

# The shifted harmonic approximation and asymptotic SU(2) and SU(1,1) Clebsch–Gordan coefficients

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## Abstract

Clebsch–Gordan coefficients of SU(2) and SU(1,1) are defined as eigenfunctions of a linear operator acting on the tensor product of the Hilbert spaces for two irreps of these groups. The *shifted harmonic approximation* is then used to solve these equations in asymptotic limits in which these eigenfunctions approach harmonic oscillator wavefunctions and thereby derive asymptotic expressions for these Clebsch–Gordan coefficients.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

The shifted harmonic approximation (SHA) is an approximation to a technique for realizing a set of operators, defined initially as linear transformations of a finite vector space, as differential operators. It was introduced for the purpose of understanding the nature of phase transitions in systems with  $\mathfrak{su}(2)$  or  $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$  spectrum generating algebras [1] and subsequently applied to a model with an  $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$  spectrum-generating algebra [2]. More recently, it has been developed for application to a multi-level pairing model [3], for which the spectrum-generating algebra is a direct sum of multiple  $\mathfrak{su}(2)$  algebras. It can be regarded as a technique for contracting a Lie algebra representation to that of a simpler algebra in well-defined limiting situations.

In this paper, we show how the SHA provides approximate expressions for SU(2) and SU(1,1) Clebsch–Gordan coefficients that become precise in certain asymptotic limits in which they approach harmonic oscillator wavefunctions when regarded as functions of appropriate parameters. Moreover, we show by examples that these limits are approached very rapidly and provide remarkably accurate approximations more generally.

In addition to the examples given above, the groups SU(2) and SU(1,1) and the coupling of their irreducible representations have numerous applications in quantum optics [4].<sup>3</sup> In this

<sup>3</sup> A highly non-exhaustive list contains, for instance, [5].

context, asymptotic SU(2) and SU(1,1) CG coefficients are of huge potential value in problems in which a prohibitive amount of time is often needed to compute the required coefficients numerically from the known analytical formulae. In contrast, the asymptotic coefficients given here can be calculated in fractions of a second. Asymptotic SU(2) CG coefficients have also been connected to a tight-binding model of a one-dimensional potential [6].

Clebsch–Gordan (CG) coefficients  $(s_1, M - m, s_2, m | SM)$  for the group SU(2) are required when two systems of spin  $\vec{s}_1$  and  $\vec{s}_2$  are coupled to the total spin  $\vec{S}$ :

$$|SM\rangle = \sum_m |s_2 m\rangle \otimes |s_1, M - m\rangle (s_1, M - m, s_2, m | SM). \quad (1)$$

In equation (1), the projections along the common quantization axis of the spins  $\vec{s}_1, \vec{s}_2$  and  $\vec{S}$  are  $M - m, m$  and  $M$ , respectively. Equation (1) is also applicable to a wide variety of other quasi-spin systems having states that carry su(2) irreps.

Because of the connection between SU(2) and SO(3), CG coefficients also appear in problems where products of spherical harmonics and other related special functions occur naturally. Furthermore, the ubiquity of su(2) as a subalgebra of other Lie algebras makes the SU(2) CG coefficients (and those of SU(1,1) for non-compact groups) an important ingredient in the computation of coupling coefficients for higher groups.

Much is known about SU(2) CG coefficients: any coefficient can be obtained in the closed form using an expression containing a sum of square roots of rational factors. However, for small values of  $m$  and  $M$ , the complexity of this sum increases rapidly with  $S$ , and numerous asymptotic estimates for large  $S$  have been developed to understand and quickly evaluate CG coefficients in this regime.

A first consideration of asymptotic limits for SU(2) coefficients is due to Wigner [7]. He approached the problem from a semi-classical perspective and obtained average expressions for the coefficients when all momenta were large; unfortunately, Wigner’s result did not capture the essentially oscillatory nature of asymptotic CG coefficients and left the door open for further studies. Brussaard and Toelhoek [8] used the WKB method to connect reduced SU(2)–Wigner functions with asymptotic CG coefficients. Their result, which applies to an asymptotic limit in which just two angular momenta  $s_1$  and  $S$  are large, significantly improved on Wigner’s because it correctly gave the sign of the coefficient and turns out to be reasonably accurate, even for modest values of  $s_1$  and  $S$ . Finally, Ponzano and Regge [9], in their authoritative work grounded almost entirely on geometrical arguments, expanded on [7] and [8] by providing accurate expressions for CG coefficients valid for three large angular momenta.

The insights provided by [8] and [9] have been a source of inspiration for many subsequent authors. A detailed survey of the literature pre-1988 can be found in [10]. Comparatively recent work, limited to asymptotic SU(2) CG coefficients with all three momenta large, include, for instance, [11, 12], wherein a detailed and systematic review of the results of Ponzano and Regge can be found, with emphasis on closing loose ends in some of their ‘heuristic’ arguments.

In this paper we show that for asymptotically large values of  $s_1$  and  $s_2$ , and for finite values of  $n$  and  $M$ , the SU(2) CG coefficients rapidly approach the asymptotic expressions

$$(s_1, M - m, s_2 m | s_1 + s_2 - n, M) \sim (-1)^n \left( \frac{a}{\sqrt{\pi} 2^n n!} \right)^{\frac{1}{2}} H_n(a(m - x_0)) e^{-\frac{1}{2} a^2 (m - x_0)^2}, \quad (2)$$

where  $H_n$  is a Hermite polynomial, with parameters defined by a simple SHA algorithm. It is also shown that these asymptotic results have analytic approximations which remain precise in the asymptotic limit but are approached somewhat less rapidly; they are given by the explicit

expression

$$(s_1 m_1 s_2 m_2 | s_1 + s_2 - n, M) \sim (-1)^n \left( \frac{a}{\sqrt{\pi} 2^n n!} \right)^{\frac{1}{2}} H_n \left( a \frac{\sigma_1 m_2 - \sigma_2 m_1}{\sigma_1 + \sigma_2} \right) e^{-\frac{1}{2} a^2 (m - x_0)^2}, \quad (3)$$

with

$$a^4 = \frac{(\sigma_1 + \sigma_2)^4}{\sigma_1^2 \sigma_2^2 [(\sigma_1 + \sigma_2)^2 - M^2]}, \quad x_0 = \frac{\sigma_2 M}{\sigma_1 + \sigma_2}, \quad (4)$$

$$\sigma_1 = \sqrt{s_1(s_1 + 1)}, \quad \sigma_2 = \sqrt{s_2(s_2 + 1)}. \quad (5)$$

Basis states for unitary irreps of the positive discrete series of  $\mathfrak{su}(1,1)$  within the tensor product of two such irreps are also given by linear combinations

$$|Kn\rangle = \sum_{n_1 n_2} |k_2 n_2\rangle \otimes |k_1 n_1\rangle \langle k_1 k_2 Nm | Kn\rangle, \quad (6)$$

where  $\langle k_1 k_2 Nm | Kn\rangle$  is an  $\text{SU}(1,1)$  CG coefficient related to a non-vanishing coefficient in the notation of Van der Jeugt [13], by

$$\langle k_1 k_2 Nm | Kn\rangle = C(k_1, n_1, k_2, n_2; K, n) \quad (7)$$

with  $N = n_1 + n_2$  and  $m = n_2 - n_1$ . Note that an  $\text{SU}(1,1)$  CG coefficient vanishes unless  $K + n = k_1 + n_1 + k_2 + n_2 = k_1 + k_2 + N$ . The notation is clarified in section 3.

Various closed form expressions for the  $\text{SU}(1,1)$  coefficients can be found in the literature. The exact expressions of immediate relevance to our work are given, for instance, in [13, 14]. In addition, several other authors [15] have published expressions spanning a variety of couplings of representations (not only of two positive discrete series) and a number of different bases.

In spite of these exact results, there is limited knowledge of the asymptotic behaviour of the coefficients, although some basic results valid for the coupling of two irreps of the positive discrete series with  $k_1 \rightarrow \infty$  and  $k_2$  finite can be found in [16], a result that proves to be useful in constructing states in odd nuclei [17] within the context of the nuclear symplectic model.

