JOURNAL OF MATHEMATICAL PHYSICS **54**, 023508 (2013)

The Poincaré algebra in rank 3 simple Lie algebras

Andrew Douglas,[1,a)] Hubert de Guise,[2,b)] and Joe Repka[3,c)]
[1]*CUNY Graduate Center and New York City College of Technology, City University of New York 11201, USA*
[2]*Department of Physics, Lakehead University, Canada*
[3]*Department of Mathematics, University of Toronto, Canada*

(Received 27 August 2012; accepted 17 January 2013; published online 15 February 2013)

We classify embeddings of the Poincaré algebra $\mathfrak{p}(3, 1)$ into the rank 3 simple Lie algebras. Up to inner automorphism, we show that there are exactly two embeddings of $\mathfrak{p}(3, 1)$ into $\mathfrak{sl}(4, \mathbb{C})$, which are, however, related by an outer automorphism of $\mathfrak{sl}(4, \mathbb{C})$. Next, we show that there is a unique embedding of $\mathfrak{p}(3, 1)$ into $\mathfrak{so}(7, \mathbb{C})$, up to inner automorphism, but no embeddings of $\mathfrak{p}(3, 1)$ into $\mathfrak{sp}(6, \mathbb{C})$. All embeddings are explicitly described. As an application, we show that each irreducible highest weight module of $\mathfrak{sl}(4, \mathbb{C})$ (not necessarily finite-dimensional) remains indecomposable when restricted to $\mathfrak{p}(3, 1)$, with respect to any embedding of $\mathfrak{p}(3, 1)$ into $\mathfrak{sl}(4, \mathbb{C})$.
 [http://dx.doi.org/10.1063/1.4790415]

I. INTRODUCTION

The Poincaré group $P(3, 1)$, also known as the inhomogeneous Lorentz group, is the Lie group of isometries of Minkowski space-time. It is the noncompact, semidirect product group $P(3, 1) \cong O(3, 1) \ltimes \mathbb{R}^4$. The complexification of the Lie algebra of the Poincaré group is the Poincaré algebra $\mathfrak{p}(3, 1)$:

$$\mathfrak{p}(3, 1) \cong (\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})) \in (\mathbb{C}^2 \otimes \mathbb{C}^2). \tag{1}$$

The Abelian subalgebra $\mathbb{C}^2 \otimes \mathbb{C}^2$ of $\mathfrak{p}(3, 1)$ is an irreducible representation of the semisimple Lie algebra $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$.

In this article, we classify embeddings of the Poincaré algebra $\mathfrak{p}(3, 1)$ into the rank 3 simple Lie algebras. Up to inner automorphism, we show that there are exactly two embeddings of $\mathfrak{p}(3, 1)$ into $\mathfrak{sl}(4, \mathbb{C})$, which are, however, related by an outer automorphism of $\mathfrak{sl}(4, \mathbb{C})$ (Sec. VII). We then show that there is a unique embedding of $\mathfrak{p}(3, 1)$ into $\mathfrak{so}(7, \mathbb{C})$, up to inner automorphism (Sec. VIII), but no embeddings of $\mathfrak{p}(3, 1)$ into $\mathfrak{sp}(6, \mathbb{C})$ (Sec. IX).

In Sec. VII, we also show that each irreducible highest weight module of $\mathfrak{sl}(4, \mathbb{C})$ (not necessarily finite-dimensional) remains indecomposable when restricted to $\mathfrak{p}(3, 1)$, with respect to any embedding of $\mathfrak{p}(3, 1)$ into $\mathfrak{sl}(4, \mathbb{C})$: We thus create a large family of indecomposable representations of $\mathfrak{p}(3, 1)$. There has been considerable work on infinite-dimensional, irreducible representations of $\mathfrak{p}(3, 1)$ (see, for instance, Refs. 12 and 13), but very little on finite-dimensional, indecomposable representations of $\mathfrak{p}(3, 1)$ (one example is Ref. 11).

In Secs. III and IV we describe the semisimple Lie algebra $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, and the rank 3 simple Lie algebras, respectively. Section V contains additional definitions and notation that are used in the following sections. In Sec. VI, we classify the embeddings of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ into the rank 3 simple Lie algebras, which will be used in the classification of embeddings of $\mathfrak{p}(3, 1)$ into the rank 3 simple Lie algebras. We begin with a discussion of the role of the Poincaré group and algebra in physics in Sec. II.

---

a)E-mail: adouglas2@gc.cuny.edu.
b)E-mail: hubert.deguise@lakeheadu.ca.
c)E-mail: repka@math.toronto.edu.

We end the section by noting that there is some interesting work on embeddings of the (real) Lie algebra Lie($P(3, 1)$) of the Poincaré group. Specifically, Doebner and Hennig[3] classified embeddings of Lie($P(3, 1)$) into the real Lie algebra $\mathfrak{so}(4, 2)$, up to inner automorphism.

Complexifying a representative of each equivalence class of embeddings of Lie($P(3, 1)$) into $\mathfrak{so}(4, 2)$ gives us a set of embeddings of $\mathfrak{p}(3, 1)$ into $\mathfrak{sl}(4, \mathbb{C})$, which may or may not be pairwise inequivalent. This list, of course, is not necessarily a complete list of embeddings of $\mathfrak{p}(3, 1)$ into $\mathfrak{sl}(4, \mathbb{C})$: For instance, the complexification of any embedding of any real form of $\mathfrak{p}(3, 1)$ into any real form of $\mathfrak{sl}(4, \mathbb{C})$ will produce an embedding of $\mathfrak{p}(3, 1)$ into $\mathfrak{sl}(4, \mathbb{C})$, which may be inequivalent to those obtained from Lie($P(3, 1)$) and $\mathfrak{so}(4, 2)$.

II. THE POINCARÉ GROUP AND ALGEBRA IN PHYSICS

The Poincaré group is the basic symmetry group of special relativity. Applying the usual argument of isotropy and homogeneity of spacetime, one concludes that Lagrangians (and the theories they engender) should be invariant under Poincaré transformations. In addition to the basic invariant $x^2 + y^2 + z^2 - (ct)^2$, the (Abelian) generators of spacetime translations can also be made into the scalar $p_x^2 + p_y^2 + p_z^2 - (E/c)^2 = -(mc)^2$, with m the rest mass of a particle (from which the famous formula follows for particles at rest, with $p_x = p_y = p_z = 0$).

Unitary representations of the Poincaré group, studied using induced representation theory, were the topic of the seminal work of Wigner.[17] In an heroic paper,[16] Ström obtained representations of the Poincaré group by means of contractions of representations of the de Sitter group $O(4, 1)$ (see also Ref. 15).

The connections between pseudo-orthogonal groups and $P(3, 1)$ have a long history. On the one hand, $P(3, 1)$ is not only a contraction of $O(4, 1)$ but also of $O(3, 2)$. On the other, $P(3, 1)$ is a subgroup of the conformal group $SO(4, 2)$ (locally isomorphic to $SU(2, 2)$, the largest symmetry group of Maxwell's equations from which basic postulate of special relativity, the invariance of the speed of light c, follows). Infinite-dimensional, irreducible representations of $SO(4, 2)$ and $SU(2, 2)$ which remain irreducible when restricted to $P(3, 1)$ were studied in Refs. 12 and 13 and found to describe 0-mass particles.

There is renewed interest in spacetime symmetries and the role of the Poincaré group (or rather, the Poincaré algebra) in the context of the so–called "Doubly Special Relativity" (DSR) theory (see the recent reports,[1,2,8]). The premise behind DSR is that, in addition to the speed of light c, there should exist another fundamental invariant related to Planck–scale physics, which would lead to a deformation of the Poincaré algebra.

III. THE SEMISIMPLE LIE ALGEBRA $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ AND ITS REPRESENTATIONS

The special linear algebra $\mathfrak{sl}(2, \mathbb{C})$ is the Lie algebra of traceless 2×2 matrices with complex entries. It is the Lie algebra of type A_1. Recall that for each nonnegative integer n there is an $(n+1)$-dimensional, irreducible $\mathfrak{sl}(2, \mathbb{C})$-module $V(n)$ (for a description of $V(n)$ see Ref. 5 or Ref. 7). Further, every finite-dimensional, irreducible $\mathfrak{sl}(2, \mathbb{C})$-module is equivalent to $V(n)$ for some nonnegative integer n.

The representations of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ are constructed from those of $\mathfrak{sl}(2, \mathbb{C})$. If V_1 and V_2 are $\mathfrak{sl}(2, \mathbb{C})$-modules, then $V_1 \otimes V_2$ is an $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$-module with action

$$(L_1, L_2) \cdot (v_1 \otimes v_2) = (L_1 \cdot v_1) \otimes v_2 + v_1 \otimes (L_2 \cdot v_2). \tag{2}$$

We have the following well-known theorem classifying the finite dimensional, irreducible representations of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ (see, for instance, Ref. 6).

Theorem 3.1: *The finite-dimensional* $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ *representation* V *is irreducible if and only if* $V \cong V(n) \otimes V(m)$ *for some* n*, and* $m \in \mathbb{Z}_{\geq 0}$*, uniquely determined by* V*.*

Remark 3.2: Let V be a finite-dimensional $\mathfrak{p}(3, 1)$-module and consider V restricted to the subalgebra $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Since the category of finite-dimensional $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$-modules is semisimple, V decomposes into irreducible $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$-modules.

IV. THE RANK 3 SIMPLE LIE ALGEBRAS AND THEIR REPRESENTATIONS

The special linear algebra $\mathfrak{sl}(4, \mathbb{C})$ is the 15-dimensional Lie algebra of traceless 4×4 matrices with complex entries. It is the simple Lie algebra of type A_3. The special orthogonal algebra $\mathfrak{so}(7, \mathbb{C})$, of type B_3 is the 21-dimensional simple Lie algebra of complex 7×7 matrices N satisfying $N^{tr} = -N$. The symplectic algebra $\mathfrak{sp}(6, \mathbb{C})$ is the 21-dimensional Lie algebra of 6×6 complex matrices N satisfying $K X^{\mathrm{tr}} K = X$ where K is the 6×6 matrix

$$K = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{3}$$

It is the simple Lie algebra of type C_3. Note that the simple Lie algebra of type D_3, namely $\mathfrak{so}(6, \mathbb{C})$, is isomorphic to $\mathfrak{sl}(4, \mathbb{C})$.

