero components). This follows simply because a $U(2)$ weight has only two components and the two subgroups, $U(n)$ and $U(2)$, being each other's duals, have highest weight states in common. This results in an enormous simplification in the analysis of a multi-mode interferometer. [Note that, as usual, the $SU(n)$ labels are obtained by taking differences of $U(n)$ labels, so that the $U(n)$ irrep $(\lambda_1,\lambda_2,0,...)$ restricts to the $SU(n)$ irrep $(\lambda_1-\lambda_2,\lambda_2,0,...)$.]

An important application of $SU(3)$ interferometry is the experimental test of Bell's theorem without inequalities, known as the GHZ test.[24] Standard tests of Bell's theorem, designed to test the hypotheses of local realism against quantum theory, involve spacelike-separated measurements of two polarization-correlated fields, and local realism establishes an upper bound on the possible degree of correlations between the two fields. The GHZ test, in its ideal form, yields one experimental result for local realism and an entirely different result for quantum theory. Thus, a particular observation determines which theory is correct, and an inequality is not necessary. In the context of $SU(3)$ Wigner functions, the important aspect of the GHZ test is that three polarization-correlated fields are used, and therefore $U(3)\times U(2)$, accounting for three fields and two polarizations, is appropriate here.

Consider, for example, the $SU(n)$ transformations of a one-rowed irrep, $(\lambda,0,...)$, by a system which ignores the polarization. For such an irrep, the highest weight state can be identified with the state

$$|\phi(\lambda,0,...)\rangle=(a_{11}^{\dagger})^{\lambda}|0\rangle. \tag{44}$$

of maximum polarization. Hence, all states of the $SU(n)$ irrep with this highest weight state have maximum polarization. Thus, the $SU(2)$ coupling becomes trivial and basis states for the irrep are labelled simply and uniquely by their weights. It follows that the basis states of the generalized version of the theorem of Sec. III C are simply the states

$$|\nu\rangle=\frac{(a_{11}^{\dagger})^{\nu_1}}{\sqrt{\nu_1!}}\frac{(a_{21}^{\dagger})^{\nu_2}}{\sqrt{\nu_2!}}\cdots\frac{(a_{n1}^{\dagger})^{\nu_n}}{\sqrt{\nu_n!}}|0\rangle. \tag{45}$$

The elements of the Weyl group are seen to act on such states by simply permuting the components $\{\nu_i\}$ of the weights.

For the general two-rowed irreps one must include explicit SU(2) coupling, as shown for SU(3) in Sec. III C. For example, basis states for a two-rowed irrep of U(4) are highest weight states of the dual algebra U(2) and have the general form

$$[|\tfrac{1}{2}\nu_1\rangle\otimes[|\tfrac{1}{2}\nu_2\rangle\otimes[|\tfrac{1}{2}\nu_3\rangle\otimes|\tfrac{1}{2}\nu_4\rangle]^I]^J]^{\lambda/2}_{\lambda/2}. \tag{46}$$

Thus, computing matrix elements of Weyl group elements for any two-rowed SU(n) irrep never involves more than SU(2) recoupling.

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APPENDIX: FACTORIZATION OF AN SU(3) ELEMENT

Claim: Any element $g\in \mathrm{SU}(3)$ can be parametrized and expressed as a product

$$g(\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\gamma_3)=R_{23}(\alpha_1,\beta_1,\gamma_1)R_{12}(\alpha_2,\beta_2,\alpha_2)R_{23}(\alpha_3,\beta_3,\gamma_3), \tag{A1}$$

where $R_{23}(\alpha,\beta,\gamma)\in \mathrm{SU}(2)_{23}$, $R_{12}(\alpha,\beta,\alpha)\in \mathrm{SU}(2)_{12}$, and the $\{\mathrm{SU}(2)_{ij}\}$ are the subgroups of SU(3) defined by the subsystems of roots shown in Fig. 1.

Proof: First observe that any SU(3) matrix can be brought to the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \tag{A2}$$

by an $\mathrm{SU}(2)_{23}$ transformation; viz.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & Y^* & Z^* \\ 0 & -Z & Y \end{pmatrix}\begin{pmatrix} x & * & * \\ y & * & * \\ z & * & * \end{pmatrix}=\begin{pmatrix} x & * & * \\ \sqrt{1-|x|^2|} & * & * \\ 0 & * & * \end{pmatrix}, \tag{A3}$$

where $Y=y(1-|x|^2)^{-\frac{1}{2}}$, and $Z=z(1-|x|^2)^{-\frac{1}{2}}$ and we have used the fact that $|x|^2+|y|^2+|z|^2=1$. A subsequent $\mathrm{SU}(2)_{12}$ transformation then brings the matrix to $\mathrm{SU}(2)_{23}$ form; i.e.,

$$\begin{pmatrix} x^* & \sqrt{1-|x|^2|} & 0 \\ -\sqrt{1-|x|^2|} & x & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x & * & * \\ \sqrt{1-|x|^2|} & * & * \\ 0 & * & * \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}. \tag{A4}$$

Thus, we determine that

$$\begin{pmatrix} x^* & \sqrt{1-|x|^2|} & 0 \\ -\sqrt{1-|x|^2|} & x & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & Y^* & Z^* \\ 0 & -Z & Y \end{pmatrix}\begin{pmatrix} x & * & * \\ y & * & * \\ z & * & * \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}. \tag{A5}$$

Inversion of this equation gives

$$\begin{pmatrix} x & * & * \\ y & * & * \\ z & * & * \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & Y & -Z^* \\ 0 & Z & Y^* \end{pmatrix}\begin{pmatrix} x & -\sqrt{1-|x|^2} & 0 \\ \sqrt{1-|x|^2} & x^* & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \tag{A6}$$

which proves the claim with suitably chosen parameter values; e.g.,

$$x=e^{-i\alpha_2}\cos(\beta_2/2),\quad \sqrt{1-|x|^2}=\sin(\beta_2/2). \tag{A7}$$

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