The SHA approach is applied in section 3 to derive asymptotic expressions for the  $\text{SU}(1,1)$  CG coefficients of equation (7) in an altogether different regime to that studied in [17]. The result, similar to that given above for a class of  $\text{SU}(2)$  coefficients, is that

$$\langle k_1 k_2 Nm | Kn\rangle \sim (-1)^{N+m} \left( \frac{a}{\sqrt{\pi} 2^n n!} \right)^{\frac{1}{2}} H_n(a(m - x_0)) e^{-\frac{1}{2} a^2 (m - x_0)^2}, \quad (8)$$

when  $N = n_1 + n_2$  becomes asymptotical large and  $n$  remains finite, and with parameters defined by a simple SHA algorithm given in section 3.1. It is also shown that precise asymptotic expressions which, however, are approached somewhat less rapidly are given by equation (8) with the explicit parameter values

$$a^2 = \frac{N + 4\sqrt{\kappa_1 \kappa_2}}{2N\sqrt{\kappa_1 \kappa_2}}, \quad x_0 = \frac{N(\kappa_1 - \kappa_2)}{N + 4\sqrt{\kappa_1 \kappa_2}}, \quad (9)$$

where

$$\kappa_1 = k_1 + \frac{N}{4}, \quad \kappa_2 = k_2 + \frac{N}{4}. \quad (10)$$

The SHA method described in this paper complements other approaches where the asymptotic behaviour is examined using WKB methods. As mentioned briefly in section 5, the SHA technique can be regarded as a procedure for obtaining the contraction of a Lie algebra to a simpler Lie algebra that is appropriate in certain limiting situations. Such contractions

are known to lead to valuable insights and useful approximation procedures in physics as evidenced, for example, in the approach of quantum mechanics to classical mechanics as  $\hbar$  or some other scale parameter approaches zero, in the approximation of fermion pair algebras by boson algebras in the random-phase approximation of many-body theory, and in the bosonic behaviour of large atomic samples [18].

## 2. Asymptotic SU(2) Clebsch–Gordan coefficients

Let  $\{\hat{S}_+, \hat{S}_-, \hat{S}_0\}$  satisfy the usual commutation relations of the complex extension of the su(2) Lie algebra:

$$[\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad [\hat{S}_+, \hat{S}_-] = 2\hat{S}_0. \quad (11)$$

Basis states,  $|sm\rangle$ , for an su(2) irrep are then defined by the equations

$$\hat{S}_0|sm\rangle = m|sm\rangle, \quad (12)$$

$$\hat{S}_\pm|sm\rangle = \sqrt{(s \mp m)(s \pm m + 1)}|s, m \pm 1\rangle. \quad (13)$$

*Coupled* basis states for irreps of su(2) within the tensor product of two such irreps are given by

$$|SM\rangle = \sum_{m_1 m_2} |s_2 m_2\rangle \otimes |s_1 m_1\rangle (s_1 m_1 s_2 m_2 | SM), \quad (14)$$

where  $(s_1 m_1 s_2 m_2 | SM)$  is an SU(2) CG coefficient. These coefficients are equal to the overlaps of coupled and uncoupled tensor product states

$$(s_1 m_1 s_2 m_2 | SM) = [\langle s_1 m_1 | \otimes \langle s_2 m_2 |] | SM\rangle. \quad (15)$$

For convenience, we denote the tensor product states by

$$|s_1 s_2 M m\rangle \equiv |s_2 m_2\rangle \otimes |s_1 m_1\rangle, \quad (16)$$

with  $M = m_1 + m_2$  and  $m = m_2$ . The SU(2) CG coefficients are then the overlaps

$$(s_1 m_1 s_2 m_2 | SM) = \langle s_1 s_2 M m | SM\rangle = [\langle s_1 m_1 | \otimes \langle s_2 m_2 |] | SM\rangle. \quad (17)$$

They are defined (to within arbitrary phase factors) by the requirement that the states  $\{|SM\rangle\}$  satisfy the eigenvalue equations

$$\hat{S}_0 | SM\rangle = M | SM\rangle, \quad (18)$$

$$\hat{S}_+ \hat{S}_- | SM\rangle = [S(S+1) - M(M-1)] | SM\rangle, \quad (19)$$

with

$$\hat{S}_0 = \hat{S}_0^1 + \hat{S}_0^2, \quad \hat{S}_\pm = \hat{S}_\pm^1 + \hat{S}_\pm^2. \quad (20)$$

### 2.1. The shifted harmonic approximation

We now determine the CG coefficients using the SHA and show them to be precise in the limit of asymptotically large values of  $s_1, s_2$  and  $S$  and finite values of  $M$  and  $n = s_1 + s_2 - S$ .

The desired CG coefficients are first expressed as overlap functions, in the form

$$\psi_n^{s_1 s_2 M}(m) = \langle s_1 s_2 M m | s_1 + s_2 - n, M\rangle. \quad (21)$$

This notation is introduced with the intention that for given values of  $s_1, s_2, M$  and  $n$ , a set of CG coefficients can be regarded as the values of a function,  $\psi_n^{s_1 s_2 M}$ , of the discrete

variable  $m$ . Moreover, the state  $|SM\rangle$  with  $S = s_1 + s_2 - n$  is completely determined by the values,  $\{\psi_n^{s_1 s_2 M}(m), m = -s_2, \dots, s_2\}$ , which implies that  $\psi_n^{s_1 s_2 M}$  can be interpreted as a wavefunction for the state  $|s_1 + s_2 - n, M\rangle$ .

The operators  $\hat{S}_0, \hat{S}_\pm$  are now mapped to operators on these wavefunctions defined by

$$\begin{aligned} \hat{S}_v \psi_n^{s_1 s_2 M}(m) &\equiv \langle s_1 s_2 M m | \hat{S}_v | s_1 + s_2 - n, M \rangle \\ &= \sum_{M' p} \langle s_1 s_2 M m | \hat{S}_v | s_1 s_2 M' p \rangle \psi_n^{s_1 s_2 M'}(p). \end{aligned} \quad (22)$$

Therefore, because the state  $|SM\rangle$  is an eigenstate of  $\hat{S}_0$  and  $\hat{S}_+ \hat{S}_-$ , the function  $\psi_n^{s_1 s_2 M}$  should likewise be an eigenfunction of  $\hat{S}_0$  and  $\hat{S}_+ \hat{S}_-$  with the same eigenvalues. With the expansions of equations (16) and (20), we obtain

$$\hat{S}_+ \hat{S}_- \psi_n^{s_1 s_2 M}(m) = f_0(m) \psi_n^{s_1 s_2 M}(m) + f_1(m) \psi_n^{s_1 s_2 M}(m+1) + f_{-1}(m) \psi_n^{s_1 s_2 M}(m-1), \quad (23)$$

where

$$\begin{aligned} f_0(m) &= \langle s_1 s_2 M m | \hat{S}_+ \hat{S}_- | s_1 s_2 M m \rangle, \\ f_1(m) &= \langle s_1 s_2 M m | \hat{S}_+ \hat{S}_- | s_1 s_2 M, m+1 \rangle, \\ f_{-1}(m) &= \langle s_1 s_2 M m | \hat{S}_+ \hat{S}_- | s_1 s_2 M, m-1 \rangle = f_1(m-1). \end{aligned} \quad (24)$$

Equation (23) can now be expressed in terms of finite difference operators, defined by

$$\hat{\Delta} \psi(m) \equiv \frac{1}{2} (\psi(m+1) - \psi(m-1)), \quad (25)$$

$$\hat{\Delta}^2 \psi(m) \equiv \psi(m+1) - 2\psi(m) + \psi(m-1), \quad (26)$$

with the result that

$$f_1(m) \psi(m+1) = f_1(m) \left(1 + \Delta + \frac{1}{2} \hat{\Delta}^2\right) \psi(m), \quad (27)$$

$$f_{-1}(m-1) \psi(m-1) = \left(1 - \Delta + \frac{1}{2} \hat{\Delta}^2\right) [f_{-1}(m) \psi(m)]. \quad (28)$$

Equation (23) then becomes

$$\begin{aligned} \hat{S}_+ \hat{S}_- \psi_n^{s_1 s_2 M}(m) &= f_0(m) \psi_n^{s_1 s_2 M}(m) + f_1(m) \left[1 + \hat{\Delta} + \frac{1}{2} \hat{\Delta}^2\right] \psi_n^{s_1 s_2 M}(m) \\ &\quad + \left[1 - \hat{\Delta} + \frac{1}{2} \hat{\Delta}^2\right] [f_{-1}(m) \psi_n^{s_1 s_2 M}(m)], \end{aligned} \quad (29)$$

and gathering terms leads to the expression of  $\hat{S}_+ \hat{S}_-$  as the difference operator

$$\hat{S}_+ \hat{S}_- = F(m) + \hat{\Delta} f_1(m) \hat{\Delta}, \quad (30)$$

where

$$F(m) = f_0(m) + f_1(m) + f_{-1}(m-1). \quad (31)$$

With  $\hat{S}_+ \hat{S}_- = (\hat{S}_+^1 + \hat{S}_+^2)(\hat{S}_-^1 + \hat{S}_-^2)$ , we also determine that

$$f_0(m) = \sigma_1^2 - (M-m)(M-m-1) + \sigma_2^2 - m(m-1), \quad (32)$$

$$f_1(m) = \sqrt{[\sigma_1^2 - (M-m)(M-m-1)][\sigma_2^2 - m(m+1)]}, \quad (33)$$

where

$$\sigma_i^2 = s_i(s_i + 1), \quad i = 1, 2. \quad (34)$$

To determine asymptotic expressions for the functions  $\psi_n^{s_1 s_2 N}$  as eigenfunctions of  $\hat{S}_+ \hat{S}_-$ , we now make the *continuous variable approximation* of extending these functions of the discrete variable  $m$  to functions  $\Psi_n^{s_1 s_2 N}$  of a continuous variable  $x$  with the property that

$$\Psi_n^{s_1 s_2 N}(x) = \psi_n^{s_1 s_2 N}(x), \quad (35)$$

whenever  $x$  is in the domain of the discrete variable  $m$ . In this approximation, which is valid in the asymptotic limits in which  $\Psi_n^{s_1 s_2 N}$  becomes a smooth function, the difference operators can be replaced by differential operators:

$$\hat{\Lambda} \rightarrow \hat{D} \equiv \frac{d}{dx}, \quad \hat{\Lambda}^2 \rightarrow \hat{D}^2 \equiv \frac{d^2}{dx^2}. \quad (36)$$