We may define the rank 3 simple Lie algebra $\mathfrak{g}$ by a set of generators $\{H_i, X_i, Y_i\}_{1 \leqslant i \leqslant 3}$ together with the Chevalley-Serre relations:[7]

$$\begin{aligned} &[H_i, H_j] = 0, && [H_i, X_j] = M^{\mathfrak{g}}_{ji} X_j, \\ &[H_i, Y_j] = -M^{\mathfrak{g}}_{ji} Y_j, && [X_i, Y_j] = \delta_{ij} H_i, \\ &(\mathrm{ad}X_i)^{1-M^{\mathfrak{g}}_{ji}}(X_j) = 0, && (\mathrm{ad}Y_i)^{1-M^{\mathfrak{g}}_{ji}}(Y_j) = 0, \quad \text{when } i \neq j, \end{aligned} \tag{4}$$

where $1 \leqslant i, j \leqslant 3$, and $M^{\mathfrak{g}}$ is the Cartan matrix of $\mathfrak{g}$ (see Ref. 7). The X_i, for $1 \leqslant i \leqslant 3$, correspond to the simple roots.

We now consider the representations of the simple Lie algebra $\mathfrak{g}$.

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ with basis H_1, H_2, H_3, define $\alpha_i, \lambda_i \in \mathfrak{h}^*$ by $\alpha_i(H_j) = M^{\mathfrak{g}}_{ji}$, and $\lambda_i(H_j) = \delta_{ij}$. The λ_i are the *fundamental weights*.

For each $\lambda = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 \in \mathfrak{h}^*$, with nonnegative integers m_1, m_2, m_3, there exists a finite-dimensional, irreducible $\mathfrak{g}$-module denoted $V_{\mathfrak{g}}(\lambda)$, with highest weight λ, and every finite-dimensional irreducible $\mathfrak{g}$-module is of this form, for some λ. The representations $V_{\mathfrak{g}}(\lambda_i)$ for $1 \leqslant i \leqslant 3$ are the *fundamental representations*.

The representation $V_{\mathfrak{g}}(\lambda)$ is realized as the quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ by the left ideal, J_λ, generated by X_i, $H_i - \lambda(H_i)$, $Y_i^{1+\lambda(H_i)}$, $1 \leqslant i \leqslant 3$ (here the action of $\mathcal{U}(\mathfrak{g})$ on itself and on $V_{\mathfrak{g}}(\lambda)$ is given by left multiplication). We will denote the element $1 + J_\lambda$ of $V_{\mathfrak{g}}(\lambda)$ by $\widetilde{u}$. Then one can show that $V_{\mathfrak{g}}(\lambda)$ is generated by $\{Y_{i_1} \cdots Y_{i_l}\widetilde{u} : l \in \mathbb{N}_0, i_1, \ldots, i_l \in \{1, 2, 3\}\}$. The weight of $Y_{i_1} \cdots Y_{i_l}\widetilde{u}$ is $\lambda - \Sigma_{j=1}^{l}\alpha_{i_j}$.

V. ADDITIONAL DEFINITIONS AND NOTATION

The following definitions and notation will be used in this article. Let $\mathfrak{g}$ be a simple Lie algebra.

- Let X_{a_i} correspond to a simple root of $\mathfrak{g}$, for $1 \leq a_i \leq \mathrm{rank}(\mathfrak{g})$. We then define

$$X_{a_1,a_2,a_3,\ldots,a_k} \equiv [[\ldots[[X_{a_1}, X_{a_2}], X_{a_3}], \ldots], X_{a_k}].$$

  $Y_{a_1,a_2,a_3,\ldots,a_k}$ is defined analogously.
- Let $\mathfrak{g}'$ be a subalgebra of $\mathfrak{g}$, and $W \in \mathfrak{g}$. We then define $[W]_{\mathfrak{g}'}$ to be the $\mathfrak{g}'$-submodule inside $\mathfrak{g}$ generated by the vector W. Given an embedding $\varphi : \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{g}$, we will be interested in the subalgebra $\varphi(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}))$ of $\mathfrak{g}$. We shall abbreviate $[W]_{\varphi(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))}$ to $[W]_\varphi$ when no ambiguity arises.
- Let $W_1, \ldots, W_m \in \mathfrak{g}$. Then,

$$\langle W_1, \ldots, W_m \rangle$$

  is the subalgebra of $\mathfrak{g}$ generated by $W_1, \ldots, W_m$.

- A *lift* of the embedding $\varphi : \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{g}$ to $\mathfrak{p}(3,1)$ is an embedding $\widetilde{\varphi} : \mathfrak{p}(3,1) \hookrightarrow \mathfrak{g}$ such that $\widetilde{\varphi}$ restricted to $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ is equal to φ. That is, $\widetilde{\varphi}|_{\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})} = \varphi$.
- Let φ and ϱ be Lie algebra embeddings of $\mathfrak{g}'$ into $\mathfrak{g}$. Then φ and ϱ are *equivalent* if there is an inner automorphism $\rho : \mathfrak{g} \to \mathfrak{g}$ such that $\varphi = \rho\circ\varrho$, and we write

$$\varphi \sim \varrho.$$

  Hence, our classification in this article is up to equivalence.
- Two embeddings φ and ϱ of $\mathfrak{g}'$ into $\mathfrak{g}$ are *linearly equivalent* if for each representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ the induced $\mathfrak{g}'$-representations $\pi\circ\varphi$, $\pi\circ\varrho$ are equivalent, and we write

$$\varphi \sim_L \varrho.$$

We define equivalence and linear equivalence of subalgebras much as we did for embeddings.

- Two subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$ of $\mathfrak{g}$ are equivalent if there is an inner automorphism ρ of $\mathfrak{g}$ such that $\rho(\mathfrak{g}') = \mathfrak{g}''$.
- Two subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$ of $\mathfrak{g}$ are linearly equivalent if for every representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ the subalgebras $\rho(\mathfrak{g}')$, $\rho(\mathfrak{g}'')$ of $\mathfrak{gl}(V)$ are conjugate under $GL(V)$.

  Clearly equivalence implies linear equivalence (for embeddings or subalgebras), but the converse is not in general true.

VI. CLASSIFICATION OF EMBEDDINGS OF $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ INTO THE RANK 3 SIMPLE LIE ALGEBRAS

As an intermediate step to classifying the embeddings of $\mathfrak{p}(3,1)$ into the rank 3 simple Lie algebras, we begin by classifying the embeddings of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ into these same Lie algebras. First, we explicitly construct several embeddings of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ into each of the rank 3 simple Lie algebras. In Theorem 6.6 below, we will show that these embeddings are pairwise inequivalent, and form the basis for our classification of embeddings of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ into the rank 3 simple Lie algebras.

Let $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ be generated by positive root vectors E, E' and negative root vectors F, F', so that $\mathfrak{sl}(2,\mathbb{C}) \cong \langle E, F\rangle \cong \langle E', F'\rangle$; and $\langle E, F\rangle \cap \langle E', F'\rangle = 0$. We first construct embeddings of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{sl}(4,\mathbb{C})$:

$$\begin{array}{rccc} \varphi_A : & \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) & \hookrightarrow & \mathfrak{sl}(4,\mathbb{C}) \\ & E & \mapsto & X_1 \\ & F & \mapsto & Y_1 \\ & E' & \mapsto & X_3 \\ & F' & \mapsto & Y_3, \end{array} \tag{5}$$

$$\begin{array}{rccc} \varrho_A : & \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) & \hookrightarrow & \mathfrak{sl}(4,\mathbb{C}) \\ & E & \mapsto & X_1 + X_3 \\ & F & \mapsto & Y_1 + Y_3 \\ & E' & \mapsto & X_4 + X_5 \\ & F' & \mapsto & Y_4 + Y_5, \end{array} \tag{6}$$

where

$$X_4 = X_{2,1}, \quad X_5 = X_{3,2}, \quad Y_4 = -Y_{2,1}, \quad Y_5 = -Y_{3,2}. \tag{7}$$

We remind the reader that, for instance, $X_{2,\,1} = [X_2, X_1]$, as defined in Sec. V.

The following are embeddings of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ into $\mathfrak{so}(7, \mathbb{C})$.

$$\begin{array}{rccc} \varphi_B: & \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) & \hookrightarrow & \mathfrak{so}(7,\mathbb{C}) \\ & E & \mapsto & X_1 \\ & F & \mapsto & Y_1 \\ & E' & \mapsto & X_3 \\ & F' & \mapsto & Y_3, \end{array} \tag{8}$$

$$\begin{array}{rccc} \varrho_B: & \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) & \hookrightarrow & \mathfrak{so}(7,\mathbb{C}) \\ & E & \mapsto & X_1 \\ & F & \mapsto & Y_1 \\ & E' & \mapsto & X_9 \\ & F' & \mapsto & Y_9, \end{array} \tag{9}$$

$$\begin{array}{rccc} \vartheta_B: & \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) & \hookrightarrow & \mathfrak{so}(7,\mathbb{C}) \\ & E & \mapsto & X_7 + Y_2 \\ & F & \mapsto & X_2 + Y_7 \\ & E' & \mapsto & X_6 \\ & F' & \mapsto & Y_6, \end{array} \tag{10}$$

$$\begin{array}{rccc} \varsigma_B: & \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) & \hookrightarrow & \mathfrak{so}(7,\mathbb{C}) \\ & E & \mapsto & X_4 + X_5 \\ & F & \mapsto & Y_4 + Y_5 \\ & E' & \mapsto & Y_8 \\ & F' & \mapsto & X_8, \end{array} \tag{11}$$

where

$$\begin{array}{lll} X_4 = X_{1,2}, & X_5 = X_{2,3}, & X_6 = X_{3,2,1}, \\ X_7 = \frac{-1}{2} X_{3,2,3}, & X_8 = \frac{1}{2} X_{3,2,3,1}, & X_9 = \frac{-1}{2} X_{3,2,1,3,2}, \\ Y_8 = \frac{-1}{2} Y_{3,2,3,1}, & Y_9 = \frac{-1}{2} Y_{3,2,1,3,2}. & \end{array} \tag{12}$$

