

Letter

***D*-functions and immanants of unitary matrices and submatrices**

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Received 12 October 2015

Accepted for publication 14 December 2015

Published 21 January 2016



CrossMark

Abstract

Motivated by recent results in multiphoton interferometry, we expand a result of Kostant on immanants of an arbitrary $m \times m$ unitary matrix $T \in \text{SU}(m)$ to the submatrices of T . Specifically, we show that immanants of principal submatrices of a unitary matrix T are a sum $\sum_t D_t^{(\lambda)}(\Omega)$ of the diagonal D -functions of group element Ω , with t determined by the choice of submatrix, and the irrep (λ) determined by the immanant under consideration. We also provide evidence that this result extends to some submatrices that are not principal diagonal, and we discuss how this result can be extended to cases where T carries an $\text{SU}(m)$ representation that is different from the defining representation.

Keywords: immanants, unitary matrices, Wigner D -functions

1. Introduction and basic result

The immanant $\text{Imm}^{(\tau)}(T)$ of the matrix $m \times m$ matrix T , associated with the partition $\{\tau\}$ of S_m , is defined in [1] by

$$\begin{aligned} \text{Imm}^{(\tau)}(T) &:= \sum_{\sigma} \chi^{(\tau)}(\sigma) P(\sigma) [T_{11} T_{22} \dots T_{mm}], \\ &\equiv \sum_{\sigma} \chi^{(\tau)}(\sigma) T_{1\sigma(1)} T_{2\sigma(2)} \dots T_{m\sigma(m)}. \end{aligned} \quad (1)$$

where $\sigma \in S_m$ permutes k to $\sigma(k)$, and $\chi^{(\tau)}(\sigma)$ is the character of σ in the irrep $\{\tau\}$ of S_m . In this paper we explore the connection between immanants and group functions (or D -functions) for the unitary groups, and extend a result of Kostant [2] to submatrices of the fundamental representation of these groups.

Our work is motivated in part by intense renewed interest in immanants of submatrices of unitary matrices and, in particular, of permanants [3] of submatrices and their use in multi-photon interferometry [4–8]. Thus, our results are interesting within the paradigm of the BosonSampling problem [9, 10] and its deep link to issues in computational complexity theory.

Immanants of totally non-negative and of Hermitian matrices have been studied in [11–13]; our results instead are applicable to unitary matrices and depend on the well-known duality between representations of the unitary and of the symmetric groups [14, 15].

This duality identifies irreps of $SU(m)$ with irreps of S_N with $N \leq m$. If $\{\lambda\} = \{\lambda_1, \lambda_2, \dots, \lambda_N\}$ is a partition of $\lambda = \sum_k \lambda_k$ labelling an irrep of S_N , we choose to label irreps of $SU(m)$ using the round brackets (λ) with $m - 1$ entries defined by $(\lambda_1 - \lambda_2, \dots, \lambda_{N-1} - \lambda_N)$, and trailing zeroes omitted. Thus, the irrep $\{21\}$ of S_3 corresponds to the $SU(4)$ irrep $(110) \sim (11)$, the $SU(5)$ irrep $(1100) \sim (11)$ etc.

Group functions (or Wigner D -functions) are defined as the overlap between two basis states of the same irrep of $SU(m)$, one of which has been transformed by an element $\Omega \in SU(m)$. If $|\psi_\ell^{(\lambda)}\rangle, |\psi_t^{(\tau)}\rangle$ are any two basis states in irreps (λ) and (τ) respectively and $T^\lambda(\Omega)$ is the matrix representing element $\Omega \in SU(m)$ in the irrep (λ) , then

$$D_{t\ell}^{(\lambda)}(\Omega) := \langle \psi_\ell^{(\lambda)} | T^\lambda(\Omega) | \psi_t^{(\tau)} \rangle \delta_{\lambda,\tau}. \tag{2}$$

Kostant [2] has shown a simple connection between immanants (defined in equation (1)) of the fundamental representation T of $SU(m)$ group elements and group functions $D_{rt}^{(\lambda)}$ of $SU(m)$ with t running over each of the zero-weight states in irrep (λ) . Specifically, let $\Omega \in SU(m)$ and $T(\Omega)$ (no superscript) be the defining $m \times m$ representation of Ω . Further define the matrix $\mathcal{D}^{(\tau)}(\Omega)$ by

$$(\mathcal{D}^{(\tau)}(\Omega))_{rt} = D_{rt}^{(\tau)}(\Omega) \tag{3}$$

with r, t restricted to labelling zero-weight states in the irrep (τ) . Then we have [2]

$$\text{Imm}^{[\tau]}(T(\Omega)) = \text{Tr}[\mathcal{D}^{(\tau)}(\Omega)]. \tag{4}$$

For $SU(2)$, this result simply states that the permanent of the matrix

$$T(\Omega) = \begin{pmatrix} e^{-\frac{1}{2}i(\alpha+\gamma)} \cos\left(\frac{\beta}{2}\right) & -e^{-\frac{1}{2}i(\alpha-\gamma)} \sin\left(\frac{\beta}{2}\right) \\ e^{\frac{1}{2}i(\alpha-\gamma)} \sin\left(\frac{\beta}{2}\right) & e^{\frac{1}{2}i(\alpha+\gamma)} \cos\left(\frac{\beta}{2}\right) \end{pmatrix}, \tag{5}$$

where $\Omega = (\alpha, \beta, \gamma) \in SU(2)$ and is the $SU(2)$ function $\text{Imm}^{[2]}(T(\Omega)) = D_{00}^1(\alpha, \beta, \gamma) = \cos \beta$ while the determinant $\text{Imm}^{[1,1]}(T(\Omega)) = D_{00}^0(\alpha, \beta, \gamma) = 1$. The trace of equation (4) contains a single term in both $SU(2)$ cases as the zero-weight subspaces in irreps $J = 1$ and $J = 0$ are both one-dimensional. Here and henceforth we follow the physics notation of labelling $SU(2)$ irreps using the angular momentum label $J = \frac{1}{2}\lambda$, such that $2J$ is an integer. Thus, $D_{00}^1(\Omega)$ is an $SU(2)$ D -function in the three-dimensional irrep $J = 1$.