The functions  $F$ ,  $f_0$  and  $f_1$  of equations (31)–(33) are similarly extended to the continuous variable  $x$ , and the operator  $\hat{S}_+ \hat{S}_-$  becomes the differential operator

$$\hat{S}_+ \hat{S}_- \rightarrow F(x) + \hat{D} f_1(x) \hat{D}. \quad (37)$$

Provided the extension of  $\psi_n^{s_1 s_2 M}(m)$  to the function  $\Psi_n^{s_1 s_2 M}(x)$  does not require the latter to be non-zero for any  $x$  that is outside of the limits for  $m$ , it is seen that  $f_1(x)$  is real for all  $x$  for which  $\Psi_n^{s_1 s_2 M}(x)$  is non-zero. The limits on the values of  $m$  are seen, from equation (16), to be such that  $-s_2 \leq m \leq s_2$  and  $-s_1 \leq M - m \leq s_1$ . We will denote the upper and lower limits on the value of  $m$  by  $m_{\max}$  and  $m_{\min}$ , respectively. Then, because the norm of the function  $\psi_n^{s_1 s_2 M}$  is given by

$$\|\psi_n^{s_1 s_2 M}\|^2 = \sum_{m=m_{\min}}^{m_{\max}} |\psi_n^{s_1 s_2 M}(m)|^2, \quad (38)$$

it follows that the corresponding smooth function  $\Psi_n^{s_1 s_2 M}(x)$  should have the norm given by

$$\|\Psi_n^{s_1 s_2 M}\|^2 = \int_{m_{\min}}^{m_{\max}} |\Psi_n^{s_1 s_2 M}(x)|^2 dx. \quad (39)$$

It is also seen that when  $\Psi_n^{s_1 s_2 M}(x)$  is zero for all  $x > m_{\max}$  and all  $x < m_{\min}$ , this integral can be extended to the range  $-\infty < x < \infty$ . The operator  $\hat{D} = d/dx$  is then seen to be skew Hermitian and  $\hat{S}_+ \hat{S}_-$  is Hermitian.

Now if the function  $\Psi_n^{s_1 s_2 M}$  is sufficiently smooth, is non-zero over a sufficiently narrow region of  $x$  within the limits  $m_{\min} < x < m_{\max}$  and is centred about a value  $x_0$ , we can make the so-called SHA [1, 2] which, in addition to the continuous variable approximation, consists of dropping all terms in  $F$ ,  $f_0$  and  $f_1$  for which the expansion of the operator  $\hat{S}_+ \hat{S}_-$  will be more than bilinear in  $x - x_0$  and  $d/dx$ . The conditions under which the SHA are valid are shown, in the following, to be well satisfied, for finite values of  $n$ , in the asymptotic limit as  $s_1 \rightarrow \infty$  and  $s_2 \rightarrow \infty$ . Thus, the SHA gives the asymptotic expression

$$\hat{S}_+ \hat{S}_- \approx E + \frac{1}{2} A \frac{d^2}{dx^2} + C(x - x_0) - \frac{1}{2} B(x - x_0)^2 + \dots, \quad (40)$$

where

$$E = F(x_0), \quad A = 2f_1(x_0), \quad C = F'(x_0), \quad B = -F''(x_0). \quad (41)$$

An examination of the values of these parameters reveals that  $A > 0$  and  $B > 0$  in situations of interest. Thus, we consider  $\mathcal{H} \approx -\hat{S}_+ \hat{S}_-$  and determine  $x_0$  to be the value for which  $C = 0$ , bringing  $\mathcal{H}$  to the form

$$\mathcal{H} = -E + \left[ -\frac{1}{2a^2} \frac{d^2}{dx^2} + \frac{1}{2} a^2 (x - x_0)^2 \right] \hbar \omega, \quad (42)$$

where

$$(\hbar\omega)^2 = AB, \quad a^4 = \frac{B}{A}. \quad (43)$$

The eigenfunctions for this Hamiltonian are harmonic oscillator wavefunctions, and the eigenvalues are

$$E_n = -E + \left(n + \frac{1}{2}\right)\hbar\omega. \quad (44)$$

Thus, we obtain the desired asymptotic Clebsch–Gordan coefficients in the form given by equation (2). Note that the eigenfunctions of a Hamiltonian are only ever defined to within an arbitrary phase. There are also arbitrary phases in the definition of CG coefficients. The phases in equation (2) were chosen to reproduce the standard phases of the Condon and Shortley CG coefficients [19].

### 2.2. Simplified analytical expressions for asymptotic SU(2) CG coefficients

The SHA results given above are easily calculated and give accurate results in the limit as  $s_1 \rightarrow \infty$  and  $s_2 \rightarrow \infty$  but  $M$  and  $n$  remain finite. Simpler analytic asymptotic expressions are obtained if we also neglect terms that go to zero as  $s_1 \rightarrow \infty$  and  $s_2 \rightarrow \infty$ .

In these limits

$$\mathcal{H} \rightarrow \text{const.} - \frac{1}{2}A_0 \frac{d^2}{dx^2} + C_0(x - x_0) + \frac{1}{2}B_0(x - x_0)^2, \quad (45)$$

with

$$A_0 \sim 2\sigma_1\sigma_2 \left(1 - \frac{M^2}{(\sigma_1 + \sigma_2)^2}\right), \quad (46)$$

$$C_0 \sim \frac{2(\sigma_1 + \sigma_2)}{\sigma_1}M, \quad (47)$$

$$B_0 \sim \frac{2(\sigma_1 + \sigma_2)^2}{\sigma_1\sigma_2}. \quad (48)$$

Thus, we can evaluate the asymptotic Clebsch–Gordan coefficients from equation (2) with

$$a^4 = \frac{B_0}{A_0} = \frac{(\sigma_1 + \sigma_2)^4}{\sigma_1^2\sigma_2^2[(\sigma_1 + \sigma_2)^2 - M^2]}, \quad (49)$$

$$x_0 = \frac{C_0}{B_0} = \frac{\sigma_2 M}{\sigma_1 + \sigma_2}, \quad (50)$$

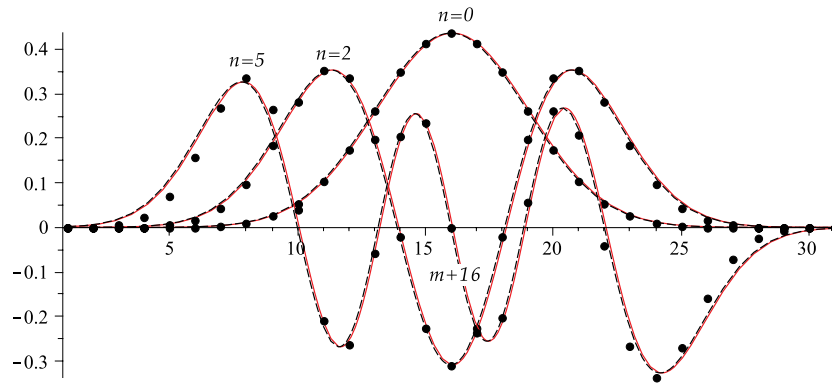
to obtain the expression

$$(s_1, M - m, s_2 m | s_1 + s_2 - n, M) \sim (-1)^n \left(\frac{a}{\sqrt{\pi} 2^n n!}\right)^{\frac{1}{2}} H_n(a(m - x_0)) e^{-\frac{1}{2}a^2(m - x_0)^2}, \quad (51)$$

and that of equation (3).