The following are embeddings of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ into $\mathfrak{sp}(6, \mathbb{C})$.

$$\begin{array}{rccc} \varphi_C: & \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) & \hookrightarrow & \mathfrak{sp}(6,\mathbb{C}) \\ & E & \mapsto & X_1 \\ & F & \mapsto & Y_1 \\ & E' & \mapsto & X_3 \\ & F' & \mapsto & Y_3, \end{array} \tag{13}$$

$$\begin{array}{rccc} \varrho_C: & \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) & \hookrightarrow & \mathfrak{sp}(6,\mathbb{C}) \\ & E & \mapsto & X_1 + X_3 \\ & F & \mapsto & Y_1 + Y_3 \\ & E' & \mapsto & X_4 + X_5 \\ & F' & \mapsto & 2Y_4 + 2Y_5, \end{array} \tag{14}$$

$$\begin{array}{rcccl} \vartheta_C: & \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}) & \hookrightarrow & \mathfrak{sp}(6,\mathbb{C}) \\ & E & \mapsto & X_3 \\ & F & \mapsto & Y_3 \\ & E' & \mapsto & X_9 \\ & F' & \mapsto & Y_9, \end{array} \tag{15}$$

$$\begin{array}{rcccl} \varsigma_C: & \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}) & \hookrightarrow & \mathfrak{sp}(6,\mathbb{C}) \\ & E & \mapsto & X_3 \\ & F & \mapsto & Y_3 \\ & E' & \mapsto & X_1+X_7 \\ & F' & \mapsto & 3Y_1+4Y_7, \end{array} \tag{16}$$

where

$$\begin{array}{llll} X_4 = X_{2,1}, & X_5 = X_{3,2}, & X_7 = \frac{1}{2}X_{3,2,2}, & X_9 = \frac{1}{2}X_{1,2,3,2,1}, \\ Y_4 = -Y_{2,1}, & Y_5 = -Y_{3,2}, & Y_7 = \frac{1}{2}Y_{3,2,2}, & Y_9 = \frac{1}{2}Y_{1,2,3,2,1}. \end{array} \tag{17}$$

The classification of embeddings of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ into the rank 3 simple Lie algebras will follow largely from Refs. 10 (see also Ref. 9) and 14:

Theorem 6.1 (Refs. 9 and 10): *Up to equivalence, there are exactly two subalgebras isomorphic to* $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ *inside* $\mathfrak{sl}(4,\mathbb{C})$*, four subalgebras isomorphic to* $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ *inside* $\mathfrak{so}(7,\mathbb{C})$*, and four subalgebras isomorphic to* $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ *inside* $\mathfrak{sp}(6,\mathbb{C})$.

Theorem 6.2 (Theorem 2 of Ref. 14): *Let* $\mathfrak{g}$ *be a classical simple Lie algebra,* $\mathfrak{h}$ *semisimple, and* $\varphi_i:\mathfrak{h}\hookrightarrow\mathfrak{g}$*,* $i=1,2$*, two embeddings. Also let* ω *be the standard representation of* $\mathfrak{g}$*. If* $\mathfrak{g}\ncong\mathfrak{so}(2n,\mathbb{C})$*, then* $\varphi_1\sim_L\varphi_2\Leftrightarrow\omega\circ\varphi_1\sim\omega\circ\varphi_2$.

Theorem 6.3 (Theorem 3 of Ref. 14): *Let* $\mathfrak{g}\cong\mathfrak{sl}(n,\mathbb{C}),\mathfrak{so}(2n+1,\mathbb{C})$*, or* $\mathfrak{sp}(2n,\mathbb{C})$*;* $\mathfrak{h}$ *semisimple; and* $\varphi_i:\mathfrak{h}\hookrightarrow\mathfrak{g}$*,* $i=1,2$*, two embeddings. Then* $\varphi_1\sim_L\varphi_2\Leftrightarrow\varphi_1\sim\varphi_2$.

Theorem 6.4: *Let* ϱ *and* φ *be embeddings of* $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ *into* $\mathfrak{sl}(4,\mathbb{C})$*,* $\mathfrak{so}(7,\mathbb{C})$*, or* $\mathfrak{sp}(6,\mathbb{C})$*. The subalgebras* $\varrho(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$ *and* $\varphi(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$ *are inequivalent if and only if* $\varrho\circ\alpha\not\sim\varphi$ *for all automorphisms* α *of* $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$.

Proof: ($\Rightarrow$) Suppose there is an automorphism α of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ such that $\varrho\circ\alpha\sim\varphi$. Then, there is an inner automorphism β of $\mathfrak{sl}(4,\mathbb{C})$, $\mathfrak{so}(7,\mathbb{C})$, or $\mathfrak{sp}(6,\mathbb{C})$, respectively, such that $\varrho(\alpha(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})))=\beta(\varphi(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})))$, as sets. Hence, the subalgebras $\varrho(\alpha(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})))=\varrho(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$, and $\varphi(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$ are equivalent.

($\Leftarrow$) Suppose that the subalgebras $\varrho(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$ and $\varphi(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$ are equivalent. Then, there exists an inner automorphism β of $\mathfrak{sl}(4,\mathbb{C})$, $\mathfrak{so}(7,\mathbb{C})$, or $\mathfrak{sp}(6,\mathbb{C})$, respectively, such that $\varrho(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))=\beta(\varphi(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$, as sets. Then, $\varrho^{-1}\circ\beta\circ\varphi$, with ϱ^{-1} the inverse of ϱ: $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\to\varrho(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$, is an automorphism of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$, and $\varrho\circ(\varrho^{-1}\circ\beta\circ\varphi)=\beta\circ\varphi$. Hence, $\varrho\circ(\varrho^{-1}\circ\beta\circ\varphi)\sim\varphi$. □

It is well-known that, up to inner automorphisms, the outer automorphisms of a semisimple Lie algebra correspond to the automorphism group of its Dynkin diagram (Proposition D.40 of

TABLE I. Classification of embeddings of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ into the rank 3 simple Lie algebras up to equivalence.

Rank 3 simple Lie algebra $\mathfrak{g}$	Embeddings $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\hookrightarrow\mathfrak{g}$
$\mathfrak{sl}(4,\mathbb{C})$	φ_A, ϱ_A
$\mathfrak{so}(7,\mathbb{C})$	φ_B, $\varphi_B\circ\varepsilon$, ϱ_B, ϑ_B, ς_B, $\varsigma_B\circ\varepsilon$
$\mathfrak{sp}(6,\mathbb{C})$	φ_C, $\varphi_C\circ\varepsilon$, ϱ_C, $\varrho_C\circ\varepsilon$, ϑ_C, ς_C, $\varsigma_C\circ\varepsilon$

Ref. 5). Hence, any outer automorphism of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ interchanges $\mathfrak{sl}(2,\mathbb{C})$ components. More specifically, we have the following theorem.

Theorem 6.5: *Up to inner automorphism, there is a unique outer automorphism of* $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$*, given by*

$$\begin{array}{rccc} \varepsilon: & \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}) & \to & \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}) \\ & E & \mapsto & E' \\ & F & \mapsto & F' \\ & E' & \mapsto & E \\ & F' & \mapsto & F. \end{array} \tag{18}$$

Using the constructed embeddings and the five theorems above we can now classify the embeddings of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ into the rank 3 simple Lie algebras.

Theorem 6.6: *A complete set of inequivalent embeddings of* $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ *into* $\mathfrak{sl}(4,\mathbb{C})$ *is* φ_A *and* ϱ_A*. A complete set of inequivalent embeddings of* $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ *into* $\mathfrak{so}(7,\mathbb{C})$ *is* φ_B, $\varphi_B\circ\varepsilon$, ϱ_B, ϑ_B, ς_B*, and* $\varsigma_B\circ\varepsilon$*. A complete set of inequivalent embeddings of* $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ *into* $\mathfrak{sp}(6,\mathbb{C})$ *is* φ_C, $\varphi_C\circ\varepsilon$, ϱ_C, $\varrho_C\circ\varepsilon$, ϑ_C, ς_C*, and* $\varsigma_C\circ\varepsilon$*. For clarity, this classification is summarized in Table I.*

Proof: Let ω be the standard representation of $\mathfrak{sl}(4,\mathbb{C})$. We then have the following decompositions:

$$\begin{aligned} \omega\circ\varphi_A &\cong (V(1)\otimes V(0))\oplus(V(0)\otimes V(1)), \\ \omega\circ\varphi_A\circ\varepsilon &\cong (V(0)\otimes V(1))\oplus(V(1)\otimes V(0)), \\ \omega\circ\varrho_A &\cong V(1)\otimes V(1), \\ \omega\circ\varrho_A\circ\varepsilon &\cong V(1)\otimes V(1). \end{aligned} \tag{19}$$

Hence, the subalgebras $\varphi_A(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$ and $\varrho_A(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$ are inequivalent by Theorems 6.4 and 6.5. By Theorem 6.1 they form a complete list of inequivalent $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ subalgebras in $\mathfrak{sl}(4,\mathbb{C})$.

Obtaining a classification of embeddings from the classification of subalgebras is achieved by considering any embedding into each of these subalgebras together with their compositions with the unique outer automorphism of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$. Hence, considering Eq. (19) and Theorems 6.2 and 6.3, φ_A and ϱ_A is a complete set of inequivalent embeddings of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{sl}(4,\mathbb{C})$.