2. Notational details

We first introduce a basis for $\mathbb{H}_p^{(1)}$, which is the p 'th copy of the carrier space for fundamental irrep $\{1\} \equiv (1)$ of $SU(m)$. We write this basis in terms of harmonic oscillator states according to

$$\mathbb{H}_p^{(1)} = \text{Span} \{a_k^\dagger(\omega_p)|0\rangle, k = 1, \dots, m\}. \tag{6}$$

The label ω_p can be thought of as an internal degree of freedom, say the frequency, of the p 'th oscillator. We introduce the (reducible) Hilbert space $\mathbb{H}^{(N)} := \mathbb{H}_1^{(1)} \otimes \mathbb{H}_2^{(1)} \dots \otimes \mathbb{H}_N^{(1)}$, ($N \leq m$), which is spanned by the set of harmonic oscillator states of the type

$$a_k^\dagger(\omega_1)a_r^\dagger(\omega_2)\dots a_s^\dagger(\omega_N)|0\rangle, \quad k = 1, \dots, m; \quad r = 1, \dots, m, \text{ etc.} \tag{7}$$

Another ingredient we need is the action of the permutation group S_N on $\mathbb{H}^{(N)}$. The action of $P(\sigma)$ is defined as

$$\begin{aligned} P(\sigma)a_k^\dagger(\omega_1)a_r^\dagger(\omega_2)\dots a_s^\dagger(\omega_N)|0\rangle \\ = a_k^\dagger(\omega_{\sigma^{-1}(1)})a_r^\dagger(\omega_{\sigma^{-1}(2)})\dots a_s^\dagger(\omega_{\sigma^{-1}(N)})|0\rangle. \end{aligned} \tag{8}$$

Alternatively, one may consider each of the sets $\{a_k^\dagger(\omega_p); k = 1, \dots, m\}$, which are labelled by p , as a tensor operator that carries the defining irrep (1) of $SU(m)$. $|0\rangle$ is invariant under the action of S_N and $SU(m)$ elements.

The algebra $u(m)$ is spanned by the S_N -invariant operators

$$\hat{C}_{ij} = \sum_{k=1}^N a_i^\dagger(\omega_k)a_j(\omega_k) \quad i, j = 1, \dots, m. \tag{9}$$

The $su(m)$ subalgebra is obtained from $u(m)$ by removing the diagonal operator $\sum_{k=1}^m \hat{C}_{ii}$, so the Cartan subalgebra of $su(m)$ is spanned by the traceless diagonal operators

$$\hat{h}_i := \hat{C}_{ii} - \hat{C}_{i+1,i+1}, \quad i = 1, \dots, m - 1. \tag{10}$$

A basis for the irrep (λ) of $SU(m)$ is given in terms of the harmonic oscillator occupation number n according to

$$|\psi^{(\lambda)}n; \Lambda\rangle = |(\lambda)n_1n_2, \dots, n_m; (\lambda')\dots(J)\rangle, \tag{11}$$

where $n := (n_1, n_2, \dots, n_m)$ and n_k indicates the number of excitations in mode $k \leq m$. The weight of this state is equals the $(m - 1)$ -tuple $[n_1 - n_2, n_2 - n_3, \dots, n_{m-1} - n_m]$. Finally, the multi-index $\Lambda := (\lambda'), \dots, (J)$ refers to a collection of indices, each of which labels irreps in the subalgebra chain

$$\begin{matrix} su(m) \supset su(m - 1) \supset \dots \supset su(2) \\ (\lambda) \quad \quad (\lambda') \quad \quad \quad (J) \end{matrix} \tag{12}$$

and is needed to fully distinguish states having the same weight. The representation labels are all integers, except as mentioned above for the half-integered $SU(2)$ label J . We take the subalgebra $su(k - 1) \subset su(k)$ to be spanned by the subset of the $k \times k$ hermitian traceless matrices of the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & * \end{pmatrix}, \tag{13}$$

where $*$ denote possible non-zero entries in $su(k - 1)$.

Using this notation, we can write the matrix representation of $\Omega \in SU(3)$ in the fundamental representation (which is denoted by (1) once the trailing 0 has been eliminated) of $SU(3)$ as

$$T(\Omega) = \begin{pmatrix} D_{100(0);100(0)}^{(1)}(\Omega) & D_{100(0);010(\frac{1}{2})}^{(1)}(\Omega) & D_{100(0);001(\frac{1}{2})}^{(1)}(\Omega) \\ D_{010(\frac{1}{2});100(0)}^{(1)}(\Omega) & D_{010(\frac{1}{2});010(\frac{1}{2})}^{(1)}(\Omega) & D_{010(\frac{1}{2});001(\frac{1}{2})}^{(1)}(\Omega) \\ D_{001(\frac{1}{2});100(0)}^{(1)}(\Omega) & D_{001(\frac{1}{2});010(\frac{1}{2})}^{(1)}(\Omega) & D_{001(\frac{1}{2});001(\frac{1}{2})}^{(1)}(\Omega) \end{pmatrix}, \quad (14)$$

with $\Omega \in \text{SU}(3)$. The result of Kostant [2] applied to $\text{SU}(3)$ then states that

$$\begin{aligned} \text{Per}(T(\Omega)) &= \text{Imm}^{[3]}(T(\Omega)) = D_{111(1);111(1)}^{(3)}(\Omega), \\ \text{Imm}^{[21]}(T(\Omega)) &= D_{111(1);111(1)}^{(11)}(\Omega) + D_{111(0);111(0)}^{(11)}(\Omega), \\ \text{Det}(T(\Omega)) &= \text{Imm}^{[111]}(T(\Omega)) = D_{000(0);000(0)}^{(0)}(\Omega) = 1. \end{aligned} \quad (15)$$

For convenience, we lighten notation and use the symbols T and Ω to respectively denote matrices and elements in different $\text{SU}(m)$ without indicating m ; this does not affect our conclusions as our results apply to any m .