### 2.3. Numerical results for su(2) CG coefficients

We have ascertained and the following examples illustrate that the SU(2) CG coefficients satisfy the conditions for the validity of the SHA in the specified asymptotic limit as  $s_1, s_2 \rightarrow \infty$  for finite values of  $n$  and  $M$ . The following results show that in addition to being



**Figure 1.** The Clebsch–Gordan coefficients  $(20, -m, 15, m|35 - n, 0)$  shown as a function of  $m$  for three values of  $n$ . Exact values are shown as dots, full SHA values as continuous (red) lines and simplified SHA values as dashed lines.

precise in these limits, the SHA and simplified SHA wavefunctions also give remarkably accurate values for modest values of  $s_1$  and  $s_2$  and surprising large values of  $M$ .

The following figures compare the values of exactly computed SU(2) CG coefficients  $(s_1, M - m, s_2, m|s_1 + s_2 - n, M)$ , with the SHA expressions given by equation (2). The continuous (red) lines are those of the full SHA approximation with the parameters as defined in section 2.1. The dashed (black) lines are those of the simplified SHA coefficients with the parameters given by equations (49)–(50).

Figure 1 illustrates that for  $s_1$  and  $s_2$  as small as 20 and 15, respectively, both the full and simplified SHA yield approximate SU(2) CG coefficients for  $n \leq 5$  that are almost indistinguishable from each other. The figure shows that inaccuracies become visible for  $n = 5$  coefficients when  $|m| \gtrsim 8$ . The results become increasingly accurate for larger values of  $s_1$  and  $s_2$  and are precise in the asymptotic limit.

Figure 2 illustrates that for  $s_1 = 60$ ,  $s_2 = 40$  and  $n = 0$  the simplified SHA is very accurate for  $|M| \lesssim 30$  whereas the full SHA is accurate for all the values of  $M$  shown.

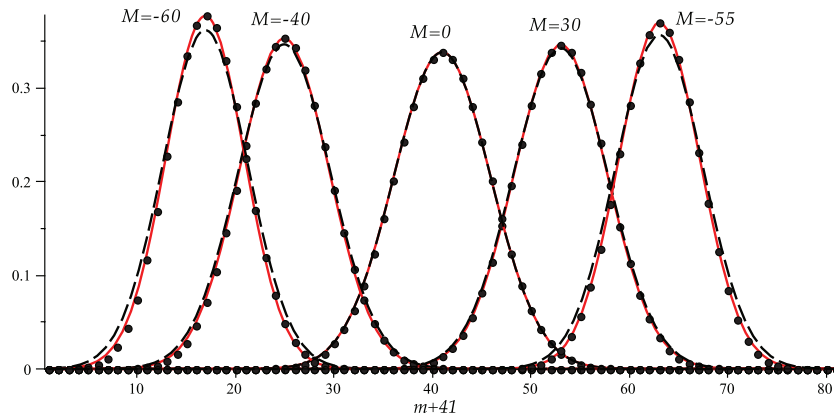
Figure 3 illustrates how accurate the SHA and simplified SHA Clebsch–Gordan coefficients can be for quite small values of  $s_1$  and  $s_2$  provided  $n$  and  $|M|$  are kept even smaller. The region in which the results are at their worst is for values of  $m$  close to its boundary values, especially in situations in which the asymptotic expressions extend beyond these boundaries.

### 3. Asymptotic SU(1,1) Clebsch–Gordan coefficients

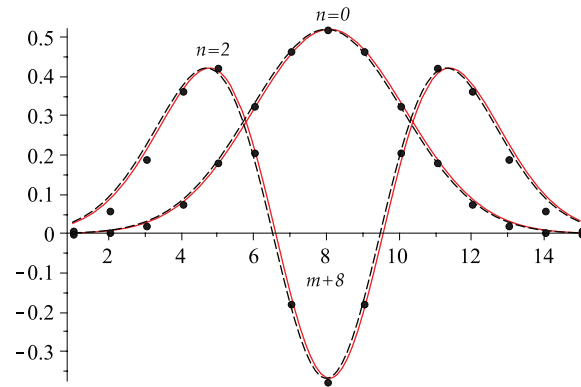
We now consider the operators  $\{\hat{K}_+, \hat{K}_-, \hat{K}_0\}$  that satisfy the commutation relations of the complex extension of the  $\mathfrak{su}(1,1)$  Lie algebra:

$$[\hat{K}_0, \hat{K}_\pm] = \pm \hat{K}_\pm, \quad [\hat{K}_-, \hat{K}_+] = 2\hat{K}_0. \tag{52}$$





**Figure 2.** The Clebsch–Gordan coefficients  $(60, M - m, 40, m | 100, M)$  shown as a function of  $m$  for a range of  $M$  values. Exact values are shown as dots, full SHA values as continuous (red) lines and simplified SHA values as dashed lines.



**Figure 3.** The Clebsch–Gordan coefficients  $(10, -m, 7, m | 17 - n, 0)$  shown as a function of  $m$ . Exact values are shown as full dots, full SHA values as continuous (red) lines and simplified SHA values as dashed lines.

Basis states  $\{|kn\rangle, n = 0, 1, 2, \dots\}$  for a unitary  $su(1,1)$  irrep of the positive discrete series are defined by the equations

$$\begin{aligned}
 \hat{K}_0|kn\rangle &= (k+n)|kn\rangle, \\
 \hat{K}_+|kn\rangle &= \sqrt{(2k+n)(n+1)}|k, n+1\rangle, \\
 \hat{K}_-|kn\rangle &= \sqrt{(2k+n-1)n}|k, n-1\rangle, \\
 \hat{K}_+\hat{K}_-|kn\rangle &= (2k+n-1)n|kn\rangle.
 \end{aligned}
 \tag{53}$$

Basis states for irreps of  $su(1,1)$  within the tensor product of two such irreps are given by linear combinations

$$|Kn\rangle = \sum_{n_1 n_2} |k_2 n_2\rangle \otimes |k_1 n_1\rangle C(k_1, n_1, k_2, n_2; K, n),
 \tag{54}$$

where  $C(k_1, n_1, k_2, n_2; K, n)$  is an SU(1,1) CG coefficient in the notation of Van der Jeugt [13], equal to the overlap of coupled and uncoupled tensor product states

$$C(k_1, n_1, k_2, n_2; K, n) = [\langle k_1, n_1 | \otimes \langle k_2, n_2 | ] |Kn\rangle. \quad (55)$$

Thus, if we put  $n_1 = \frac{1}{2}N - m$  and  $n_2 = \frac{1}{2}N + m$  and denote an uncoupled tensor product state by

$$|k_1 k_2 Nm\rangle \equiv |k_2, \frac{1}{2}N + m\rangle \otimes |k_1, \frac{1}{2}N - m\rangle, \quad (56)$$

we obtain the more useful expression of a CG coefficient as an overlap

$$C(k_1, n_1, k_2, n_2; K, n) = \langle k_1 k_2 Nm | Kn\rangle. \quad (57)$$

These coefficients are defined (to within arbitrary phase factors) by the requirement that the states  $\{|Kn\rangle\}$  satisfy the eigenvalue equations

$$\hat{K}_0 |Kn\rangle = (K + n) |Kn\rangle, \quad (58)$$

$$\hat{K}_+ \hat{K}_- |Kn\rangle = (2K + n - 1) n |Kn\rangle, \quad (59)$$

with

$$\hat{K}_0 = \hat{K}_0^1 + \hat{K}_0^2, \quad \hat{K}_\pm = \hat{K}_\pm^1 + \hat{K}_\pm^2, \quad (60)$$

and the understanding that

$$\hat{K}_v^1 (|k_2 m_2\rangle \otimes |k_1 m_1\rangle) = |k_2 m_2\rangle \otimes \hat{K}_v^1 |k_1 m_1\rangle, \quad (61)$$

$$\hat{K}_v^2 (|k_2 m_2\rangle \otimes |k_1 m_1\rangle) = \hat{K}_v^2 |k_2 m_2\rangle \otimes |k_1 m_1\rangle. \quad (62)$$

### 3.1. SU(1,1) CG coefficients in the shifted harmonic approximation

We now determine asymptotic limits to these CG coefficients in the SHA and show them to be precise for large values of  $N$  and finite values of  $n$  and  $|k_1 - k_2|$ .

Before embarking on an SHA calculation of asymptotic SU(1,1) CG coefficients, we first examine some coefficients to see if they satisfy the necessary conditions for the validity of the SHA. From exact calculations in the phase convention of Van der Jeugt<sup>4</sup>, it is determined that for large values of  $N$  and small values of  $n$ , the SU(1,1) CG coefficients  $\langle k_1 k_2 Nm | Kn\rangle$ , when multiplied by the phase factor  $(-1)^{N+m}$  and regarded as functions of  $m$ , approach standard harmonic oscillator wavefunctions. Thus, we define an overlap function of the discrete variable  $m$  by

$$\psi_n^{k_1 k_2 N}(m) = (-1)^{N+m} \langle k_1 k_2 Nm | Kn\rangle \quad (63)$$

with  $K = N - n + k_1 + k_2$ .