Let ω be the standard representation of $\mathfrak{so}(7,\mathbb{C})$. We have the following decompositions:

$$\begin{aligned}
\omega\circ\varphi_B &\cong (V(0)\otimes V(2))\oplus(2V(1)\otimes V(0)),\\
\omega\circ\varphi_B\circ\varepsilon &\cong (V(2)\otimes V(0))\oplus(2V(0)\otimes V(1)),\\
\omega\circ\varrho_B &\cong (V(1)\otimes V(1))\oplus(3V(0)\otimes V(0)),\\
\omega\circ\varrho_B\circ\varepsilon &\cong (V(1)\otimes V(1))\oplus(3V(0)\otimes V(0)),\\
\omega\circ\vartheta_B &\cong (V(2)\otimes V(0))\oplus(V(0)\otimes V(2))\\
&\oplus (V(0)\otimes V(0)),\\
\omega\circ\vartheta_B\circ\varepsilon &\cong (V(0)\otimes V(2))\oplus(V(2)\otimes V(0))\\
&\oplus (V(0)\otimes V(0)),\\
\omega\circ\varsigma_B &\cong (V(2)\otimes V(0))\oplus(V(1)\otimes V(1)),\\
\omega\circ\varsigma_B\circ\varepsilon &\cong (V(0)\otimes V(2))\oplus(V(1)\otimes V(1)).
\end{aligned}\tag{20}$$

Using reasoning as in the above case, φ_B, $\varphi_B\circ\varepsilon$, ϱ_B, ϑ_B, ς_B, and $\varsigma_B\circ\varepsilon$ is a complete set of inequivalent embeddings of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{so}(7,\mathbb{C})$.

Let ω be the standard representation of $\mathfrak{sp}(6,\mathbb{C})$. We have the following decompositions:

$$\begin{aligned}
\omega\circ\varphi_C &\cong (2V(1)\otimes V(0))\oplus(V(0)\otimes V(1)),\\
\omega\circ\varphi_C\circ\varepsilon &\cong (2V(0)\otimes V(1))\oplus(V(1)\otimes V(0)),\\
\omega\circ\varrho_C &\cong (V(1)\otimes V(2)),\\
\omega\circ\varrho_C\circ\varepsilon &\cong (V(2)\otimes V(1)),\\
\omega\circ\vartheta_C &\cong (V(1)\otimes V(0))\oplus(V(0)\otimes V(1))\\
&\oplus (2V(0)\otimes V(0)),\\
\omega\circ\vartheta_C\circ\varepsilon &\cong (V(0)\otimes V(1))\oplus(V(1)\otimes V(0))\\
&\oplus (2V(0)\otimes V(0)),\\
\omega\circ\varsigma_C &\cong (V(0)\otimes V(3))\oplus(V(1)\otimes V(0)),\\
\omega\circ\varsigma_C\circ\varepsilon &\cong (V(3)\otimes V(0))\oplus(V(0)\otimes V(1)).
\end{aligned}\tag{21}$$

Using reasoning as in the above case, the result for $\mathfrak{sp}(6,\mathbb{C})$ follows. □

VII. CLASSIFICATION OF EMBEDDINGS OF $\mathfrak{p}(3,1)$ INTO $\mathfrak{sl}(4,\mathbb{C})$

In this section we classify the embeddings of $\mathfrak{p}(3,1)$ into $\mathfrak{sl}(4,\mathbb{C})$, up to equivalence. We will use the results of Sec. VI as summarized in Theorem 6.6.

We first consider the decomposition of $\mathfrak{sl}(4,\mathbb{C})$ with respect to the adjoint action of $\varphi_A(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$:

$$\begin{aligned}
\mathfrak{sl}(4,\mathbb{C}) \cong_{\varphi_A} &[X_1]_{\varphi_A} \oplus [X_3]_{\varphi_A} \oplus\\
&[X_{3,2,1}]_{\varphi_A} \oplus [Y_2]_{\varphi_A} \oplus\\
&[H_1+2H_2+H_3]_{\varphi_A}\\
\cong_{\varphi_A} &V(2)\otimes V(0) \oplus V(0)\otimes V(2) \oplus\\
&2V(1)\otimes V(1) \oplus V(0)\otimes V(0).
\end{aligned}\tag{22}$$

Lemma 7.1: The $\mathfrak{sl}(4,\mathbb{C})$-subspaces $[X_{3,2,1}]_{\varphi_A}$ and $[Y_2]_{\varphi_A}$ are Abelian subalgebras of $\mathfrak{sl}(4,\mathbb{C})$. However, $[X_{3,2,1}+\beta Y_2]_{\varphi_A}$ is not Abelian for any $\beta\in\mathbb{C}^*$.

Proof: The following is a basis of $[X_{3,2,1}]_{\varphi_A}$:

$$X_{3,2,1},\quad [Y_1,X_{3,2,1}],\quad [Y_3,[Y_1,X_{3,2,1}]],\quad [Y_3,X_{3,2,1}].\tag{23}$$

Direct calculation shows

$$[[X_{3,2,1}]_{\varphi_A}, X_{3,2,1}] = 0. \tag{24}$$

Equation (24) together with the Jacobi identity imply

$$[[X_{3,2,1}]_{\varphi_A}, [X_{3,2,1}]_{\varphi_A}] = \{0\}, \tag{25}$$

where $[[X_{3,2,1}]_{\varphi_A}, [X_{3,2,1}]_{\varphi_A}]$ is interpreted to be the set of all products $[L, L']$ such the $L, L' \in [X_{3,2,1}]_{\varphi_A}$. Hence, $[X_{3,2,1}]_{\varphi_A}$ is an Abelian subalgebra of $\mathfrak{sl}(4, \mathbb{C})$. In a similar fashion, we show $[Y_2]_{\varphi_A}$ is Abelian.

Note that $[Y_1, X_{3,2,1} + \beta Y_2] \in [X_{3,2,1} + \beta Y_2]_{\varphi_A}$. Since

$$[X_{3,2,1} + \beta Y_2, [Y_1, X_{3,2,1} + \beta Y_2]] = 2\beta X_3, \tag{26}$$

we have that $[X_{3,2,1} + \beta Y_2]_{\varphi_A}$ is not Abelian if $\beta \neq 0$. □

Note that $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \cong V(2) \otimes V(0) \oplus V(0) \otimes V(2)$, and $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong V(1) \otimes V(1)$ as $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$-representations. Hence, by Lemma 7.1, the following define all lifts of φ_A to $\mathfrak{p}(3, 1)$,

$$\begin{array}{rccc} \widetilde{\varphi}^{\alpha}_{A,X_{3,2,1}} : & \mathfrak{p}(3,1) & \hookrightarrow & \mathfrak{sl}(4,\mathbb{C}) \\ & u & \mapsto & \alpha X_{3,2,1}, \\ \widetilde{\varphi}^{\alpha}_{A,Y_2} : & \mathfrak{p}(3,1) & \hookrightarrow & \mathfrak{sl}(4,\mathbb{C}) \\ & u & \mapsto & \alpha Y_2, \end{array} \tag{27}$$

where u is a highest weight vector of $\mathbb{C}^2 \otimes \mathbb{C}^2$, and $\alpha \in \mathbb{C}^*$.

Consider

$$\begin{array}{rclclc} \mathfrak{sl}(4,\mathbb{C}) & \cong_{\varrho_A} & [X_1 + X_3]_{\varrho_A} & \oplus & [X_4 + X_5]_{\varrho_A} & \oplus \\ & & [X_6]_{\varrho_A} & & & \\ & \cong_{\varrho_A} & V(2) \otimes V(0) & \oplus & V(0) \otimes V(2) & \oplus \\ & & V(2) \otimes V(2). & & & \end{array} \tag{28}$$

The decomposition is computed by identifying highest weight vectors, and then considering dimensions of $\mathfrak{sl}(4, \mathbb{C})$ and of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$-modules. Since $V(1) \otimes V(1)$ does not occur in the decomposition, ϱ_A cannot be lifted to an embedding of $\mathfrak{p}(3, 1)$ into $\mathfrak{sl}(4, \mathbb{C})$. Hence, by Theorem 6.6, $\widetilde{\varphi}^{\alpha}_{A,X_{3,2,1}}$ and $\widetilde{\varphi}^{\alpha}_{A,Y_2}$ define all embeddings of $\mathfrak{p}(3, 1)$ into $\mathfrak{sl}(4, \mathbb{C})$, where $\alpha \in \mathbb{C}^*$.

Theorem 7.2: *A complete set of inequivalent embeddings of* $\mathfrak{p}(3, 1)$ *into* $\mathfrak{sl}(4, \mathbb{C})$ *is given by* $\widetilde{\varphi}^{1}_{A,X_{3,2,1}}$ *and* $\widetilde{\varphi}^{1}_{A,Y_2}$. *Although* $\widetilde{\varphi}^{1}_{A,X_{3,2,1}}$ *and* $\widetilde{\varphi}^{1}_{A,Y_2}$ *are not equivalent, they are related by an outer automorphism of* $\mathfrak{sl}(4, \mathbb{C})$*: That is, there exits an* $\mathfrak{sl}(4, \mathbb{C})$ *outer automorphism* ρ *such that* $\rho \circ \widetilde{\varphi}^{1}_{A,Y_2} = \widetilde{\varphi}^{1}_{A,X_{3,2,1}}$.

Proof: For all $\alpha, \beta \in \mathbb{C}^*$, define an inner automorphism $\rho_{\alpha,\beta} : \mathfrak{sl}(4, \mathbb{C}) \to \mathfrak{sl}(4, \mathbb{C})$ as follows

$$\begin{array}{ll} X_1 \mapsto X_1, & Y_1 \mapsto Y_1, \\ X_2 \mapsto \frac{\beta}{\alpha} X_2, & Y_2 \mapsto \frac{\alpha}{\beta} Y_2, \\ X_3 \mapsto X_3, & Y_3 \mapsto Y_3. \end{array} \tag{29}$$

We then have $\rho_{\alpha,\beta} \circ \widetilde{\varphi}^{\alpha}_{A,X_{3,2,1}} = \widetilde{\varphi}^{\beta}_{A,X_{3,2,1}}$, so that $\widetilde{\varphi}^{\alpha}_{A,X_{3,2,1}} \sim \widetilde{\varphi}^{\beta}_{A,X_{3,2,1}}$. Similarly $\widetilde{\varphi}^{\alpha}_{A,Y_2} \sim \widetilde{\varphi}^{\beta}_{A,Y_2}$.