The novel contribution of this paper is to extend result of [2] encapsulated in equation (4) to submatrices of the fundamental representations. In addition, using outer plethysms, we can extend the theorem to immanants of matrix representations of $\text{SU}(m)$ beyond the fundamental representation and to matrix representations of subgroups of $\text{SU}(m)$.

3. Proving the theorem: the case $N = m$

We consider the state $|\Psi_{123\dots m}\rangle = a_1^\dagger(\omega_1)a_2^\dagger(\omega_2)\dots a_m^\dagger(\omega_m)|0\rangle$, which lives in the (reducible) tensor product space $\mathbb{H}^{(m)} = \mathbb{H}_1^{(1)} \otimes \mathbb{H}_2^{(1)} \dots \otimes \mathbb{H}_m^{(1)}$. The first lemma deals with the weight of this state.

Lemma 1. *The $\text{SU}(m)$ weight of $|\Psi_{123\dots m}\rangle$ is $[0]$.*

Proof. This is immediate since every mode is occupied once, so $n_i = 1 \forall i$. Since the component k of the weight is $n_k - n_{k+1}$, $\hat{h}_i|\Psi_{123\dots m}\rangle = 0 \forall i$. □

From lemma 1, we write $|\Psi_{123\dots m}\rangle$ as an expansion over zero-weight states in all irrep occurring in $\mathbb{H}^{(m)}$ according to

$$|\Psi_{123\dots m}\rangle = \sum_{\alpha\lambda\ell} \tilde{c}_\ell^{(\lambda)_\alpha} |\psi_\ell^{(\lambda)_\alpha}\rangle, \quad \tilde{c}_\ell^{(\lambda)_\alpha} = \langle \psi_\ell^{(\lambda)_\alpha} | \Psi_{123\dots m} \rangle, \quad (16)$$

where $(\lambda)_\alpha$ is the α 'th copy of the irrep $(\lambda)_\alpha$ of $\text{SU}(m)$ and ℓ labels those basis states that have 0-weight in the irrep $(\lambda)_\alpha$ of $\text{SU}(m)$.

Lemma 2. *With the notation above:*

$$\sum_\sigma \chi^{(\tau)}(\sigma) P(\sigma) |\Psi_{123\dots m}\rangle = \frac{m!}{\text{dim}(\tau)} \sum_{\alpha t} \tilde{c}_t^{(\tau)_\alpha} |\psi_t^{(\tau)_\alpha}\rangle. \quad (17)$$

The proof of lemma 2 relies on the duality between representations of the symmetric and the unitary groups. From duality, the basis states $\{|\psi_t^{(\tau)_\alpha}\rangle\}$ are also basis states for the irrep $\{\tau\}$ of S_m . Hence, using equation (8) we obtain

$$P(\sigma) |\Psi_{123\dots m}\rangle = \sum_{\alpha\lambda\ell} |\psi_\ell^{(\lambda)_\alpha}\rangle \langle \psi_\ell^{(\lambda)_\alpha} | P(\sigma) |\Psi_{123\dots m}\rangle, \quad (18)$$

$$= \sum_{\alpha\ell\lambda k} |\psi_\ell^{(\lambda)_\alpha}\rangle \Gamma_{\ell k}^{(\lambda)}(\sigma) \langle \psi_k^{(\lambda)_\alpha} | \Psi_{123\dots m}\rangle, \tag{19}$$

$$= \sum_{\alpha\ell\lambda k} |\psi_\ell^{(\lambda)_\alpha}\rangle \Gamma_{k\ell}^{(\lambda)}(\sigma^{-1}) \tilde{c}_k^{(\lambda)_\alpha}, \tag{20}$$

where $\Gamma^{(\lambda)}$ is the unitary irrep $\{\lambda\}$ of S_m . Writing $\chi^{(\tau)}(\sigma) = \sum_t \Gamma_t^{(\tau)}(\sigma)$ gives us

$$\sum_\sigma \chi^{(\tau)}(\sigma) P(\sigma) |\Psi_{123\dots m}\rangle = \sum_{\alpha k \lambda \ell} \tilde{c}_\ell^{(\lambda)_\alpha} |\psi_\ell^{(\lambda)_\alpha}\rangle \left[\sum_{\sigma t} \Gamma_t^{(\tau)}(\sigma) \Gamma_{\ell k}^{(\lambda)}(\sigma^{-1}) \right], \tag{21}$$

$$= \frac{m!}{\dim(\tau)} \sum_{\alpha t} \tilde{c}_t^{(\tau)_\alpha} |\psi_t^{(\tau)_\alpha}\rangle, \tag{22}$$

Proof. where we have used the orthogonality of characters to arrive at equation (22). \square