Because the state  $|Kn\rangle$  is an eigenstate of the operator  $\hat{K}_+ \hat{K}_-$ , it follows that the representation of this state by the function  $\psi_n^{k_1 k_2 N}$  of the discrete variable  $m$  is an eigenfunction of the operator  $\hat{K}_+ \hat{K}_-$  defined by

$$\begin{aligned} \hat{K}_+ \hat{K}_- \psi_n^{k_1 k_2 N}(m) &\equiv (-1)^{N+m} \langle k_1 k_2 Nm | \hat{K}_+ \hat{K}_- | Kn\rangle \\ &= \sum_p (-1)^{m-p} \langle k_1 k_2 Nm | \hat{K}_+ \hat{K}_- | k_1 k_2 Np\rangle \psi_n^{k_1 k_2 N}(p). \end{aligned} \quad (64)$$

<sup>4</sup> Exact su(1,1) CG coefficients were calculated with a Maple code supplied by J Van der Jeugt.

Thus, we obtain an equation for  $\psi_n^{k_1 k_2 N}$  of the identical form to that of equation (23) for the SU(2) CG coefficients, given here by

$$\hat{\mathcal{K}}_+ \hat{\mathcal{K}}_- \psi_n^{k_1 k_2 N}(m) = f_0(m) \psi_n^{k_1 k_2 N}(m) + f_1(m) \psi_n^{k_1 k_2 N}(m+1) + f_1(m-1) \psi_n^{k_1 k_2 N}(m-1), \quad (65)$$

but now with

$$\begin{aligned} f_0(m) &= \langle k_1 k_2 N m | \hat{\mathcal{K}}_+ \hat{\mathcal{K}}_- | k_1 k_2 N m \rangle \\ &= (2k_1 + \frac{1}{2}N - m - 1) (\frac{1}{2}N - m) + (2k_2 + \frac{1}{2}N + m - 1) (\frac{1}{2}N + m), \end{aligned} \quad (66)$$

$$\begin{aligned} f_1(m) &= -\langle k_1 k_2 N m | \hat{\mathcal{K}}_+ \hat{\mathcal{K}}_- | k_1 k_2 N, m+1 \rangle \\ &= -\left[ (2k_1 + \frac{1}{2}N - m - 1) (\frac{1}{2}N - m) (2k_2 + \frac{1}{2}N + m) (\frac{1}{2}N + m + 1) \right]^{\frac{1}{2}}. \end{aligned} \quad (67)$$

To determine asymptotic expressions for the functions  $\psi_n^{k_1 k_2 N}(m)$ , as eigenfunctions of  $\hat{\mathcal{K}}_+ \hat{\mathcal{K}}_-$ , we extend these functions of the discrete variable  $m$  to functions  $\Psi_n^{k_1 k_2 N}$  of a continuous variable  $x$  with the property that

$$\Psi_n^{k_1 k_2 N}(x) = \psi_n^{k_1 k_2 N}(x), \quad (68)$$

whenever  $x$  is in the domain of the discrete variable  $m$ . Thus, as for SU(2), we obtain an expression for  $\hat{\mathcal{K}}_+ \hat{\mathcal{K}}_-$  as the differential operator

$$\hat{\mathcal{K}}_+ \hat{\mathcal{K}}_- = F(x) + \hat{D} f_1(x) \hat{D}, \quad (69)$$

where  $F(x) = f_0(x) + f_1(x) + f_1(x-1)$  and  $\hat{D} = d/dx$ . With  $\kappa_1$  and  $\kappa_2$  defined by

$$2\kappa_1 := 2k_1 + \frac{1}{2}N, \quad 2\kappa_2 := 2k_2 + \frac{1}{2}N, \quad (70)$$

we also obtain the expressions

$$f_0(x) = (2\kappa_1 - x - 1) (\frac{1}{2}N - x) + (2\kappa_2 + x - 1) (\frac{1}{2}N + x) \quad (71)$$

$$f_1(x) = -\left[ (2\kappa_1 - x - 1) (\frac{1}{2}N - x) (2\kappa_2 + x) (\frac{1}{2}N + x + 1) \right]^{\frac{1}{2}}. \quad (72)$$

Provided the extension of  $\psi_n^{k_1 k_2 N}(m)$  to the smooth function  $\Psi_n^{k_1 k_2 N}(x)$  does not require the latter to be non-zero for any  $x$  that is outside of the limits for  $m$ , it is seen that  $f_1(x)$  is real for all  $x$  for which  $\Psi_n^{s_1 s_2 M}(x)$  is non-zero. The limits on the values of  $m$  are seen, from equation (56), to be such that  $-N/2 \leq m \leq N/2$ . Then, because the norm of the function  $\psi_n^{k_1 k_2 N}(m)$  is given by

$$\|\psi_n^{k_1 k_2 N}(m)\|^2 = \sum_{m=-N/2}^{N/2} |\psi_n^{k_1 k_2 N}(m)|^2, \quad (73)$$

it follows that the corresponding smooth function  $\Psi_n^{k_1 k_2 N}(x)$  should have the norm given by

$$\|\Psi_n^{k_1 k_2 N}\|^2 = \int_{-N/2}^{N/2} |\Psi_n^{k_1 k_2 N}(x)|^2 dx. \quad (74)$$

Also when evaluated without approximation,  $\Psi_n^{k_1 k_2 N}(x)$  is zero for all  $x > N/2$  and all  $x < -N/2$  and this integral can be extended to the range  $-\infty < x < \infty$ . The operator  $\hat{D} = d/dx$  is then seen to be skew Hermitian and  $\hat{\mathcal{K}}_+ \hat{\mathcal{K}}_-$  is Hermitian.

Now if the function  $\Psi_n^{k_1 k_2 N}(x)$  is sufficiently smooth, is non-zero over a narrow region of  $x$  within the limits  $-N/2 < x < N/2$  and is centred about a value  $x_0$ , we can again invoke the

SHA of dropping all terms that are more than bilinear in  $x - x_0$  and  $d/dx$  in an expansion of the operator  $\hat{K}_+\hat{K}_-$ . This gives

$$\hat{K}_+\hat{K}_- \approx E - \frac{1}{2}A \frac{d^2}{dx^2} + C(x - x_0) + \frac{1}{2}B(x - x_0)^2, \quad (75)$$

where

$$E = F(x_0), \quad A = -2f_1(x_0), \quad C = F'(x_0), \quad B = F''(x_0). \quad (76)$$

This approximation becomes precise, in the  $N \rightarrow \infty$  asymptotic limit, provided the parameters  $n, k_1$  and  $k_2$  remain finite. In fact, it turns out that  $k_1+k_2$  can also be large provided the difference  $|k_1 - k_2|$  remains small in comparison to  $N$ . The manner in which the SHA ceases to be valid for large values of  $n$  and  $|k_1 - k_2|$ , relative to  $N$ , is shown in section 3.3. Thus, we consider the SHA Hamiltonian

$$\mathcal{H} = E - \frac{1}{2}A \frac{d^2}{dx^2} + C(x - x_0) + \frac{1}{2}B(x - x_0)^2. \quad (77)$$

The appropriate value for  $x_0$  is that for which  $C = 0$  and

$$\mathcal{H} = E + \left[ -\frac{1}{2a^2} \frac{d^2}{dx^2} + \frac{1}{2}a^2(x - x_0)^2 \right] \hbar\omega, \quad (78)$$

with

$$(\hbar\omega)^2 = AB, \quad a^4 = \frac{B}{A}. \quad (79)$$

The eigenfunctions for this Hamiltonian are harmonic oscillator wavefunctions, and the eigenvalues are given by

$$E_n = E + \left(n + \frac{1}{2}\right)\hbar\omega. \quad (80)$$

With a standard choice of phase for the harmonic oscillator eigenfunctions, we then obtain the desired asymptotic Clebsch–Gordan coefficients, as given in equation (8).