It now only remains to show $\widetilde{\varphi}^{1}_{A,X_{3,2,1}} \nsim \widetilde{\varphi}^{1}_{A,Y_2}$. By way of contradiction, suppose $\widetilde{\varphi}^{1}_{A,X_{3,2,1}} \sim \widetilde{\varphi}^{1}_{A,Y_2}$. Let $\rho : \mathfrak{sl}(4, \mathbb{C}) \to \mathfrak{sl}(4, \mathbb{C})$ be an inner automorphism of $\mathfrak{sl}(4, \mathbb{C})$ such that

$$\rho \circ \widetilde{\varphi}^{1}_{A,Y_2} = \widetilde{\varphi}^{1}_{A,X_{3,2,1}}, \tag{30}$$

which implies

$$\rho(Y_2) = X_{3,2,1}. \tag{31}$$

The automorphism ρ must send a highest (respectively lowest) weight vector to a highest (respectively, lowest) weight vector of the same weight with respect to the adjoint action of $\varphi_A(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$. Thus

$$\begin{aligned}&\rho(H_1+2H_2+H_3)=\beta(H_1+2H_2+H_3),\\&\rho(X_2)=\gamma Y_{3,2,1}+\gamma' X_2,\end{aligned}\tag{32}$$

for $\beta\in\mathbb{C}^*$, and γ, γ' not both zero. Consider

$$\begin{aligned}\rho([H_1+2H_2+H_3, Y_2]) &= -2\rho(Y_2)\\&= -2X_{3,2,1},\\ [\rho(H_1+2H_2+H_3), \rho(Y_2)] &= [\beta(H_1+2H_2+H_3), X_{3,2,1}]\\&= 2\beta X_{3,2,1},\end{aligned}\tag{33}$$

hence, since ρ is a Lie algebra homomorphism, $\beta = -1$. Thus

$$\rho(H_1+2H_2+H_3)=-(H_1+2H_2+H_3).\tag{34}$$

Consider

$$\begin{aligned}\rho([H_1+2H_2+H_3, X_2]) &= 2\gamma Y_{3,2,1}+2\gamma' X_2,\\ [\rho(H_1+2H_2+H_3), \rho(X_2)] &= 2\gamma Y_{3,2,1}-2\gamma' X_2.\end{aligned}\tag{35}$$

Hence $\gamma' = 0$ so that

$$\rho(X_2)=\gamma Y_{3,2,1}.\tag{36}$$

Consider

$$\begin{aligned}\rho([X_2, Y_2]) &= \rho(H_2),\\ [\rho(X_2), \rho(Y_2)] &= [\gamma Y_{3,2,1}, X_{3,2,1}] = -\gamma(H_1+H_2+H_3),\end{aligned}\tag{37}$$

so that $\rho(H_2) = -\gamma(H_1 + H_2 + H_3)$. Consider

$$\begin{aligned}\rho([H_2, X_2]) &= 2\gamma Y_{3,2,1}\\ [\rho(H_2), \rho(X_2)] &= [-\gamma(H_1+H_2+H_3), \gamma Y_{3,2,1}]\\&= 2\gamma^2 Y_{3,2,1},\end{aligned}\tag{38}$$

so that $\gamma = 1$. Hence ρ is defined by

$$\begin{aligned}\rho:\quad &X_1\mapsto X_1, &&Y_1\mapsto Y_1,\\ &X_2\mapsto Y_{3,2,1}, &&Y_2\mapsto X_{3,2,1},\\ &X_3\mapsto X_3, &&Y_3\mapsto Y_3.\end{aligned}\tag{39}$$

The outer automorphisms of $\mathfrak{sl}(4,\mathbb{C})$ correspond to the group of automorphisms of its Dynkin diagram (Proposition D.40 of Ref. 5). More specifically, $\mathfrak{sl}(4,\mathbb{C})$ has a unique outer automorphism ρ' that may defined by

$$\begin{aligned}\rho':\quad &X_1\mapsto X_3, &&Y_1\mapsto Y_3,\\ &X_2\mapsto X_2, &&Y_2\mapsto Y_2,\\ &X_3\mapsto X_1, &&Y_3\mapsto Y_1.\end{aligned}\tag{40}$$

Let ω be the standard representation of $\mathfrak{sl}(4,\mathbb{C})$; then, using Eqs. (39) and (40), we have the following

$$\begin{aligned}\omega\circ\rho &\cong V(\lambda_3),\\ \omega\circ\rho' &\cong V(\lambda_3).\end{aligned}\tag{41}$$

Hence, Theorems 6.2 and 6.3 imply that $\rho \sim \rho'$, which implies that ρ is an outer automorphism, a contradiction. Hence, $\widetilde{\varphi}^1_{A,X_{3,2,1}} \nsim \widetilde{\varphi}^1_{A,Y_2}$. However, we have also shown that $\rho \circ \widetilde{\varphi}^1_{A,Y_2} = \widetilde{\varphi}^1_{A,X_{3,2,1}}$, for the outer automorphism ρ of $\mathfrak{sl}(4,\mathbb{C})$ defined in Eq. (40). □

We close the section by considering the irreducible representations of $\mathfrak{sl}(4,\mathbb{C})$ restricted to $\mathfrak{p}(3,1)$ under any embedding in the following theorem.

Theorem 7.3: *Every irreducible highest weight module of* $\mathfrak{sl}(4,\mathbb{C})$ *(not necessarily finite-dimensional) remains indecomposable when restricted to* $\mathfrak{p}(3,1)$ *with respect to any embedding of* $\mathfrak{p}(3,1)$ *into* $\mathfrak{sl}(4,\mathbb{C})$.

Proof: The theorem follows immediately from the observation that the image of $\mathfrak{p}(3,1)$ under any embedding into $\mathfrak{sl}(4,\mathbb{C})$, all of which were identified in Theorem 7.2, contains all positive or all negative root vectors of $\mathfrak{sl}(4,\mathbb{C})$ (see, for instance, Ref. 4); thus its proof is omitted. □

VIII. CLASSIFICATION OF EMBEDDINGS OF $\mathfrak{p}(3,1)$ INTO $\mathfrak{so}(7,\mathbb{C})$

In this section we classify the embeddings of $\mathfrak{p}(3,1)$ into $\mathfrak{so}(7,\mathbb{C})$, up to equivalence. We will use the results of Sec. VI as summarized in Theorem 6.6.

We first consider the decomposition of $\mathfrak{so}(7,\mathbb{C})$ with respect to the adjoint action of $\varphi_B(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$, $\varphi_B\circ\varepsilon(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$, and $\vartheta(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$, respectively:

$$\begin{aligned}
\mathfrak{so}(7,\mathbb{C}) \cong_{\varphi_B} & [X_1]_{\varphi_B} \oplus [X_3]_{\varphi_B} \oplus \\
& [X_8]_{\varphi_B} \oplus [Y_2]_{\varphi_B} \oplus \\
& [X_9]_{\varphi_B} \oplus [Y_9]_{\varphi_B} \oplus \\
& [H]_{\varphi_B} \\
\cong_{\varphi_B} & V(2)\otimes V(0) \oplus V(0)\otimes V(2) \oplus \\
& 2V(1)\otimes V(2) \oplus 3V(0)\otimes V(0),
\end{aligned} \tag{42}$$

$$\begin{aligned}
\mathfrak{so}(7,\mathbb{C}) \cong_{\varphi_B\circ\varepsilon} & [X_1]_{\varphi_B\circ\varepsilon} \oplus [X_3]_{\varphi_B\circ\varepsilon} \oplus \\
& [X_8]_{\varphi_B\circ\varepsilon} \oplus [Y_2]_{\varphi_B\circ\varepsilon} \oplus \\
& [X_9]_{\varphi_B\circ\varepsilon} \oplus [Y_9]_{\varphi_B\circ\varepsilon} \oplus \\
& [H]_{\varphi_B} \\
\cong_{\varphi_B\circ\varepsilon} & V(0)\otimes V(2) \oplus V(2)\otimes V(0) \oplus \\
& 2V(2)\otimes V(1) \oplus 3V(0)\otimes V(0),
\end{aligned} \tag{43}$$

where $H = X_1 + 2X_2 + X_3$,

$$\begin{aligned}
\mathfrak{so}(7,\mathbb{C}) \cong_{\vartheta_B} & [X_8]_{\vartheta_B} \oplus [X_7]_{\vartheta_B} \oplus \\
& [Y_2]_{\vartheta_B} \oplus [Y_1+Y_9]_{\vartheta_B} \oplus \\
& [X_6]_{\vartheta_B} \\
\cong_{\vartheta_B} & V(2)\otimes V(2) \oplus 2V(2)\otimes V(0) \oplus \\
& 2V(0)\otimes V(2).
\end{aligned} \tag{44}$$

Since there is no $\mathbb{C}^2\otimes\mathbb{C}^2 \cong V(1)\otimes V(1)$ component in any of the above decompositions, we may not lift the embedding φ_B, $\varphi_B\circ\varepsilon$, or ϑ_B to $\mathfrak{p}(3,1)$.

Next we consider the decomposition of $\mathfrak{so}(7,\mathbb{C})$ with respect to the adjoint action of $\varsigma_B(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$, and $\varsigma_B\circ\varepsilon(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$:

$$\begin{aligned}\mathfrak{so}(7,\mathbb{C}) &\cong_{\varsigma_B} [X_5]_{\varsigma_B} \oplus [X_4]_{\varsigma_B} \oplus \\ &\quad [Y_8]_{\varsigma_B} \oplus [2Y_1+Y_3]_{\varsigma_B} \oplus \\ &\quad [X_2]_{\varsigma_B} \\ &\cong_{\varsigma_B} 2V(2)\otimes V(0) \oplus V(0)\otimes V(2) \oplus \\ &\quad V(1)\otimes V(1) \oplus V(3)\otimes V(1),\end{aligned} \tag{45}$$

$$\begin{aligned}\mathfrak{so}(7,\mathbb{C}) &\cong_{\varsigma_B\circ\varepsilon} [X_5]_{\varsigma_B\circ\varepsilon} \oplus [X_4]_{\varsigma_B\circ\varepsilon} \oplus \\ &\quad [Y_8]_{\varsigma_B\circ\varepsilon} \oplus [2Y_1+Y_3]_{\varsigma_B\circ\varepsilon} \oplus \\ &\quad [X_2]_{\varsigma_B\circ\varepsilon} \\ &\cong_{\varsigma_B\circ\varepsilon} 2V(0)\otimes V(2) \oplus V(2)\otimes V(0) \oplus \\ &\quad V(1)\otimes V(1) \oplus V(1)\otimes V(3).\end{aligned} \tag{46}$$

Lemma 8.1: The $\mathfrak{so}(7,\mathbb{C})$ subspaces $[2Y_1+Y_3]_{\varsigma_B}$ and $[2Y_1+Y_3]_{\varsigma_B\circ\varepsilon}$ are not Abelian.