Because the action of $\Omega \in \text{SU}(m)$ commutes with the action of $\sigma \in S_m$, we have

$$\text{Imm}^{(\tau)}(T(\Omega)) = \sum_\sigma \chi^{(\tau)}(\sigma) P(\sigma) [T_{11}(\Omega) T_{22}(\Omega) \dots T_{mm}(\Omega)], \tag{23}$$

$$= \langle \Psi_{123\dots m} | [T(\Omega) \otimes T(\Omega) \dots \otimes T(\Omega)] \left[\sum_\sigma \chi^{(\tau)}(\sigma) P(\sigma) \right] | \Psi_{123\dots m}\rangle, \tag{24}$$

$$= \sum_{\alpha t} (\tilde{c}_t^{(\tau)_\alpha})^* \tilde{c}_t^{(\tau)_\alpha} \frac{m!}{\dim(\tau)} D_{tt}^{(\tau)}(\Omega). \tag{25}$$

Introducing the scaled coefficients $c_t^{(\tau)_\alpha} = \tilde{c}_t^{(\tau)_\alpha} \sqrt{\frac{m!}{\dim(\tau)}}$, we finally obtain

$$\text{Imm}^{(\tau)}(T(\Omega)) = \sum_{rt} \left[\sum_\alpha (c_r^{(\tau)_\alpha})^* c_t^{(\tau)_\alpha} \right] D_{rt}^{(\tau)}(\Omega), \tag{26}$$

where the sums over t and r is a sum over zero-weight states in $(\tau)_\alpha$.

This result is not unexpected as the operator

$$\hat{\Pi}^{(\tau)} = \left[\sum_\sigma \chi^{(\tau)}(\sigma) P(\sigma) \right], \quad \sigma \in S_m \tag{27}$$

is a projector to that subspace of S_m which has permutation symmetry $\{\tau\}$, and hence (by duality) is a projector to a subspace that carries (possibly multiple copies of) the irrep (τ) of $\text{SU}(m)$ in the m -fold product $(1)^{\otimes m}$.

Theorem 3. (Kostant [2]) $\text{Imm}^{(\tau)}(T(\Omega)) = \sum_t D_t^{(\tau)}(\Omega)$.

Proof. We present a proof that will eventually allow us to dispense with the requirements that $N = m$ and that states have zero-weight. Construct the matrix

$$W_{rt}^{(\tau)} = \sum_\alpha c_{\alpha t}^{(\tau)} (c_{\alpha r}^{(\tau)})^*. \tag{28}$$

Equation (26) then becomes

$$\text{Imm}^{(\tau)}(T(\Omega)) = \sum_{rt} W_{rt}^{(\tau)} D_{rt}^{(\tau)}(\Omega) = \text{Tr}[W^{(\tau)} \mathcal{D}^{(\tau)}(\Omega)], \quad (29)$$

with $\mathcal{D}^{(\tau)}(\Omega)$ defined in equation (3). Our objective is to prove that $W^{(\tau)}$ is the unit matrix.

Any immanant has the property of invariance under conjugation by elements in S_m i.e., the immanant of any matrix satisfies

$$\begin{aligned} \text{Imm}^{(\tau)}(T(\Omega)) &= \sum_{\sigma} \chi^{(\tau)}(\sigma) P(\sigma) [T_{11}(\Omega) T_{22}(\Omega) \dots T_{mm}(\Omega)] \\ &= \sum_{\sigma} \chi^{(\tau)}(\sigma) P^{-1}(\bar{\sigma}) P(\sigma) P(\bar{\sigma}) [T_{11}(\Omega) T_{22}(\Omega) \dots T_{mm}(\Omega)], \end{aligned} \quad (30)$$

with $\sigma, \bar{\sigma} \in S_m$. Under conjugation by $\bar{\sigma}$, equation (26) becomes

$$\begin{aligned} \text{Imm}^{(\tau)}(T(\Omega)) &= \text{Tr}[\Gamma^{(\tau)}(\bar{\sigma}) W^{(\tau)} \Gamma^{(\tau)}(\bar{\sigma}^{-1}) \mathcal{D}^{(\tau)}(\Omega)], \\ &= \text{Tr}[W^{(\tau)} \mathcal{D}^{(\tau)}(\Omega)]. \end{aligned} \quad (31)$$

Since $\mathcal{D}^{(\tau)}(\Omega)$ is certainly *not* the unit matrix for arbitrary Ω , it follows that

$$\Gamma^{(\tau)}(\bar{\sigma}) W^{(\tau)} \Gamma^{(\tau)}(\bar{\sigma}^{-1}) = W^{(\tau)}, \quad (32)$$

i.e. the matrix $W^{(\tau)}$ is invariant under any permutation. By Schur's lemma $W^{(\tau)}$ must therefore be proportional to the unit matrix, i.e. we have $W_{ts}^{(\tau)} = \xi \delta_{ts}$ with ξ the relevant constant of proportionality. The immanant thus takes the form

$$\text{Imm}^{(\tau)}(T(\Omega)) = \xi \left(\sum_t D_t^{(\tau)}(\Omega) \right). \quad (33)$$

To determine ξ , choose $\Omega = \mathbf{1}$. Then $T(\mathbf{1})$ is the $m \times m$ unit matrix, and $T_{k,\sigma(k)}(\mathbf{1})$ is zero unless $\sigma = \mathbf{1} \in S_m$. The immanant for $\Omega = \mathbf{1}$ is then just the dimension of the irrep $\{\tau\}$ and we have

$$\text{Imm}^{(\tau)}(T(\mathbf{1})) = \xi \chi^{(\tau)}(\mathbf{1}) = \dim(\tau) \quad \xi \left(\sum_t \mathbf{1} \right) = \xi \dim(\{\tau\}) \quad (34)$$

since $D_t^{(\tau)}(\mathbf{1}) = 1$. Hence, $\xi = 1$ and the theorem is proved. \square

