Dynamical symmetry reduction and discrete tomography of a Ξ atom

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Abstract
When implemented using a reasonable Hamiltonian, the tomography of a three-level Ξ atom
is complicated by the equidistant energy levels of the atom. This restricts the possible
transformations to those in the SO(3) subgroup of SU(3). Although complete reconstruction is
possible for a single Ξ atom using a continuous set of tomograms, the discrete optimal set of
tomograms, related to mutually unbiased bases in dimension 3, are not accessible by time
evolution. We discuss here the search for an optimal set of discrete basis states compatible
with the reduced SO(3) symmetry of the system.

1. Introduction: general comments on tomographic reconstruction

For an n-dimensional quantum system, ρ is an $n \times n$ matrix specified by $n^2 - 1$ real parameters. Thus, any attempt to determine ρ requires at least $n^2 - 1$ measurements. Tomographic reconstruction is based on measurements of the populations of an appropriately 'rotated' density matrix ρ'. These measurements usually produce linear combinations of the original parameters of ρ. Obviously the measurements must be chosen so that they result in a non-singular linear system of equations that can be inverted.

More specifically, the diagonal elements

$$\rho_{kk} = \mathrm{tr}(\rho|k\rangle\langle k|) = \langle k|\rho|k\rangle \tag{1}$$

of ρ are directly accessible provided we can measure the population of levels k. To obtain off-diagonal matrix elements, we need a 'rotated basis' $U(\Omega)|k\rangle$, where Ω parameterizes an $n \times n$ basis transformation U. Careful selection of a set of Ω_i should yield a system of invertible equations of the form

$$\omega(\Omega_i) = \langle k|U^{\dagger}(\Omega_i)\rho\, U(\Omega_i)|k\rangle. \tag{2}$$

If there are no constraints, i.e. if any $U(\Omega)$ is achievable experimentally, the optimal (in the sense of [1]) set of $U(\Omega_i)$ are those that transform basis states $\{|k\rangle\}$ into a set of basis states $\{|k'\rangle\}$ that is unbiased with respect to the original $\{|k\rangle\}$ used to measure the populations [1][4].

The questions we wish to address here are the following: (i) What is the next optimal choice of Ω_i when there are constraints so that some of the optimal $U(\Omega)$ described in [1] cannot be achieved experimentally? (ii) How does one define an optimal choice when there are constraints? In this contribution, we will discuss this question in the context of tomography of a single Ξ atom.

2. The case of SU(2)

Let us first review the procedure for a quantum system with two outcomes with no constraint, where the possible transformations $U(\Omega)$ are elements of the group SU(2).

It is particularly convenient to write the density matrix ρ in the form

$$\rho = \begin{pmatrix} \frac{1}{2} + z & x + \mathrm{i}y \\ x - \mathrm{i}y & \frac{1}{2} - z \end{pmatrix} = \frac{1}{2}\mathbb{1}_{2\times 2} + z\sigma_z + x\sigma_x + y\sigma_y. \tag{3}$$

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[4] For a more mathematical description see Petz [1].

The tomograms $\omega(\alpha, \beta)$ are obtained using

$$U(\alpha, \beta, \gamma) = e^{-i\alpha\sigma_z/2} e^{-i\beta\sigma_y/2} e^{-i\gamma\sigma_z/2}, \quad |1\rangle = (1, 0)^{\mathrm{T}} \quad (4)$$

(where T is transposed so that $|1\rangle$ is a column vector). Explicitly,

$$\omega(\alpha, \beta) = \langle 1|U^{\dagger}(\alpha, \beta, \gamma) \rho U(\alpha, \beta, \gamma)|1\rangle \quad (5)$$

$$= \tfrac{1}{2} + z \cos \alpha - x \cos \beta \sin \alpha + y \sin \alpha \sin \beta. \quad (6)$$

The angle γ does not enter this expression and so can be chosen at our convenience.

One can solve for the three unknowns x, y and z in a number of ways: it is enough to choose the angles in equation (6) so as to generate three linearly independent equations. However, when acting on $|1\rangle$, the choices $U_z = U(0, 0, 0)$, $U_x = U(0, \tfrac{1}{2}\pi, 0)$ and $U_y = U(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi, -\tfrac{1}{2}\pi)$ produce eigenstates of σ_z, σ_x and σ_y, respectively; these eigenstates have been shown to be optimal [1] in the sense that they are unbiased, e.g. $|\langle \psi_x^{\pm} | \chi_y^{\pm} \rangle|^2 = \tfrac{1}{2}$.

The transformations U_x, U_y and U_z producing optimal vectors have an interesting property: they are elements of a finite group G of order 48 which map the eigenvectors of the Pauli matrices into themselves.