Proof: Note that $[X_8, 2Y_1 + Y_3] \in [2Y_1+Y_3]_{\varsigma_B}$ and $[2Y_1+Y_3]_{\varsigma_B\circ\varepsilon}$. Since

$$[[X_8, 2Y_1+Y_3], 2Y_1+Y_3] = 2X_4 - 4X_5 \neq 0, \tag{47}$$

we have that $[2Y_1+Y_3]_{\varsigma_B}$ and $[2Y_1+Y_3]_{\varsigma_B\circ\varepsilon}$ are not Abelian. □

Since, by Lemma 8.1, the $V(1)\otimes V(1)$ component in each of the above decompositions is not Abelian, we cannot lift either ς_B or $\varsigma_B\circ\varepsilon$ to $\mathfrak{p}(3,1)$.

Finally, we consider the decomposition of $\mathfrak{so}(7,\mathbb{C})$ with respect to the adjoint action of $\varrho_B(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$:

$$\begin{aligned}\mathfrak{so}(7,\mathbb{C}) &\cong_{\varrho_B} [X_1]_{\varrho_B} \oplus [X_3]_{\varrho_B} \oplus \\ &\quad [Y_3]_{\varrho_B} \oplus [H_3]_{\varrho_B} \oplus \\ &\quad [X_4]_{\varrho_B} \oplus [X_6]_{\varrho_B} \oplus \\ &\quad [X_8]_{\varrho_B} \oplus [X_9]_{\varrho_B} \\ &\cong_{\varrho_B} V(2)\otimes V(0) \oplus 3V(0)\otimes V(0) \oplus \\ &\quad 3V(1)\otimes V(1) \oplus V(0)\otimes V(2).\end{aligned} \tag{48}$$

Lemma 8.2: The $\mathfrak{so}(7,\mathbb{C})$-subspace $[\beta X_4+\beta' X_6+\beta'' X_8]_{\varrho_B}$ is Abelian if and only if $(\beta')^2 = \beta''\beta$, where $\beta, \beta', \beta'' \in \mathbb{C}$.

Proof: The following is a basis of $[\beta X_4+\beta' X_6+\beta'' X_8]_{\varrho_B}$:

$$\begin{aligned}&\beta X_4+\beta' X_6+\beta'' X_8, && [Y_1, \beta X_4+\beta' X_6+\beta'' X_8], \\ &[Y_9,[Y_1,\beta X_4+\beta' X_6+\beta'' X_8]], && [Y_9, \beta X_4+\beta' X_6+\beta'' X_8].\end{aligned} \tag{49}$$

Direct calculation shows that a basis of $[[\beta X_4+\beta' X_6+\beta'' X_8]_{\varrho_B}, \beta X_4+\beta' X_6+\beta'' X_8]$ is

$$\begin{aligned}&(\beta\beta'' - (\beta')^2)X_9, \\ &2((\beta')^2-\beta''\beta)(H_1+H_2) + ((\beta')^2-\beta\beta'')H_3, \\ &(\beta\beta''-(\beta')^2)X_1.\end{aligned} \tag{50}$$

Hence,

$$\begin{aligned}&[[\beta X_4+\beta' X_6+\beta'' X_8]_{\varrho_B}, \beta X_4+\beta' X_6+\beta'' X_8] = 0 \\ &\iff (\beta')^2 = \beta''\beta.\end{aligned} \tag{51}$$

By the Jacobi identity,

$$[[\beta X_4+\beta' X_6+\beta'' X_8]_{\varrho_B},\beta X_4+\beta' X_6+\beta'' X_8]=0$$
$$\Longleftrightarrow [[\beta X_4+\beta' X_6+\beta'' X_8]_{\varrho_B},[\beta X_4+\beta' X_6+\beta'' X_8]_{\varrho_B}]=0. \tag{52}$$

By Eqs. (51) and (52) the result follows. □

Note that $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\cong V(2)\otimes V(0)\oplus V(0)\otimes V(2)$, and $\mathbb{C}^2\otimes\mathbb{C}^2\cong V(1)\otimes V(1)$ as $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$-representations. Hence, by Theorem 6.6 and Lemma 8.2, the following define all embeddings of $\mathfrak{p}(3,1)$ into $\mathfrak{so}(7,\mathbb{C})$,

$$\begin{array}{rccc}\widetilde{\varrho}_B^{\,\beta,\beta',\beta''}: & \mathfrak{p}(3,1) & \hookrightarrow & \mathfrak{so}(7,\mathbb{C}) \\ & u & \mapsto & \beta X_4+\beta' X_6+\beta'' X_8,\end{array} \tag{53}$$

where u is a highest weight vector of $\mathbb{C}^2\otimes\mathbb{C}^2$, and $\beta\beta''=(\beta')^2$, with β,β',β'' not all zero.

Lemma 8.3: Every embedding of $\mathfrak{p}(3,1)$ *into* $\mathfrak{so}(7,\mathbb{C})$ *is equivalent to* $\widetilde{\varrho}_B^{\,1,\gamma,\gamma^2}$ *for some* $\gamma\in\mathbb{C}$.

Proof: Since we cannot lift φ_B, $\varphi\circ\varepsilon$, ϑ_B, ς_B, or $\varsigma_B\circ\varepsilon$ to $\mathfrak{p}(3,1)$, Theorem 6.6 implies that each embedding of $\mathfrak{p}(3,1)$ into $\mathfrak{so}(7,\mathbb{C})$ must come from a lift of ϱ_B. That is, considering Lemma 8.2, each lift of $\mathfrak{p}(3,1)$ into $\mathfrak{so}(7,\mathbb{C})$ is equivalent to $\varrho_B^{\beta,\beta',\beta''}$ for some $\beta,\beta',\beta''\in\mathbb{C}$ such that $\beta\beta''=(\beta')^2$.

The result will follow by showing that if $\beta\neq 0$, then $\varrho_B^{\beta,\beta',\beta''}\sim\varrho_B^{1,\gamma,\gamma^2}$ for some $\gamma\in\mathbb{C}$; $\varrho_B^{0,0,\beta''}\sim\varrho_B^{1,0,0}$; and by noting that an inner automorphism of $\mathfrak{so}(7,\mathbb{C})$ will send an Abelian subalgebra to an Abelian subalgebra.

We first show that if $\beta\neq 0$, then $\varrho_B^{\beta,\beta',\beta''}\sim\varrho_B^{1,\gamma,\gamma^2}$ for some $\gamma\in\mathbb{C}$. Define an inner automorphism ρ of $\mathfrak{so}(7,\mathbb{C})$ as follows:

$$\begin{array}{ll} X_1\mapsto X_1, & Y_1\mapsto Y_1,\\ X_2\mapsto \frac{1}{\beta}X_2, & Y_2\mapsto \beta Y_2,\\ X_3\mapsto \beta X_3, & Y_3\mapsto \frac{1}{\beta}Y_3.\end{array} \tag{54}$$

We then have $\rho\circ\widetilde{\varrho}_B^{\,\beta,\beta',\beta''}=\widetilde{\varrho}_B^{\,1,\beta',\beta\beta''}=\widetilde{\varrho}_B^{\,1,\beta',(\beta')^2}$.

We now show that $\varrho_B^{0,0,\beta''}\sim\varrho_B^{1,0,0}$. Define an inner automorphism ρ of $\mathfrak{so}(7,\mathbb{C})$ as follows:

$$\begin{array}{ll} X_1\mapsto X_1, & Y_1\mapsto Y_1,\\ X_2\mapsto \beta'' X_7, & Y_2\mapsto \frac{1}{\beta''}Y_7,\\ X_3\mapsto \frac{1}{\beta''}Y_3, & Y_3\mapsto \beta'' X_3.\end{array} \tag{55}$$

It follows that $\rho\circ\widetilde{\varrho}_B^{\,0,0,\beta''}=\widetilde{\varrho}_B^{\,1,0,0}$. □

In Proposition 8.5 below we show that $\widetilde{\varrho}_B^{\,1,\lambda,\lambda^2}\sim\widetilde{\varrho}_B^{\,1,\gamma,\gamma^2}$. This will complete our classification, as recorded in Theorem 8.6. In order to facilitate the proof of Proposition 8.5, we need an additional proposition, and an explicit realization of $\mathfrak{so}(7,\mathbb{C})$. We first introduce terminology.

For $\mathbf{u},\mathbf{v}\in\mathbb{C}^3$, $\mathbf{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$, we consider the "dot product" $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^{tr}\mathbf{v}=\sum_i u_iv_i$. A "null vector" is a vector $\mathbf{u}\in\mathbb{C}^3$ such that $\mathbf{u}\cdot\mathbf{u}=0$.

Proposition 8.4: If $\mathbf{u},\mathbf{v}\in\mathbb{C}^3$ *are nonzero null vectors, then there exists* $\omega\in SO(3)_{\mathbb{C}}$ *such that* $\omega\mathbf{u}=\mathbf{v}$.

Proof: It will suffice to prove that given any nonzero null vector $\mathbf{u} \in \mathbb{C}^3$, then there exists $\omega \in SO(3)_{\mathbb{C}}$ such that $\omega \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathbf{u}$. This shows that the nonzero null vectors in $\mathbb{C}^3$ comprise a single $SO(3)_{\mathbb{C}}$ orbit.

We write $\mathbf{u} = \mathbf{x} + i\mathbf{y}$, with $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. Then $0 = \mathbf{u} \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{x} - \mathbf{y} \cdot \mathbf{y} + 2i\mathbf{x} \cdot \mathbf{y}$. In particular, we must have $\mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y}$ and $\mathbf{x} \cdot \mathbf{y} = 0$. Suppose $\mathbf{x} \cdot \mathbf{x} = a^2$, for some $0 \neq a \in \mathbb{R}$.