4. Results on submatrices: the case $N < m$.

We now consider the submatrices of T . In multiphoton inteferometry, such submatrices describe the unitary scattering from an input state of the form

$$|\Psi_{k_1 \dots k_p}\rangle = a_{k_1}^\dagger(\omega_1) a_{k_2}^\dagger(\omega_2) \dots a_{k_p}^\dagger(\omega_p) |0\rangle, \quad p < m \quad (35)$$

to an output state $|\Psi_{\ell_1 \dots \ell_p}\rangle$, which need not be identical to $|\Psi_{k_1 \dots k_p}\rangle$. Both input and output live in the reducible Hilbert space $\mathbb{H}^{(p)}$, and have expansions of the form

$$|\Psi_{k_1 \dots k_p}\rangle = \sum_{\alpha \lambda \ell} \tilde{d}_\ell^{(\lambda)\alpha} |\psi_\ell^{(\lambda)\alpha}\rangle, \quad (36)$$

where $|\psi_\ell^{(\lambda)\alpha}\rangle$ has weight $[k_1 - k_2, k_2 - k_3, \dots, k_{p-1} - k_p]$.

4.1. Principal coaxial submatrices

First we select from $T(\Omega)$ a principal submatrix $\bar{T}(\Omega)_k$, i.e., $\bar{T}(\Omega)_k$ is obtained by keeping rows and columns $k = (k_1, k_2, \dots, k_p)$ with $p < m$. In such a case, the input and output states

are identical. The permutation group S_p shuffles the p indices k_1, k_2, \dots, k_p amongst themselves. Although the submatrix $\bar{T}(\Omega)_k$ is not unitary, the proof of theorem 3 does not depend on the unitarity of $T(\Omega)$ and so can be copied to show.

Corollary 4. *The immanant $\text{Imm}_k^{(\lambda)}(T(\Omega))$ of a submatrix $\bar{T}(\Omega)_k$, which is a principal submatrix of T , is given by*

$$\text{Imm}_k^{(\lambda)}(T(\Omega)) = \sum_r D_{rr}^{(\lambda)}(\Omega), \tag{37}$$

where (λ) is the irrep of $SU(m)$ corresponding to the partition $\{\lambda\}$, and where the sum over r is a sum over all the states in (λ) with weight $[k_1 - k_2, k_2 - k_3, \dots, k_{p-1} - k_p]$; following equation (36) this is the weight of $|\Psi_{k_1 \dots k_p}\rangle$ in equation (35) and need not be zero. \square

As an example, one can verify that, if we strip the 5×5 fundamental matrix representation of $SU(5)$ from its third and fifth rows and columns, then the states entering in the sum of equation (37) are linear combinations of terms of the form

$$P(\sigma)[a_1^\dagger(\omega_1)a_2^\dagger(\omega_2)a_4^\dagger(\omega_3)|0\rangle] \tag{38}$$

with weight $[0, 1, -1, 1]$. Using the $\text{su}(k) \downarrow \text{su}(k - 1)$ branching rules [16, 17] to label basis states, the $\{2, 1\}$ immanant of this submatrix is the sum

$$\begin{aligned} \text{Imm}_{24}^{\{2,1\}}(T(\Omega)) &= D_{11010(2)(1)(\frac{1}{2}); 11010(2)(1)(\frac{1}{2})}^{(1,1)}(\Omega) \\ &\quad + D_{11010(0,1)(1)(\frac{1}{2}); 11010(0,1)(1)(\frac{1}{2})}^{(1,1)}(\Omega), \end{aligned} \tag{39}$$

where the labels $(2)(1)(\frac{1}{2})$ and $(0, 1)(1)(\frac{1}{2})$ refer to the $\text{su}(4) \supset \text{su}(3) \supset \text{su}(2)$ chains of irreps (recall that trailing 0s are omitted).

Littlewood [1] has established a number of relations between immanants of a matrix and sums of products of immanants of principal coaxial submatrices. For instance, the equality for Schur functions $\{3\}\{1\} = \{3,1\} + \{4\}$ yields the immanant relation

$$\begin{aligned} \sum_{ijkl} (\text{Imm}_{ijk}^{\{3\}}(T(\Omega))) (\text{Imm}_\ell^{\{1\}}(T(\Omega))) \\ = \text{Imm}^{\{3,1\}}(T(\Omega)) + \text{Imm}^{\{4\}}(T(\Omega)), \end{aligned} \tag{40}$$

where the sum over $ijkl$ is a sum over complementary coaxial submatrices, i.e.

$$\begin{array}{cc|cc} ijk & \ell & ijk & \ell \\ \hline 123 & 4 & 124 & 3 \\ 134 & 2 & 234 & 1 \end{array} \tag{41}$$

This expands to a sum of products of immanants of submatrices given explicitly by

$$\begin{aligned} (\text{Imm}_{23}^{\{3\}}(T(\Omega))) (\text{Imm}_4^{\{1\}}(T(\Omega))) + (\text{Imm}_{24}^{\{3\}}(T(\Omega))) (\text{Imm}_3^{\{1\}}(T(\Omega))) \\ + (\text{Imm}_{34}^{\{3\}}(T(\Omega))) (\text{Imm}_2^{\{1\}}(T(\Omega))) + (\text{Imm}_{234}^{\{3\}}(T(\Omega))) (\text{Imm}_1^{\{1\}}(T(\Omega))) \\ = \text{Imm}^{\{3,1\}}(T(\Omega)) + \text{Imm}^{\{4\}}(T(\Omega)), \end{aligned} \tag{42}$$

which becomes an equality on the corresponding products of sum of SU(4) D -functions:

$$\begin{aligned}
 & D_{1110(2)(\frac{1}{2});1110(2)(\frac{1}{2})}^{(3)}(\Omega)D_{0001(1)(\frac{1}{2});0001(1)(\frac{1}{2})}^{(1)}(\Omega) \\
 & + D_{1101(2)(\frac{1}{2});1101(2)(\frac{1}{2})}^{(3)}(\Omega)D_{0010(1)(\frac{1}{2});0010(1)(\frac{1}{2})}^{(1)}(\Omega) \\
 & + D_{1011(2)(1);1011(2)(1)}^{(3)}(\Omega)D_{0100(1)(0);0100(1)(0)}^{(1)}(\Omega) \\
 & + D_{0111(3)(1);0111(3)(1)}^{(3)}(\Omega)D_{1000(0)(0);1000(0)(0)}^{(1)}(\Omega) \\
 = & D_{111(3)(1);1111(3)(1)}^{(21)}(\Omega) + D_{1111(11)(1);1111(11)(1)}^{(21)}(\Omega) \\
 & + D_{1111(11)(0);1111(11)(0)}^{(21)}(\Omega) + D_{1111(3)(1);1111(3)(1)}^{(4)}(\Omega). \tag{43}
 \end{aligned}$$

The subgroup labels are obtained by systematically using the $\mathfrak{su}(k) \downarrow \mathfrak{su}(k - 1)$ branching rules [18].

4.2. Generic submatrices

To fix ideas, we start with the 4×4 matrix T and remove row 1 and column 2 to obtain the submatrix \bar{T} :

$$T(\Omega) \rightarrow \bar{T}(\Omega) = \begin{pmatrix} T_{21}(\Omega) & T_{23}(\Omega) & T_{24}(\Omega) \\ T_{31}(\Omega) & T_{33}(\Omega) & T_{34}(\Omega) \\ T_{41}(\Omega) & T_{43}(\Omega) & T_{44}(\Omega) \end{pmatrix}. \tag{44}$$

The immanants of 3×3 submatrix $\bar{T}(\Omega)$ are in the form

$$\text{Imm}^{\{\lambda\}}(\bar{T}(\Omega)) = \sum_{\sigma} \chi^{\{\lambda\}}(\sigma) P(\sigma) [T_{11}(\Omega)T_{22}(\Omega)T_{34}(\Omega)] \tag{45}$$

$$= \hat{\Pi}^{\{\lambda\}} [T_{11}(\Omega)T_{22}(\Omega)T_{34}(\Omega)], \tag{46}$$

where $\hat{\Pi}^{\{\lambda\}}$ is the immanant projector of equation (27) and σ permutes the triple (124).

Let $\{a_k^\dagger(\omega_k)|0\rangle, k = 1, \dots, 4\}$ be a basis for the fundamental irrep of SU(4), and define

$$|\Psi_{134}\rangle := a_1^\dagger(\omega_1)a_3^\dagger(\omega_2)a_4^\dagger(\omega_3)|0\rangle, \tag{47}$$

$$|\Phi_{234}\rangle := a_2^\dagger(\omega_1)a_3^\dagger(\omega_2)a_4^\dagger(\omega_3)|0\rangle, \tag{48}$$

as three-particle states elements of $\mathbb{H}^{\{1\} \otimes \{1\} \otimes \{1\}}$. Clearly there is $\sigma' \in S_4$ such that

$$|\Psi_{234}\rangle = P(\sigma')|\Phi_{134}\rangle. \tag{49}$$

Indeed by inspection this element is given by $P(\sigma') = P_{12}$. More generally, if

$$|\Phi_k\rangle = a_{k_1}^\dagger(\omega) a_{k_2}^\dagger(\omega_2) a_{k_3}^\dagger(\omega_3) |0\rangle, \quad k = (k_1, k_2, k_3), \tag{50}$$

$$|\Psi_q\rangle = a_{q_1}^\dagger(\omega) a_{q_2}^\dagger(\omega_2) a_{q_3}^\dagger(\omega_3) |0\rangle, \quad q = (q_1, q_2, q_3) \tag{51}$$

then there is σ_{qk} exists such that $|\Psi_q\rangle = P(\sigma_{qk})|\Phi_k\rangle$. As the action of the permutation group commutes with the action of the unitary group:

$$\begin{aligned} \text{Imm}^{(\lambda)}(\bar{T}(\Omega))_{kq} &= \langle \Phi_k | [T(\Omega) \otimes T(\Omega) \dots \otimes T(\Omega)] \hat{\Pi}^{(\lambda)} P(\sigma_{qk}) | \Phi_k \rangle, \\ &= \langle \Phi_k | \hat{\Pi}^{(\lambda)} [T(\Omega) \otimes T(\Omega) \dots \otimes T(\Omega)] P(\sigma_{qk}) | \Phi_k \rangle, \\ &= \sum_{rs\alpha} \langle \Phi_k | \hat{\Pi}^{(\lambda)} | \psi_r^{(\lambda)\alpha} \rangle, \end{aligned} \tag{52}$$

$$\sum \langle \Phi_k | \hat{\Pi}^{(\lambda)} | \psi_r^{(\lambda)\alpha} \rangle \times \langle \psi_r^{(\lambda)\alpha} | T^{(\lambda)}(\Omega) P(\sigma_{qk}) | \psi_s^{(\lambda)\alpha} \rangle \langle \psi_s^{(\lambda)\alpha} | \Phi_k \rangle. \tag{53}$$

Now, the permutation $P(\sigma_{qk})$ is represented by a unitary matrix in the carrier space $(\lambda)_\alpha$. Thus, there exist $\Omega_{qk} \in \text{SU}(4)$ and a phase ζ such that $P(\sigma_{qk}) | \psi_s^{(\lambda)\alpha} \rangle = e^{i\zeta} T(\Omega_{qk}) | \psi_s^{(\lambda)\alpha} \rangle$. This transforms our original problem back to the case of principal submatrices, but with now an element $\Omega \cdot \Omega_{qk}$ i.e.,

$$\text{Imm}^{(\lambda)}(\bar{T}(\Omega)) = \sum_t D_t^{(\lambda)}(\Omega \cdot \Omega_{qk}). \tag{54}$$