The Pauli matrices σ_x, σ_y and σ_z (or, rather, their eigenvectors) appear naturally even although the transformations U are elements of the SU(2) group rather than the related algebra. The Pauli matrices (up to factors of i) are elements in a set of eight matrices

$$\pm \mathbb{1}_{2 \times 2}, \quad \pm i\sigma_x, \quad \pm i\sigma_y, \quad \pm i\sigma_z, \quad (7)$$

which form a finite subgroup of SU(2) called the Pauli group in dimension 2 $\wp_2$ [2]. G acts by conjugation on $\wp_2$, transforming elements of $\wp_2$ into themselves.

3. The case of SU(3)

For a quantum mechanical system with three possible outcomes, the transformations U between the states of the system are SU(3) transformations. The density matrix ρ can be conveniently written in the form

$$\rho = \begin{pmatrix} \tfrac{1}{3} + d + d^* & a + \omega b + \omega^2 c & a^* + b^* + c^* \\ a^* + \omega^2 b^* + \omega c^* & \tfrac{1}{3} + \omega d + \omega^2 d^* & a + \omega^2 b + \omega c \\ a + b + c & a^* + \omega b^* + \omega^2 c^* & \tfrac{1}{3} + \omega^2 d + \omega d^* \end{pmatrix} \quad (8)$$

with $\omega = e^{2\pi i/3}$. This parameterization of the density matrix is not accidental.

Another parameterization, and one method for reconstructing ρ, is discussed in [3]. However, the prescription of [3] requires the continuous monitoring of some observables so as to extract their time evolution. Such continuous monitoring implies an infinite number of measurements.

Here, we focus instead on describing a finite number of measurements and the related finite group structure. We introduce the matrices

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \quad (9)$$

From this, we generate the following 24 matrices:

$$\begin{aligned} A_k^- &= \omega^{-k} X, & A_k &= \omega^k X^2, \\ D_k^- &= \omega^{-k} Z^2, & D_k &= \omega^k Z, \\ B_k &= \omega^k X^2 Z, & B_k^- &= \omega^{-k} X Z^2, \\ C_k &= \omega^k X^2 Z^2, & C_k^- &= \omega^{-k} X Z \end{aligned} \quad (10)$$

with $k = 0, 1, 2$. Augmented with $I_k = \omega^k \mathbb{1}_{3\times 3}$, these 27 matrices form the Pauli subgroup $\wp_3$ of SU(3). The subset $\{A_0, A_0^-, B_0, B_0^-, C_0, C_0^-, D_0, D_0^-, \mathbb{1}_{3\times 3}\}$ is a basis for 3×3 Hermitian matrices; ρ, given by equation (7), takes the form

$$\begin{aligned} \rho = \tfrac{1}{3}\mathbb{1} &+ a A_0 + a^* A_0^- + b B_0 + b^* B_0^- \\ &+ c C_0 + c^* C_0^- + d D_0 + d^* D_0^*. \end{aligned} \quad (11)$$

Operators in sets $A = \{A_0, A_1, A_2\}$ and $A^- = \{A_0^-, A_1^-, A_2^-\}$ obviously commute with one another, as do those in sets B and B^-, C and C^-, and D and D^-. Eigenvectors of A-type operators are unbiased with those eigenvectors of operators of type B, C and D, e.g. $|\langle \psi_i^{\alpha} | \chi_i^{\beta} \rangle|^2 = \tfrac{1}{3}$, where $\alpha = A_k$ or A_k^- and $\beta = B_k$ or B_k^-.

The expansion of ρ in equation (10) is equivalent to an expansion in terms of unbiased projectors, and thus the direct generalization to SU(3) of the expansion of equation (3) in terms of Pauli matrices.

Using (for instance) the eigenstate $|\psi_{B_0}^3\rangle = \tfrac{1}{\sqrt{3}}(1, -\omega, 1)^{\mathrm{T}}$ of B_0,

$$\langle \psi_{B_0}^3|\rho|\psi_{B_0}^3\rangle = \tfrac{1}{3} + (b_0 + b_0^*). \quad (12)$$

Next, using one eigenstate $|\psi_{B_0}^{-,3}\rangle = \tfrac{1}{\sqrt{3}}(\omega^2, \omega^2, 1)^{\mathrm{T}}$ of B_0^-,

$$\langle \psi_{B_0}^{-,3}|\rho|\psi_{B_0}^{-,3}\rangle = \tfrac{1}{3} - \tfrac{1}{2}(1 + i\sqrt{3})(b_0 + b_0^*) + i\sqrt{3}b_0. \quad (13)$$

We can clearly continue in this way until we solve for every unknown in ρ.

There also exists a (large) finite subgroup $\tilde{G}$ of SU(3) that, in direct generalization of G for SU(2), transforms MUB vectors into MUB vectors or, equivalently, that conjugates $\wp_3$ into itself.