Then $\frac{1}{a}\mathbf{x}$ and $\frac{1}{a}\mathbf{y}$ are orthogonal unit vectors in $\mathbb{R}^3$, so there exists $\omega' \in SO(3)_{\mathbb{R}}$ with these as its first two columns. In particular then, $\omega'(a\mathbf{e}_1 + ai\mathbf{e}_2) = \mathbf{u}$.

Now, letting $\alpha = \frac{1}{2}(a + \frac{1}{a})$, $\beta = \frac{1}{2i}(a - \frac{1}{a})$, it is easy to show that $\omega'' = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an element of $SO(3)_{\mathbb{C}}$ and that $\omega'' \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ ai \\ 0 \end{pmatrix}$.

Letting $\omega = \omega'\omega''$, we find that $\omega \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathbf{u}$, as required. □

Regarding $\mathfrak{so}(7, \mathbb{C})$ as the group of 7×7 skew symmetric complex matrices, it is possible to construct a Chevalley basis explicitly.

Define $H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} E = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, Z = \begin{pmatrix} 1 \\ i \end{pmatrix}, \bar{E} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \bar{Z} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Then, writing elements of $\mathfrak{so}(7, \mathbb{C})$ in block form, with rows and columns grouped in blocks of sizes 2, 2, 2, 1, it is possible to define various elements as follows:

$$H_1 = \begin{pmatrix} H & 0 & 0 & 0 \\ 0 & -H & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H & 0 & 0 \\ 0 & 0 & -H & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2H & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_1 = \frac{1}{2}\begin{pmatrix} 0 & E & 0 & 0 \\ -E^{tr} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & -E^{tr} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z \\ 0 & 0 & -Z^t & 0 \end{pmatrix}, \tag{56}$$

$$Y_1 = \frac{1}{2}\begin{pmatrix} 0 & -\bar{E} & 0 & 0 \\ \bar{E}^{tr} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Y_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{E} & 0 \\ 0 & \bar{E}^{tr} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{Z} \\ 0 & 0 & \bar{Z}^{tr} & 0 \end{pmatrix}.$$

In each case, the blocks are 2×2, except for the last row and last column, which are a single row and a single column. The bottom right "block" is a single entry.

The remaining X's and Y's in the Chevalley basis are formed by taking commutators of these, as partially described in Eq. (12). In particular, with $K = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $\bar{K} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$,

$$X_9 = \frac{1}{2}\begin{pmatrix} 0 & -K & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Y_9 = \frac{1}{2}\begin{pmatrix} 0 & \bar{K} & 0 & 0 \\ -\bar{K} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{57}$$

Proposition 8.5: Let $\lambda, \gamma \in \mathbb{C}$. *Then* $\widetilde{\varrho}_B^{1,\lambda,\lambda^2} \sim \widetilde{\varrho}_B^{1,\gamma,\gamma^2}$.

Proof: Using the realization above, it is easy to see that the image of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ under any of the embeddings $\widetilde{\varrho}_B^{1,\lambda,\lambda^2}$ is the 4×4 block in the upper left corner of $\mathfrak{so}(7,\mathbb{C})$.

Then, using Eq. (53), it is easy to check that, in (4, 3) block form:

$$U_\lambda = \widetilde{\varrho}_B^{1,\lambda,\lambda^2}(u) = \begin{pmatrix} 0_{4\times 4} & -A^{tr} \\ A & 0_{3\times 3} \end{pmatrix}, \tag{58}$$

where A is the 3×4 matrix $\begin{pmatrix} v_1 & iv_1 & 0 & 0 \\ v_3 & iv_2 & 0 & 0 \\ v_3 & iv_3 & 0 & 0 \end{pmatrix}$, with

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_3 \\ v_3 \end{pmatrix} = \mathbf{v}_\lambda = \begin{pmatrix} -\frac{1}{2} + \frac{1}{2}\lambda^2 \\ \frac{i}{2} + \frac{i}{2}\lambda^2 \\ -\lambda \end{pmatrix}. \tag{59}$$

Note that $\mathbf{v}_\lambda$ is a nonzero null vector, for any $\lambda \in \mathbb{C}$.

Now, given $\lambda, \gamma \in \mathbb{C}$, we know that there is $\omega \in SO(3)_\mathbb{C}$ such that $\omega \mathbf{v}_\lambda = \mathbf{v}_\gamma$. Let $\Omega = \begin{pmatrix} id_{4\times 4} & 0 \\ 0 & \omega \end{pmatrix}$. Then $\Omega \in SO(7)_\mathbb{C}$, Ω commutes with $\widetilde{\varrho}_B^{1,\lambda,\lambda^2}(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}))$, and $\Omega U_\lambda \Omega^{-1} = U_\gamma$. Hence Ω intertwines the two embeddings, proving that $\widetilde{\varrho}_B^{1,\lambda,\lambda^2} \sim \widetilde{\varrho}_B^{1,\gamma,\gamma^2}$. □

Lemma 8.3 and Proposition 8.5 imply the following theorem, which classifies the embeddings of $\mathfrak{p}(3,1)$ into $\mathfrak{so}(7,\mathbb{C})$.

Theorem 8.6: *The embedding* $\widetilde{\varrho}_B^{1,0,0}$ *is the unique embedding of* $\mathfrak{p}(3,1)$ *into* $\mathfrak{so}(7,\mathbb{C})$, *up to equivalence.*

Remark 8.7: One could consider the irreducible representations of $\mathfrak{so}(7,\mathbb{C})$ restricted to $\mathfrak{p}(3,1)$ under any embedding, much as we did in the above section with embeddings of $\mathfrak{p}(3,1)$ into $\mathfrak{sl}(4,\mathbb{C})$. It is an open question whether irreducible representation of $\mathfrak{so}(7,\mathbb{C})$ restricted to $\mathfrak{p}(3,1)$ remain indecomposable with respect to a given embedding. We note that the technique of the proof of Theorem 7.3 cannot be applied in this case since the number of positive roots of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ plus the dimension of $\mathbb{C}^2 \otimes \mathbb{C}^2$ is less than the number of positive roots of $\mathfrak{so}(7,\mathbb{C})$.

IX. NO EMBEDDINGS OF $\mathfrak{p}(3,1)$ INTO $\mathfrak{sp}(6,\mathbb{C})$

In this section we show that there are no embeddings of $\mathfrak{p}(3,1)$ into $\mathfrak{sp}(6,\mathbb{C})$. We begin by considering the decomposition of $\mathfrak{sp}(6,\mathbb{C})$ with respect to the adjoint action of $\varrho_C(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}))$,

$\varrho_C \circ \varepsilon(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}))$, $\varsigma_C(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}))$, or $\varsigma_C \circ \varepsilon(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}))$, respectively:

$$\begin{aligned} \mathfrak{sp}(6,\mathbb{C}) \cong_{\varrho_C} & [X_1+X_3]_{\varrho_C} \oplus [X_4+X_5]_{\varrho_C} \oplus \\ & [X_9]_{\varrho_C} \\ \cong_{\varrho_C} & V(2)\otimes V(0) \oplus V(0)\otimes V(2) \oplus \\ & V(2)\otimes V(4), \end{aligned} \tag{60}$$

$$\begin{aligned} \mathfrak{sp}(6,\mathbb{C}) \cong_{\varrho_C\circ\varepsilon} & [X_1+X_3]_{\varrho_C\circ\varepsilon} \oplus [X_4+X_5]_{\varrho_C\circ\varepsilon} \oplus \\ & [X_9]_{\varrho_C\circ\varepsilon} \\ \cong_{\varrho_C\circ\varepsilon} & V(0)\otimes V(2) \oplus V(2)\otimes V(0) \oplus \\ & V(4)\otimes V(2). \end{aligned} \tag{61}$$

$$\begin{aligned} \mathfrak{sp}(6,\mathbb{C}) \cong_{\varsigma_C} & [X_3]_{\varsigma_C} \oplus [X_6]_{\varsigma_C} \oplus \\ & [X_9]_{\varsigma_C} \oplus [X_1+X_7]_{\varsigma_C} \\ \cong_{\varsigma_C} & V(2)\otimes V(0) \oplus V(1)\otimes V(3) \oplus \\ & V(0)\otimes V(6) \oplus V(0)\otimes V(2), \end{aligned} \tag{62}$$

$$\begin{aligned} \mathfrak{sp}(6,\mathbb{C}) \cong_{\varsigma_C\circ\varepsilon} & [X_3]_{\varsigma_C\circ\varepsilon} \oplus [X_6]_{\varsigma_C\circ\varepsilon} \oplus \\ & [X_9]_{\varsigma_C\circ\varepsilon} \oplus [X_1+X_7]_{\varsigma_C\circ\varepsilon} \\ \cong_{\varsigma_C\circ\varepsilon} & V(0)\otimes V(2) \oplus V(3)\otimes V(1) \oplus \\ & V(6)\otimes V(0) \oplus V(2)\otimes V(0). \end{aligned} \tag{63}$$

Since $V(1)\otimes V(1)$ does not occur in any of the above decompositions, we cannot lift ϱ_C, $\varrho_C\circ\varepsilon$, ς_C, or $\varsigma_C\circ\varepsilon$ to $\mathfrak{p}(3, 1)$.

We now consider the decomposition of $\mathfrak{sp}(6,\mathbb{C})$ with respect to the adjoint action of $\vartheta_C(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}))$:

$$\begin{aligned} \mathfrak{sp}(6,\mathbb{C}) \cong_{\vartheta_C} & [X_5] \oplus [Y_2] \oplus \\ & [X_1] \oplus [X_8] \oplus \\ & [X_6] \oplus [X_9] \oplus \\ & [X_3] \oplus [X_7] \oplus \\ & [Y_7] \oplus [H_2+H_3] \\ \cong_{\vartheta_C} & 2V(1)\otimes V(0) \oplus 2V(0)\otimes V(1) \oplus \\ & V(1)\otimes V(1) \oplus V(0)\otimes V(2) \oplus \\ & V(2)\otimes V(0) \oplus 3V(0)\otimes V(0). \end{aligned} \tag{64}$$

Lemma 9.1: $[X_6]_{\vartheta_C}$ is not Abelian.