Unfortunately, the action $P(\sigma_{qk}) | \psi_s^{(\lambda)\alpha} \rangle$ is in general highly non-trivial [19–21] and it is not obvious how to find Ω_{qk} , much less $(\Omega \cdot \Omega_{qk})$. Nevertheless, we found that the sum of D -functions that occur on the right-hand side of equation (53) always contains the same number of D as the dimension of the dual irrep $\{\tau\}$, and that the coefficients of these D 's is always one. This result relied on (i) evaluating the appropriate group functions using the algorithm [18], (ii) explicitly constructing each of the immanants of all possible 4×4 submatrices and of all possible 3×3 submatrices of the fundamental irrep of $\text{SU}(5)$ and (iii) explicitly constructing the immanants of 3×3 submatrices of the fundamental irrep of $\text{SU}(4)$ or $\text{SU}(5)$.

Thus, in the specific case of the submatrix given in equation (44), we have

$$\text{Imm}^{[21]}(\bar{T}(\Omega))_{(234)(134)} = D_{0111(2)(\frac{1}{2};1011(2)(\frac{1}{2})}^{(1,1)}(\Omega) + D_{0111(11)(\frac{1}{2};1011(11)(\frac{1}{2})}^{(1,1)}(\Omega). \tag{55}$$

We also verified that a similar identity holds for all 3×3 submatrices of $T(\Omega) \in \text{SU}(4)$. For instance,

$$\text{Imm}^{[21]}(\bar{T}(\Omega))_{(234)(124)} = D_{0111(2)(\frac{1}{2};1101(2)(\frac{1}{2})}^{(1,1)}(\Omega) + D_{0111(11)(\frac{1}{2};1101(01)(\frac{1}{2})}^{(1,1)}(\Omega), \tag{56}$$

$$\text{Imm}^{[21]}(\bar{T}(\Omega))_{(134)(124)} = D_{1011(2)(\frac{1}{2};1110(2)(\frac{1}{2})}^{(1,1)}(\Omega) + D_{1011(11)(\frac{1}{2};1110(01)(\frac{1}{2})}^{(1,1)}(\Omega). \tag{57}$$

Likewise, we have, for $T(\Omega) \in \text{SU}(5)$,

$$\begin{aligned} \text{Imm}^{[21]}(\bar{T}(\Omega))_{(1345)(1235)} \\ = D_{01101(11)(2)(\frac{1}{2},10110(2)(\frac{1}{2})}^{(1,1)}(\Omega) + D_{01101(11)(01)(\frac{1}{2},10110(01)(01)(\frac{1}{2})}^{(1,1)}(\Omega), \end{aligned} \tag{58}$$

$$\begin{aligned} \text{Imm}^{[31]}(\bar{T}(\Omega))_{(235)(134)} &= D_{10111(3)(3)(1),11101(3)(2)\frac{1}{2}}^{(2,1)}(\Omega) \\ &+ D_{10111(11)(11)(1),11101(11)(2)\frac{1}{2}}^{2,1}(\Omega) + D_{10111(11)(11)(0),11101(11)(01)\frac{1}{2}}^{(2,1)}(\Omega) \end{aligned} \tag{59}$$

this last being an example of a 4×4 submatrix not principal coaxial. We thus conjecture that, even for generic submatrices, $\text{Imm}^{(\lambda)}(\bar{T}(\Omega))_{kq}$ is a sum of $\dim\{\lambda\}$ distinct D 's with coefficients equal to +1, although we cannot yet formulate a solid proof.

5. Outer plethysms and subgroups

We now consider an application of our result to immanants of unitary matrices that are not in the fundamental representation. The difficulty in this case is that the various S_N -invariant subspaces in the N -fold tensor product no longer contain a single $SU(m)$ irrep. Consider for instance $T(\Omega)$, the 4×4 ($J = 3/2$) matrix representation of $SU(2)$. Using the standard $D_{M'M}^J(\Omega)$ notation for the group functions and $|JM\rangle$ as the notation for basis states, we have

$$\text{Imm}^{\{2,2\}}(T(\Omega)) = \frac{26}{35}D_{00}^4(\Omega) + \frac{6}{7}D_{00}^2(\Omega) + \frac{2}{5}D_{00}^0(\Omega), \tag{60}$$

which is still a sum of diagonal group functions $\sum_J c_J D_{00}^J(\Omega)$. The dimension of the S_4 irrep $\{2, 2\}$ is 2 and, while the sum contains more than two terms, we still have $\sum_J c_J = \dim(\{2, 2\})$. The possible values of J entering in the sum are those that occur in the (outer) plethysm $(3/2) \otimes_{\varphi} \{2, 2\}$, and can be found using Schur function techniques. (Here, \otimes_{φ} denotes the plethysm operation.)

A more sophisticated application is to the evaluation of immanants of the 6×6 matrix representation $(2, 0)$ of $SU(3)$. $(2)^{\otimes 6}$ contains more than one $SU(3)$ irrep. For instance, one might consider various immanants of the 6×6 matrix $T(\Omega)$ of the $(2, 0)$ of $SU(3)$. We may consider this matrix as an element of the $SU(3)$ subgroup of $SU(6)$.