### 3.2. Simplified analytical expressions for asymptotic $SU(1,1)$ CG coefficients

The SHA results given above are easily calculated and give accurate results for large values of  $N \rightarrow \infty$  and small values of  $n$ . Simpler analytical expressions are obtained if we further neglect terms in the SHA Hamiltonian which go to zero in the  $N \rightarrow \infty$  asymptotic limit. In these limits

$$\mathcal{H} \rightarrow \text{const.} - \frac{1}{2}A_0 \frac{d^2}{dx^2} + C_0x + \frac{1}{2}Bx^2, \quad (81)$$

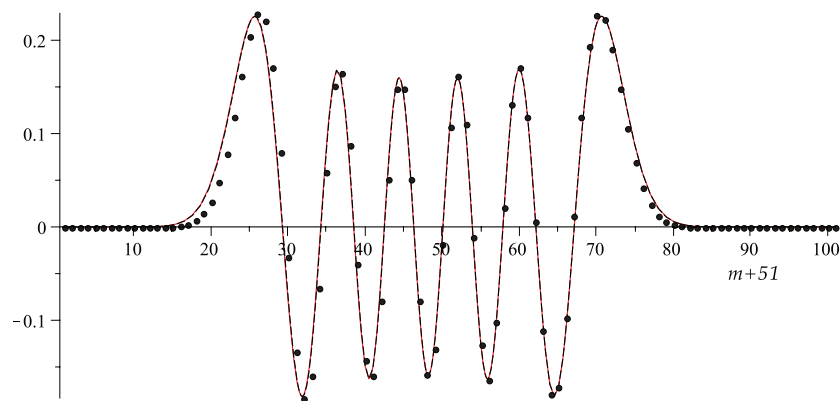
with

$$A_0 = 2N\sqrt{\kappa_1\kappa_2}, \quad (82)$$

$$C_0 = \frac{N + 4\sqrt{\kappa_1\kappa_2}}{2\sqrt{\kappa_1\kappa_2}}(\kappa_2 - \kappa_1), \quad (83)$$

$$B_0 = \frac{(N + 4\sqrt{\kappa_1\kappa_2})^2}{2N\sqrt{\kappa_1\kappa_2}}. \quad (84)$$

Thus, we can evaluate the asymptotic Clebsch–Gordan coefficients from equation (8) with  $a^2 = \sqrt{B_0/A_0}$  and  $x_0 = -C_0/B_0$  given explicitly in equation (9).



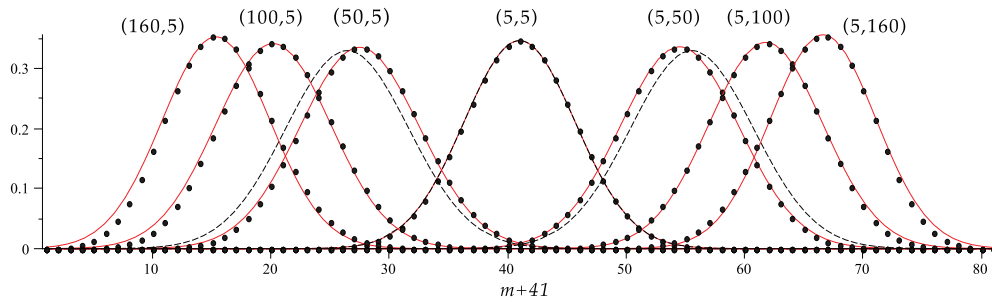
**Figure 4.** SU(1,1) Clebsch–Gordan coefficients  $(-1)^m \langle 10, 17, 100, m | 17, 10 \rangle$  shown as a function of  $m$ . Exact values are shown as full dots, full SHA values as continuous (red) lines and simplified SHA values as dashed lines.

### 3.3. Numerical results for $su(1,1)$ CG coefficients

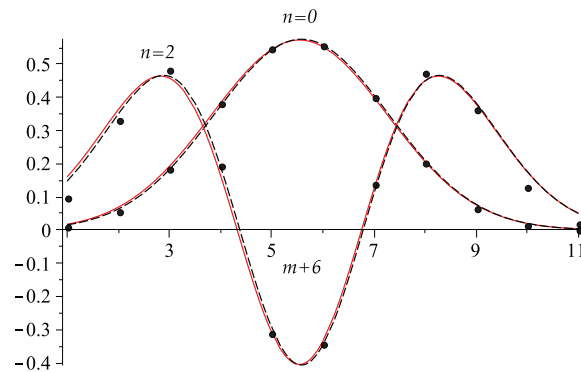
We have ascertained and the following examples illustrate that the SU(1,1) CG coefficients, when multiplied by a phase factor  $(-1)^{N+m}$ , satisfy the conditions for the validity of the SHA in the specified asymptotic limit as  $N \rightarrow \infty$  for finite values of  $n$  and  $|k_1 - k_2|$ . The restriction on the value of  $n$  is because the harmonic oscillator wavefunctions become broad and oscillate rapidly for large values of  $n$  with the result that if  $n/N$  is too large the conditions for the validity of the SHA cease to be satisfied. Likewise, the restriction on the value of  $|k_1 - k_2|$  is understood to arise from the value of  $C_0$  which for  $N \rightarrow \infty$  approaches  $C_0 \sim 3(k_2 - k_1)$ . Thus, unless  $|k_1 - k_2| \ll N$ , it is not guaranteed that the centroid,  $x_0 \approx -C_0/B_0$ , of the SHA wavefunction will be sufficiently close to the centre of the domain  $-N/2 \leq x \leq N/2$  for the SHA to be a valid approximation. However, the results below show that in addition to being precise in the asymptotic limits, the SHA (as opposed to the simplified SHA) remains remarkably accurate for relatively large values of  $|k_1 - k_2|$  when  $N$  is large.

The following figures compare the values of exactly computed SU(1,1) CG coefficients  $\langle k_1 k_2 N m | K n \rangle$  with the SHA expressions given by equation (8). The continuous (red) lines are those of the full SHA approximation with the parameters as defined in section 3.1. The (black) dashed lines are those of the simplified analytical SHA coefficients with parameters given by equation (9). The asymptotic limits require that  $n$  remains finite while  $N \rightarrow \infty$ . The simplified SHA coefficients retain the property of being precise in the  $N \rightarrow \infty$  limit but requires, in addition, that  $|k_1 - k_2|$  remains small compared to  $N/2$ . In fact, as the following figures illustrate the SHA coefficients yield surprisingly good approximations for quite modest values of  $N$  and the analytical approximations are seen to be almost indistinguishable from the full SHA expressions when  $|k_1 - k_2|$  is not too large.

Figure 4 illustrates the accuracy that can be obtained with the above-defined asymptotic SU(1,1) CG coefficients for  $N = 100$ ,  $n = 10$  and  $k_2 - k_1 = 7$ . It shows that the SHA coefficients and the analytical approximations to them are virtually indistinguishable. It also shows that the errors in the asymptotic coefficients for a large but finite value of  $N$  start to become most noticeable, for these relatively large values of  $n$  and  $|k_1 - k_2|$ , at the upper and lower reaches of  $m$ .



**Figure 5.** SU(1,1) Clebsch–Gordan coefficients  $(-1)^m \langle k_1 k_2, 80, m | K, 0 \rangle$  with  $K = 80 + k_1 + k_2$  for a range of  $(k_1, k_2)$  values shown as functions of  $m$ . Exact values are shown as full dots, full SHA values as continuous (red) lines, and simplified SHA values as dashed lines. The simplified SHA values for  $|k_1 - k_2| > 45$  are unacceptably inaccurate and are not shown.



**Figure 6.** SU(1,1) Clebsch–Gordan coefficients  $(-1)^m \langle 1/2, 3/2, 10, m | 12, 0 \rangle$  and  $\langle 1/2, 3/2, 10, m | 10, 2 \rangle$  shown as a function of  $m$ . Exact values are shown as full dots, full SHA values as continuous (red) lines and analytical asymptotic limits of the SHA values as dashed lines.

Figure 5 shows the SU(1,1) CG coefficients for a range of  $k_1 - k_2$  values. The calculations for other  $k_1 - k_2$  values show that the simplified SHA coefficients are close to those of the full SHA for  $|k_1 - k_2| \lesssim 25$  but are noticeably different for larger values of  $|k_1 - k_2|$  as seen for the  $(k_1, k_2) = (5, 50)$  and  $(50, 5)$  coefficients shown in the figure. It is also seen that, for the full SHA, errors start to become evident for large values of  $|k_1 - k_2|$  as the value of  $m$  approaches its upper or lower bounds.

The SU(1,1) CG coefficients  $\langle k_1 k_2 N m | K n \rangle$  are given exactly both in the SHA and in the simplified analytical approximation to the SHA for finite values of  $n$  and  $|k_1 - k_2|$  in the  $N \rightarrow \infty$  asymptotic limit. However, the full SHA expressions remain noticeably more accurate over a considerably larger domain. In fact, the coefficients continue to be accurate for relatively small values of  $N$  and even smaller values of  $n$  and  $|k_1 - k_2|$ . This is illustrated for  $N = 10$  in figure 6. The region in which the results are at their worst is for values of  $m$  close to their limits, especially in situations in which the asymptotic expressions extend beyond these limits.