Proof: Note that X_6, and $[Y_3, [Y_9, X_6]] \in [X_6]_{\vartheta_C}$, however $[X_6, [Y_3, [Y_9, X_6]]] = -H_1 - H_2 - 2H_3 \neq 0$. □

Since $[X_6]_{\vartheta_C} \cong V(1)\otimes V(1)$, and $V(1)\otimes V(1)$ has multiplicity 1 in the decomposition of Eq. (64), Lemma 9.1 implies that ϑ_C cannot be lifted to $\mathfrak{p}(3, 1)$.

We now consider the decomposition of $\mathfrak{sp}(6,\mathbb{C})$ with respect to the adjoint action of $\varphi_C(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$ and $\varphi_C\circ\varepsilon(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$, respectively:

$$\begin{aligned}\mathfrak{sp}(6,\mathbb{C}) \;\cong_{\varphi_C}\;& [X_1]_{\varphi_C} \oplus [X_3]_{\varphi_C} \oplus\\ & [X_9]_{\varphi_C} \oplus [Y_7]_{\varphi_C} \oplus\\ & [X_6]_{\varphi_C} \oplus [Y_2]_{\varphi_C} \oplus\\ & [H]_{\varphi_C}\\ \cong_{\varphi_C}\;& V(2)\otimes V(0) \oplus V(0)\otimes V(2) \oplus\\ & 2V(2)\otimes V(0) \oplus 2V(1)\otimes V(1) \oplus\\ & V(0)\otimes V(0),\end{aligned} \tag{65}$$

$$\begin{aligned}\mathfrak{sp}(6,\mathbb{C}) \;\cong_{\varphi_C\circ\varepsilon}\;& [X_1]_{\varphi_C\circ\varepsilon} \oplus [X_3]_{\varphi_C\circ\varepsilon} \oplus\\ & [X_9]_{\varphi_C\circ\varepsilon} \oplus [Y_7]_{\varphi_C\circ\varepsilon} \oplus\\ & [X_6]_{\varphi_C\circ\varepsilon} \oplus [Y_2]_{\varphi_C\circ\varepsilon} \oplus\\ & [H]_{\varphi_C\circ\varepsilon}\\ \cong_{\varphi_C\circ\varepsilon}\;& V(0)\otimes V(2) \oplus V(2)\otimes V(0) \oplus\\ & 2V(0)\otimes V(2) \oplus 2V(1)\otimes V(1) \oplus\\ & V(0)\otimes V(0),\end{aligned} \tag{66}$$

where

$$X_6 = X_{1,2,3}, \quad X_7 = \tfrac{1}{2}X_{3,2,2}, \quad H = H_1 + 2H_2 + 2H_3. \tag{67}$$

Lemma 9.2: Suppose α and β are not both zero, then $[\alpha X_6+\beta Y_2]_{\varphi_C}$ and $[\alpha X_6+\beta Y_2]_{\varphi_C\circ\varepsilon}$ are not Abelian.

Proof: A basis for $[\alpha X_6+\beta Y_2]_{\varphi_C}$ and for $[\alpha X_6+\beta Y_2]_{\varphi_C\circ\varepsilon}$ is given by

$$\alpha X_6+\beta Y_2, \quad \alpha X_5-\beta Y_4, \quad \alpha X_2+\beta Y_6, \quad \alpha X_4-\beta Y_5. \tag{68}$$

Consider

$$[\alpha X_6+\beta Y_2, \alpha X_2+\beta Y_6] = \alpha^2 X_8 + \alpha\beta(H_1+2H_3) + \beta^2 Y_8. \tag{69}$$

Hence $[\alpha X_6+\beta Y_2]_{\varphi_C}$ and $[\alpha X_6+\beta Y_2]_{\varphi_C\circ\varepsilon}$ are not Abelian if α and β are not both zero. □

Lemma 9.2 implies that neither φ_C nor $\varphi_C\circ\varepsilon$ can be lifted to $\mathfrak{p}(3,1)$.

We have shown that we cannot lift φ_C, $\varphi_C\circ\varepsilon$, ϱ_C, $\varrho_C\circ\varepsilon$, ϑ_C, ς_C, or $\varrho_C\circ\varsigma$ to $\mathfrak{p}(3,1)$. Since φ_C, $\varphi_C\circ\varepsilon$, ϱ_C, $\varrho_C\circ\varepsilon$, ϑ_C, ς_C, and $\varrho_C\circ\varsigma$ is a complete list of inequivalent embeddings of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{sp}(6,\mathbb{C})$ (Theorem 6.6), we have the following theorem.

Theorem 9.3: *There are no embeddings of* $\mathfrak{p}(3,1)$ *into* $\mathfrak{sp}(6,\mathbb{C})$.

X. CONCLUSIONS

We have classified the embeddings of the Poincaré algebra into the rank 3 simple Lie algebras. Up to inner automorphism, we showed that there are exactly two embeddings of $\mathfrak{p}(3,1)$ into $\mathfrak{sl}(4,\mathbb{C})$, which are, however, related by an outer automorphism of $\mathfrak{sl}(4,\mathbb{C})$ (Theorem 7.2). There is a unique embedding of $\mathfrak{p}(3,1)$ into $\mathfrak{so}(7,\mathbb{C})$, up to inner automorphism (Theorem 8.6), but no embeddings of $\mathfrak{p}(3,1)$ into $\mathfrak{sp}(6,\mathbb{C})$ (Theorem 9.3). We record the classification in Table II below.

TABLE II. Classification of embeddings of $\mathfrak{p}(3, 1)$ into the rank 3 simple Lie algebras up to equivalence.

Rank 3 simple Lie algebra $\mathfrak{g}$	Embeddings $\mathfrak{p}(3, 1) \hookrightarrow \mathfrak{g}$
$\mathfrak{sl}(4, \mathbb{C})$	$\widetilde{\varphi}^1_{A,X_{3,2,1}}, \widetilde{\varphi}^1_{A,Y_2}$
$\mathfrak{so}(7, \mathbb{C})$	$\widetilde{\varrho}^{1,0,0}_B$
$\mathfrak{sp}(6, \mathbb{C})$	None

As an application, we also showed that each irreducible highest weight module of $\mathfrak{sl}(4, \mathbb{C})$ (not necessarily finite-dimensional) remains indecomposable when restricted to $\mathfrak{p}(3, 1)$, with respect to any embedding of $\mathfrak{p}(3, 1)$ into $\mathfrak{sl}(4, \mathbb{C})$: We thus created a large family of indecomposable representations of $\mathfrak{p}(3, 1)$ (Theorem 7.3).

ACKNOWLEDGMENTS

The work of A.D. is partially supported by a research grant from the Professional Staff Congress/City University of New York (PSC/CUNY). The work of H.deG. and J.R. is partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

The authors would like to thank Andrey Minchenko and Willem de Graaf for valuable correspondence regarding semisimple subalgebras in simple Lie algebras. Finally, the authors would like to thank the anonymous reviewer for beneficial comments.

[1] G. Amelin-Camelia, "Doubly-special relativity: Facts, myths and some key open issues," Symmetry **2**, 230–271 (2010).
[2] A. Borowiec and A. Pachol, "The classical basis for the k-Poincaré Hopf algebra and doubly special relativity theories," J. Phys. A: Math. Theor. **43**, 045203 (2010).
[3] H. D. Doebner and J. D. Hennig, "Poincaré subalgebras in $\mathfrak{so}(4, 2)$ and $\mathfrak{sl}(5, \mathbb{R})$," Rep. Math. Phys. **5**(3), 321–326 (1974).
[4] A. Douglas and J. Repka, "Indecomposable representations of the Euclidean algebra $\mathfrak{e}(3)$ from irreducible representations of the symplectic algebra $\mathfrak{sp}(4, \mathbb{C})$," J. Phys.: Conf. Ser. **284**, 012022 (2011).
[5] W. Fulton and J. Harris, *Representation Theory* (Springer-Verlag, New York, 1991).
[6] B. C. Hall, *Lie Groups, Lie Algebras, and Representation Theory* (Springer-Verlag, New York, 2003).
[7] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory* (Springer-Verlag, New York, 1972).
[8] J. Kowalski-Glikman, "Introduction to doubly special relativity," *Planck Scale Effects in Astrophysics and Cosmology*, Lecture Notes in Physics Vol. 669, edited by J. Kowalski-Glikman and G. Amelino-Camelia (Springer, Berlin/Heidelberg, 2005).
[9] W. A. de Graaf, "SLA-computing with Simple Lie algebras. A GAP package," 2012; see http://www.science.unitn.it/~degraaf/sla.html.
[10] W. A. de Graaf, "Constructing semisimple subalgebras of semisimple Lie algebras," J. Algebra **325**, 416–430 (2011).
[11] R. Lenczewski and B. Gruber, "Indecomposable representations of the Poincaré algebra," J. Phys. A **19**(1), 1–20 (1986).
[12] G. Mack and I. Todorov, "Irreducibility of the ladder representations of U(2, 2) when restricted to the Poincaré subgroup," J. Math. Phys. **10**(11), 2078–2085 (1969).
[13] J. Mickelsson and J. Niederle, "On representations of the conformal group which when restricted to its Poincaré or Weyl subgroups remain irreducible," J. Math. Phys. **13**(1), 23–27 (1972).
[14] A. N. Minchenko, "The semisimple subalgebras of exceptional Lie algebras," Trans. Mosc. Math. Soc. S **0077–1554**(06), 225–259 (2006).
[15] G. S. Pogosyan and P. Winternitz, "Separation of variables and subgroup bases on n-dimensional hyperboloids," J. Math. Phys. **43**(6), 3387–3410 (2002).
[16] S. Ström, "Construction of representations of the inhomogeneous Lorentz group by means of contraction of representations of the (1+4) de Sitter group," Ark. Fys. **30**(31), 455–472 (1965).
[17] E. P. Wigner, "On unitary representations of the inhomogeneous Lorentz group," Ann. Math. **40**, 149–204 (1939).