As an example, the permanent $\text{Imm}^{\{6\}}(T(\Omega)) = \langle \Psi_{444} | R(\Omega) | \Psi_{444} \rangle$ but the state $|\Psi_{444}\rangle$ is now a linear combination of $SU(3)$ states in various irreps that occur in $(2) \otimes_{\varphi} \{6\}$. Explicitly, we have

$$\begin{aligned} |\Psi_{444}\rangle = & \sqrt{\frac{64}{385}} |(12)444(4)\rangle + \sqrt{\frac{18}{385}} |(8, 2)444(\alpha)\rangle + \sqrt{\frac{4}{21}} |(4, 4)444(\beta)\rangle \\ & + \frac{1}{3} |(6)222(2)\rangle + \sqrt{\frac{16}{63}} |(0, 6)444(2)\rangle + \sqrt{\frac{2}{45}} |(0)000(0)\rangle, \end{aligned} \tag{61}$$

where

$$|(8, 2)444(\alpha)\rangle = \sqrt{\frac{10}{21}} |(8, 2)444(4)\rangle + \sqrt{\frac{11}{21}} |(8, 2)444(2)\rangle, \tag{62}$$

$$\begin{aligned} |(4, 4)444(\beta)\rangle = & \sqrt{\frac{12}{35}} |(4, 4)444(4)\rangle + \sqrt{\frac{4}{21}} |(4, 4)444(2)\rangle \\ & + \sqrt{\frac{7}{15}} |(4, 4)444(0)\rangle. \end{aligned} \tag{63}$$

Note that both $|(8, 2)444(\alpha)\rangle$ and $|(4, 4)444(\beta)\rangle$ carry the fully symmetric $\{3\}$ irrep of S_3 , as can be verified using the matrix elements of the permutation operators given in [21]. The permanent $\text{Imm}^{\{6\}}(T(\Omega))$ can be written as a sum of diagonal D -functions:

$$\begin{aligned}
 & \text{Imm}^{[6]}(T(\Omega)) \\
 &= c_1 D_{(444;4)(444;4)}^{(12,0)} \\
 &+ c_2 D_{(444;4)(444;4)}^{(8,2)} + c_3 D_{(444;4)(444;2)}^{(8,2)} + c_4 D_{(444;2)(444;4)}^{(8,2)} + c_5 D_{(444;2)(444;2)}^{(8,2)} \\
 &+ c_6 D_{(444;4)(444;4)}^{(4,4)} + c_7 D_{(444;4)(444;2)}^{(4,4)} + c_8 D_{(444;4)(444;0)}^{(4,4)} + c_9 D_{(444;2)(444;4)}^{(4,4)} \\
 &+ c_{10} D_{(444;2)(444;2)}^{(4,4)} + c_{11} D_{(444;2)(444;0)}^{(4,4)} + c_{12} D_{(444;0)(444;4)}^{(4,4)} + c_{13} D_{(444;0)(444;2)}^{(4,4)} \\
 &+ c_{14} D_{(444;0)(444;0)}^{(4,4)} + c_{15} D_{(222;2)(222;2)}^{(6,0)} + c_{16} D_{(444;2)(444;2)}^{(0,6)} + c_{17} D_{(000;0)(000;0)}^{(0,0)} \tag{64}
 \end{aligned}$$

with solution

$c_1 = \frac{64}{385}$				
$c_2 = \frac{60}{539}$	$c_3 = \frac{6}{49} \sqrt{\frac{10}{11}}$	$c_4 = \frac{6}{49} \sqrt{\frac{10}{11}}$	$c_5 = \frac{6}{49}$	
$c_6 = \frac{16}{245}$	$c_7 = \frac{16\sqrt{5}}{147}$	$c_8 = \frac{8}{105}$	$c_9 = \frac{16\sqrt{5}}{147}$. (65)
$c_{10} = \frac{16}{441}$	$c_{11} = \frac{8}{63\sqrt{5}}$	$c_{12} = \frac{8}{105}$	$c_{13} = \frac{8}{63\sqrt{5}}$	
$c_{14} = \frac{4}{45}$	$c_{15} = \frac{1}{9}$	$c_{16} = \frac{16}{63}$	$c_{17} = \frac{2}{45}$	

The coefficients c_k are immediately seen to be related to coefficients in equations (61)–(63). As anticipated from the theorem, the sum of coefficients of diagonal D s: $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_{10} + c_{14} + c_{15} + c_{16} + c_{17} = 1$.

6. Discussion and conclusion

The relation between characters of S_N and $U(n)$ is well known and a rich source of results in mathematical physics. Our work expands these beyond characters to novel connections between immanants and the group functions proper. Results on group functions (see [21, 22] as well as [18] and references therein) are comparatively less common than those available for, say, the calculation of generator matrix elements or Clebsch–Gordan coefficients. We hope some of the results given here might be useful in providing impetus or remedy to this relative paucity of results on group functions.

Immanants are connected to the interferometry of partially distinguishable pulses [4–6]; the associated permutation symmetries lead to novel interpretations of immanants as a type of normal coordinates describing lossless passive interferometers [6]. This connection immediately provides a physical interpretation to the appropriate combinations of group functions corresponding to these immanants, and should stimulate further development of toolkits to compute group functions.

Conjectures in complexity theory regarding the behaviour of permanents of large unitary matrices may also provide an entry point towards understanding the behaviour of D -functions in similar asymptotic regimes. It remains to see if this line of thought can also be turned around: it might be possible to use results on the asymptotic behaviour of D -functions to establish some conjectures on the behaviour of immanants of large matrices.

Finally, although the Schur–Weyl duality is not directly applicable to subgroups of the unitary groups, the permutation group retains its deep connection with representations of the

classical groups, which are considered as subgroups of the unitary groups [23, 24]. Hence, it might be possible to extend the results of this paper to functions of the orthogonal or symplectic groups, thus generalising the result of section 5 on immanants associated with plethysms of representations.

Acknowledgments

This work was supported by NSERC and Lakehead University. ID acknowledges AITF, NSERC and USARO for financial support.

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