#### 4. Comparisons with the random-phase approximation (RPA)

Whereas the SHA and RPA are both harmonic oscillator approximations and both become precise in asymptotic limits, they apply in different complementary situations. In their application to the derivation of asymptotic SU(2) and SU(1,1) CG coefficients, the RPA can be viewed as a contraction of the su(2) Lie algebra to a harmonic oscillator boson algebra. Thus, we consider the possibility that the SHA might also correspond to such a contraction, albeit one that is valid in a different domain of an SU(2) representation space.

##### 4.1. SU(2) CG coefficients

We start from the observation that SU(2) CG coefficients are the eigenstates of a Hamiltonian

$$\hat{H} = \alpha \hat{S}_0 + \chi \hat{S}_+ \hat{S}_- \quad (85)$$

on the tensor product space of two su(2) irreps,  $\{s_1\} \otimes \{s_2\}$  in a basis of product states  $\{|s_1 m_1\rangle \otimes |s_2 m_2\rangle\}$ . The Hamiltonian  $\hat{H}$  has eigenstates  $\{|SM\rangle\}$ , with eigenvalues given by

$$E_{SM} = \alpha M + \chi[S(S+1) - M(M-1)]. \quad (86)$$

Inspection shows that, when  $\chi > -\alpha$  for  $\alpha > 0$ , the lowest-energy eigenstate of  $\hat{H}$  is the state with  $S = \sigma$ ,  $M = -S$ , where  $\sigma = s_1 + s_2$  and, when  $\chi < -\alpha$ , the lowest-energy eigenstate is the state with  $S = \sigma$ ,  $M = 0$ . Thus, the two situations are relevant for the calculation of asymptotic SU(2) CG coefficients  $(s_1 m_1 s_2 m_2 | SM)$  for large values of  $s_1$ ,  $s_2$  and  $S$ , with  $M$  close to either  $-S$  or 0, respectively. As we now observe, RPA gives solutions in the first scenario whereas the SHA does so in the second.

In the  $\sigma \rightarrow \infty$  asymptotic limit for  $\alpha > 0$  and  $\chi > -\alpha$ , it is convenient to relabel the low-lying states by  $|SM\rangle \rightarrow |nm\rangle$ , where  $n = \sigma - S$  and  $m = S + M$ . The energy  $\mathcal{E}_{nm} = E_{SM}$  is then given to leading order in  $n$  and  $m$  by

$$\mathcal{E}_{nm} \sim -\alpha\sigma + \alpha n + [\alpha + \chi(2\sigma + 1)]m, \quad (87)$$

which is an eigenvalue of a Hamiltonian expressed in terms of the raising and lowering operators of two simple harmonic oscillators by

$$\hat{\mathcal{H}} = -\alpha\sigma + \alpha c^\dagger c + [\alpha + \chi(2\sigma + 1)]b^\dagger b, \quad (88)$$

with

$$c^\dagger c |nm\rangle = n |nm\rangle, \quad b^\dagger b |nm\rangle = m |nm\rangle. \quad (89)$$

For asymptotically large values of  $s_1$  and  $s_2$ , this result can be obtained in the RPA by contracting the su(2) algebra in each of the  $\{s_1\}$  and  $\{s_2\}$  irreps by

$$\hat{S}_+ \sim \frac{1}{\sqrt{s}} a^\dagger, \quad \hat{S}_- \sim \frac{1}{\sqrt{s}} a, \quad \hat{S}_0 = -s + a^\dagger a \quad (90)$$

for  $s = s_1$  and  $s = s_2$ , respectively. Diagonalization of the Hamiltonian  $\hat{H}$  of equation (85) in this contraction limit, then leads to the asymptotic expression (88) and determines the  $c$  and  $b$  boson operators from which one can derive asymptotic CG coefficients. The contraction (90) which leads to this RPA result is valid in the domain of states  $\{|SM\rangle\}$  in which  $M$  is close to  $-S$  and  $S$  is close to  $s_1 + s_2$  for large values of  $s_1$  and  $s_2$ .

For  $\chi < -\alpha$ , the RPA breaks down because the ground state of the Hamiltonian suddenly flips from an  $|S, M = -S\rangle$  state to an  $|S, M = 0\rangle$  state at  $\chi = -\alpha$ . However, for  $\chi < -\alpha$  the SHA provides asymptotic solutions.

In the  $\sigma \rightarrow \infty$  asymptotic limit for  $\chi < -\alpha$ , it is convenient to relabel the low-lying states by  $|SM\rangle \rightarrow |nM\rangle$ , where  $n = \sigma - S$ . It is then found that  $\mathcal{E}_{nM} = E_{SM}$  is given to leading order in  $n$  by

$$\mathcal{E}_{nM} \approx \chi\sigma(\sigma + 1) - \chi(2\sigma + 1)n + \alpha M - \chi M(M - 1). \quad (91)$$

Thus, in this asymptotic limit, the spectrum of eigenvalues of the Hamiltonian (85) is that of a simple harmonic oscillator coupled to a U(1) rotor

$$\hat{\mathcal{H}} = \chi\sigma(\sigma + 1) - \chi(2\sigma + 1)c^\dagger c + \alpha\hat{S}_0 - \chi\hat{S}_0(\hat{S}_0 - 1), \quad (92)$$

where

$$c^\dagger c|nM\rangle = n|nM\rangle, \quad \hat{S}_0|nM\rangle = M|nM\rangle. \quad (93)$$

The SHA gives an explicit expression for a shifted harmonic oscillator that is essentially equivalent to  $\hat{\mathcal{H}}$ . We have not succeeded in deriving this Hamiltonian by a contraction of the su(2) algebra. However, it is noted that the SHA is based on a realization of the su(2) Lie algebra given by the expressions, in terms of the harmonic oscillator operators,  $\hat{x} = x$ ,  $\hat{p} = -i\hbar d/dx$

$$\hat{S}_+ \rightarrow \hat{\mathcal{S}}_+ = e^{-i\hat{p}}[s(s + 1) - \hat{x}(\hat{x} + 1)]^{1/2}, \quad (94)$$

$$\hat{S}_- \rightarrow \hat{\mathcal{S}}_- = [s(s + 1) - \hat{x}(\hat{x} + 1)]^{1/2}e^{i\hat{p}}, \quad (95)$$

$$\hat{S}_0 \rightarrow \hat{\mathcal{S}}_0 = x. \quad (96)$$

Thus, it would appear likely that the SHA Hamiltonian (92) could be obtained as a contraction of this realization.

#### 4.2. SU(1,1) CG coefficients

We now compare the complementary RPA and SHA derivations of asymptotic SU(1,1) CG coefficients by diagonalization of a Hamiltonian

$$\hat{H} = \alpha\hat{K}_0 + \chi\hat{K}_+\hat{K}_-. \quad (97)$$

The spectrum of eigenstates of this Hamiltonian are the SU(1,1) states  $\{|Kn\rangle\}$  with eigenvalues given by

$$E_{Kn} = \alpha(K + n) + \chi(2K + n - 1)n, \quad (98)$$

or, with a relabelling of states by  $|Kn\rangle \rightarrow |Nn\rangle$ , where  $N = K + n - k_1 - k_2$  by

$$\mathcal{E}_{Nn} = \alpha(k_1 + k_2 + N) + \chi(2k_1 + 2k_2 + 2N - n - 1)n. \quad (99)$$

For  $\alpha$  and  $\chi$  positive, the lowest value of  $\mathcal{E}_{Nn} = E_{Kn}$  is for  $n = N = 0$ . Then, for asymptotically large values of  $k_1$  and  $k_2$ , the low-lying eigenvalues are given to leading order in  $N$  and  $n$  by

$$\mathcal{E}_{Nn} \sim \alpha(k_1 + k_2) + \alpha N + \chi(2k_1 + 2k_2 - 1)n, \quad (100)$$

which are the eigenvalues of the Hamiltonian

$$\hat{H} \sim \alpha(k_1 + k_2) + \alpha c^\dagger c + \chi(2k_1 + 2k_2 - 1)b^\dagger b \quad (101)$$

for a two-harmonic-oscillator system with

$$c^\dagger c|Nn\rangle = N|Nn\rangle, \quad b^\dagger b|Nn\rangle = n|Nn\rangle. \quad (102)$$

The corresponding eigenvectors, in the space of coupled tensor product states  $\{|k_1n_1\rangle \otimes |k_2n_2\rangle\}$ , and hence a subset of asymptotic SU(1,1) CG coefficients, can be obtained in the RPA



by a contraction of the two  $\text{su}(1,1)$  irreps  $\{k_1\}$  and  $\{k_2\}$ . The relevant contraction is obtained by observing that for small values of  $n$ , the  $\text{su}(1,1)$  relationships

$$[\hat{K}_-, \hat{K}_+]|kn\rangle = 2\hat{K}_0|kn\rangle = 2(k+n)|kn\rangle, \quad [\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm \quad (103)$$

are satisfied in the  $k \rightarrow \infty$  asymptotic limit for finite values of  $n$  by

$$\frac{1}{\sqrt{2k}}\hat{K}_+ \sim a^\dagger, \quad \frac{1}{\sqrt{2k}}\hat{K}_- \sim a, \quad \hat{K}_0 \sim k + a^\dagger a. \quad (104)$$

Thus, by expressing the Hamiltonian of equation (97) in terms of the  $\text{su}(1,1)$  operators of the  $\{k_1\}$  and  $\{k_2\}$  sub-representations, it becomes the Hamiltonian for two coupled harmonic oscillators which, in the RPA, are decoupled by diagonalization to give the uncoupled oscillator expression of equation (101). The contraction (104) which leads to this RPA result is valid in the domain of coupled states  $\{|Kn\rangle\}$  in which  $K$  is close to  $k_1 + k_2$  and  $n$  is small.