4. The Ξ atom

In a Ξ atom, the levels are equally spaced. A resonant field will simultaneously excite both $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$ transitions, resulting in a Hermitian operator of the form [4]

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim \hat{L}_x/\sqrt{2}, \quad (14)$$

where $\hat{L}_x$ is the usual angular momentum operator. It is not possible to separately access all transitions and there is a dynamical symmetry reduction: the possible transformations form a group equivalent to the subgroup SO(3) of SU(3). As a consequence, one cannot prepare *every* MUB vector.

Reconstruction is possible using continuous SO(3) transformations [4]; as with the prescription of [3], this requires an infinite number of measurements and is not efficient.

We ask instead what is an optimal finite basis of SO(3)-related vectors for tomography of the Ξ atom. By optimal, we expect not only that the number of vectors will be minimal but also that they will minimize the error in the procedure for reconstructing ρ. We ask if we can find a finite subgroup of SO(3) that acts on these basis states in the manner that the finite subgroups G and $\tilde{G}$ of SU(2) and SU(3) respectively, described in sections 2 and 3, act on elements of a mutually unbiased basis.

We performed a numerical search for the most 'MUB-like' vectors reachable from $|1\rangle$ by SO(3) transformation. We searched for four families of two orthogonal vectors such that the distance function [5]

$$F = \sum_{1 \leqslant b \leqslant b'}^{4} \sum_{j=1}^{2} \sum_{j'=1}^{2} \left(|\langle \psi_j^b(\Omega_j^b) \,|\, \psi_{j'}^{b'}(\Omega_{j'}^{b'})\rangle| - \chi_{jj'}^{bb'} \right)^2, \quad (15)$$

$$\chi_{jj'}^{bb'} = \begin{cases} \delta_{jj'} & \text{if } b = b', \\ \frac{1}{\sqrt{3}} & \text{if } b \neq b' \end{cases}$$

is minimum. Here, we note that the function F is 0 if we input the usual MUB vectors [5]. Some intuition in the solution vectors is gained by mapping the resulting set of vectors to points in $\mathbb{R}^3$ using $(\langle L_x \rangle, \langle L_y \rangle, \langle L_z \rangle)$. We found that the resulting vectors are always at the vertices of a cube. The symmetry group of the cube is a finite group: the octahedral group, containing 24 elements. Unfortunately, these vectors connected by octahedral transformations only produce five linearly independent tomograms: we cannot reconstruct the density matrix using those vectors. (The number of linearly independent tomograms is related to the SO(3) quadrupole moments of the basis described in [3].)

An alternative to the most 'MUB-like' vectors is to search for a set of vectors that will produce the largest determinant of the resulting system of equations in the unknowns. We expect that maximizing the determinant will minimize the error, just like the usual MUB vectors also maximize this determinant for SU(2) and SU(3).

Searching for those vectors that maximize the determinant using SO(3) transformation yields a set of eight linear vectors that (by construction) allow us to recover the unknowns. However, these vectors are distinct from the eight vectors minimizing F in equation (15).

Unfortunately, this new set of vectors maximizing the determinant displays no obvious symmetry, although they could be related to the icosahedra (the solid with 12 vertices described by a finite group of 60 elements) or the dodecahedra (the solid with 20 vertices also described by a finite subgroup with 60 elements). Even if we do find some 'nice' symmetry property between those vectors that optimize the determinant

and some regular polytope, there remains the question: what is the meaning of the extra vertices? Does this imply that there is more than one choice of optimal basis?

5. Conclusion

In this contribution, we have emphasized the role of finite groups in tomography. The generalized Pauli matrices (in two and three dimensions) form a subgroup of SU(2) and SU(3), respectively. Eigenvectors of the generalized Pauli matrices are MUB vectors, and there is a finite group transforming these vectors among themselves. Besides satisfying the usual overlap condition, MUB vectors also produce a maximum in the determinant of the system associated with the tomograms.

Finite groups appear to be natural starting points for the search of optimal basis vectors, particularly in systems like Ξ, where a dynamical reduction of the symmetry from SU(3) to SO(3) prevents the implementation of the familiar MUB transformations. For atoms in the Λ configuration, the symmetry is reduced from SU(3) to SU(2) × U(1) [4], and MUBs cannot be implemented either.

In the Ξ atom, the vectors that minimize the distance proposed in [5] are at the vertices of a cube, and there is a finite group associated with this solid. Unfortunately, the vectors optimizing this distance do not produce enough linearly independent tomograms to reconstruct the density matrix. An alternative set of vectors has been found: it is one that maximizes the determinant of the associated system of tomograms. However, this set does not minimize the distance of [5]. Moreover, there are no obvious group properties associated with these vectors at the moment.

Acknowledgments

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