Other  $\text{su}(1,1)$  CG coefficients are obtained by considering the eigenvectors of the Hamiltonian (97) in the subspace of states of a fixed value of  $N$ . For asymptotically large  $N$  and small  $n$ , equation (99) then reduces to the eigenvalues

$$\mathcal{E}_{Nn} \sim \alpha(k_1 + k_2 + N) + 2\chi(k_1 + k_2 + N)n \quad (105)$$

of the Hamiltonian

$$\hat{H} \sim \alpha(k_1 + k_2 + N) + 2\chi(k_1 + k_2 + N)b^\dagger b \quad (106)$$

for a simple harmonic oscillator with  $b^\dagger b|Nn\rangle = n|Nn\rangle$ . The SHA gives an explicit expression for this Hamiltonian as a shifted harmonic oscillator Hamiltonian. We have not succeeded in deriving this expression in terms of a contraction of the  $\text{su}(1,1)$  algebra. However, the SHA is based on an approximation to a realization of the  $\text{su}(1,1)$  Lie algebra given by the expressions in terms of the harmonic oscillator operators  $\hat{x} = x$ ,  $\hat{p} = -i\hbar d/dx$ ,

$$\hat{K}_+ \rightarrow \hat{\mathcal{K}}_+ = e^{-i\hat{p}}[(2k + \hat{x})(\hat{x} + 1)]^{1/2}, \quad (107)$$

$$\hat{K}_- \rightarrow \hat{\mathcal{K}}_- = [(2k + \hat{x})(\hat{x} + 1)]^{1/2}e^{i\hat{p}}, \quad (108)$$

$$\hat{K}_0 \rightarrow \hat{\mathcal{K}}_0 = k + \hat{x}. \quad (109)$$

Thus, it would appear once again likely that the Hamiltonian (106) can be obtained as a contraction of this realization.

## 5. Discussion and conclusion

In this paper we have demonstrated that turning a three-term recursion relation into a second-order differential equation makes it easy to understand the oscillatory nature of CG coefficients. From a more numerical perspective, it has been noted that the final forms given in the paper, even the simplified expressions, are remarkably accurate even for values of the parameters that are far from the asymptotic limits in which they become precise.

It is readily ascertained that our asymptotic  $\text{SU}(2)$  CG coefficients retain the symmetry

$$(s_1 m_1 s_2 m_2 | s_1 + s_2 - n, M) = (-1)^n (s_2 m_2 s_1 m_1 | s_1 + s_2 - n, M) \quad (110)$$

under the exchange  $s_1 m_1 \leftrightarrow s_2 m_2$ . Indeed, this is seen from the simplified expression of equation (3) in which the argument of the Hermite polynomial simply changes its sign under this exchange. A parallel result is obtained for an  $\text{SU}(1,1)$  CG coefficient for which, in the sign convention used,

$$C(k_1, n_1, k_2, n_2; K, n) = (-1)^n C(k_2, n_2, k_1, n_1; K, n). \quad (111)$$

Due to the asymmetrical way in which the asymptotic limits are approached, it is not expected that the more general symmetries of these CG coefficients will be preserved. However, such symmetries as are known, e.g., for the exchange  $s_1 m_1 \leftrightarrow s_3, -m_3$  can be used to re-arrange the arguments of a given CG coefficient such that the value of  $n$  is minimized. The effective range over which the asymptotic approximations are expected to produce acceptable results is thereby increased.

More generally, the successes of the SHA in its applications to date [1–3] suggest that it is a potentially powerful technique that could be applied more generally. Note, for example, that the finite and discrete series of irreps of all semi-simple Lie algebras are characterized by the irreps of their many  $su(2)$  and/or  $su(1,1)$  subalgebras generated by raising and lowering operators. In particular, every pair of raising and lowering operators,  $X_v^\pm$ , of a semi-simple Lie algebra can be normalized to have  $SU(2)$  commutation relations  $[X_v^+, X_v^-] = h_v$ ,  $[h_v, X^\pm] = \pm 2X_v^\pm$ . Note also that many dynamical systems have low-energy collective states that are well approximated as harmonic vibrational states. As a result the RPA (random phase approximation) has become a powerful tool in many-body theory. It is also of interest to explore the possibility of mapping the algebraic structure of a problem to a rotor algebra, for example, rather than that of a harmonic vibrator.

The search for an extension of the SHA to apply to many-body systems with more general algebraic structures is worthwhile because it is known that, for a variety of many-body systems, the RPA works well for describing vibrational normal mode excitations with relatively weak interactions but breaks down when the interactions become too strongly attractive. This is the situation in which the system is understood to undergo a phase change; the frequency of one of the vibrational modes goes to zero and a deformed equilibrium state, with rotational plus vibrational degrees of freedom, emerges. Thus, we are optimistic that a generalized SHA will prove to be appropriate for such situations. Section 4 has shown the SHA to be complementary to the RPA its ability to give (asymptotic) solutions to an eigenvalue equation in regions where the RPA breaks down. Moreover, the SHA has been shown, in previous applications, to provide practical solutions of the multi-level BCS Hamiltonian that are remarkably accurate for relatively strong pairing interactions [3].

### Appendix. Special case

The special case

$$(s_1, -m; s_2, m | s_1 + s_2, 0) \sim \left( \frac{1}{\pi} \left( \frac{s_1 + s_2}{s_1 s_2} \right) \right)^{\frac{1}{4}} e^{-m^2(s_1+s_2)/2s_1s_2} \quad (\text{A.1})$$

follows by careful application of Stirling's formula

$$s! \sim \sqrt{\frac{2\pi}{s}} s^{s+1/2} e^{-s} \quad (\text{A.2})$$

to the exact expression

$$(s_1, -m, s_2, m | s_1 + s_2, 0) = \left[ \frac{(2s_1)!(2s_2)!(s_1 + s_2)!(s_1 + s_2)!}{(2s_1 + 2s_2)!(s_1 + m)!(s_1 - m)!(s_2 + m)!(s_2 - m)!} \right]^{\frac{1}{2}}. \quad (\text{A.3})$$

First, one can use equation (A.2) to show that

$$\frac{s!s!}{(s+m)!(s-m)!} \sim \sqrt{\frac{s^2 - m^2}{s^2}} \frac{s^{2s+2}}{(s+m)^{s+m+1}(s-m)^{s-m+1}}. \quad (\text{A.4})$$

With  $x = m/s$ , this expression reduces to

$$\frac{s!s!}{(s+m)!(s-m)!} \sim \frac{1}{\sqrt{1-x^2}} \left[ \frac{(1-x)^x}{(1+x)^x} \frac{1}{(1-x^2)} \right]^s \quad (\text{A.5})$$

and, for a finite value of  $m$ , reduces further in the limit as  $s \rightarrow \infty$  to

$$\frac{s!s!}{(s+m)!(s-m)!} \sim \frac{1}{(1+m^2/s^2)^s} = e^{-m^2/s}. \quad (\text{A.6})$$

Equation (A.3) can then be manipulated to directly yield equation (A.1).

As a special case we observe that

$$(s, -m; s, m|2s, 0) = \frac{(2s)!(2s)!}{\sqrt{(4s)!}(s+m)!(s-m)!}, \quad (\text{A.7})$$

$$= \left( \frac{2}{\pi s} \right)^{\frac{1}{4}} e^{-m^2/s}. \quad (\text{A.8})$$

Both equations (A.1) and (A.8) agree with equation (2) for  $n = 0$  and  $M = 0$  in the limit where  $s_1, s_2 \rightarrow \infty$ .

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