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Complementarity and phases in $SU(3)$

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Abstract

Phase operators and phase states are introduced for irreducible representations of the Lie algebra $\mathfrak{su}(3)$ using a polar decomposition of ladder operators. In contradistinction with $\mathfrak{su}(2)$, it is found that the $\mathfrak{su}(3)$ polar decomposition does not uniquely determine a Hermitian phase operator. We describe two possible ways of proceeding: one based on imposing $SU(2)$ invariance and the other based on the idea of complementarity. The generalization of these results to $SU(n)$ is sketched.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Phase is a unique concept for the proper understanding of classical optical phenomena. It is therefore surprising that, at a foundational level, quantum optics can apparently subsist without a quantum phase. One can use, for example, the better behaved field-quadrature operators [1] or be content with a pragmatic approach in which phase is a parameter that can be efficiently estimated [2–7]. It is equally possible to represent states as quasidistribution functions in phase space and specify their phase properties by classical angles [8–13]. Finally, it is also entirely reasonable to approach the problem from an operational perspective [14–18] that emphasizes the apparatus involved in measurements on the system; the phase then refers to a feature of this apparatus.

However, if one adheres to the orthodox interpretation of quantum mechanics and regards phase as a physical property, then surely it ought to be represented by a Hermitian operator. In other words, the phase variable should be subject to quantization and, for a sufficiently small number of particles, quantized phase effects should be accessible to experimentation.

Despite the difficulties borne out of the first and eminent attempts [19–21], and ultimately ascribed to the semiboundedness of the eigenvalue spectrum of the number operator, significant advances have been made in the last few years in clarifying the status of a quantum phase operator. The primary objective in this context has been the description of the phase of a single-mode field or, equivalently, of a harmonic oscillator. The progress made is manifest and the work on this subject has already been reviewed [22–25].

Although the definition of the absolute phase is in itself an interesting problem, such an absolute phase has no meaning from a practical point of view. Strictly speaking, only relative variables are of interest in physics. Most, if not all, methods of phase measurement are arrangements determining the relative phase between two different modes. One might falsely expect that the relative phase should be constructed merely as the difference of phases. Perhaps surprisingly, experience demonstrates that this is not the case: there are theoretical and experimental results that cannot be accounted for using the difference of phases [26]. In view of this, one should start a study of the relative phase without any previous assumption about single-mode phases. In particular, the conjugate variable to a relative phase is a number difference that is not bounded from below. Thus, it is reasonable to expect that the relative phase will be free of the problems arising in the one-mode case.

In the characterization of the relative phase for two-mode fields, the Stokes parameters play an important role [27]. From an experimental point of view, they are measurable quantities. An essential observation is that they are formally also elements of $\mathfrak{su}(2)$, which turns out to be the dynamical symmetry algebra of a qubit, in the modern parlance of quantum information [28]. Interesting links with finite quantum systems have been discussed in [29] and [30]. In fact, as shown a long time ago in the pioneering works of Lévy-Leblond [31] and Vourdas [32–34] (see also [35]), the polar decomposition of $\mathfrak{su}(2)$ gives rise to a *bona fide* phase operator, which is also complementary to the population difference, generalizing somehow Dirac's original idea [36–38].

One might believe—again falsely—that the passage from two-level systems and $\mathfrak{su}(2)$ to three-level systems and $\mathfrak{su}(3)$ would be immediate. We realize in this paper that this is not so: there appears to be no general phase operator for $\mathfrak{su}(3)$ that simultaneously verifies a polar decomposition, Hermiticity and adequate commutation relations.

The polar decomposition of an operator is always possible in any dimension (although it is generally not unique). On the other hand, complementarity in finite-dimensional spaces is usually implemented via finite Fourier transformations. It will be shown that the happy coincidence where both concepts occur in the same problem must be in general abandoned for $\mathfrak{su}(3)$ and more generally for $\mathfrak{su}(n)$: barring exceptional circumstances, complementarity and polar decompositions are apparently incompatible.

Because it is generalized much more easily, the definition of phase operators obtained via polar decompositions will be studied in detail, with special attention to the Lie algebra $\mathfrak{su}(3)$. For the general $\mathfrak{su}(n)$ case, it is natural to define $n-1$ relative phase operators; we find that they do not in general commute, except in very specific circumstances where the dimension of the system is n or goes to infinity. It will be shown how to quantify this lack of commutativity and how this can be related to simple counting arguments based on the geometry of $\mathfrak{su}(n)$ weight space.

Considerable insight in the structure of phase operators, emphasizing the connection between polar decomposition methods, the abstract operators and the geometrical nature of the relative phase, is gained by introducing a coherent-state realization of the generators (see [39, 40] for variations on this theme). These realizations also provide very useful calculational simplifications, particularly as representations become large and as we increase the rank $n-1$ of $\mathfrak{su}(n)$. We introduce this representation first for $\mathfrak{su}(2)$, in section 2, reserving a wealth of mathematical details for the appendix. In particular, these realizations allow us to reach some conclusions about the commutativity of phase operators for

$\mathfrak{su}(n)$ with $n \geqslant 3$, in the limit of large representations (i.e. the classical limit), thus allowing our conclusions to be checked against classical concepts associated with relative phases.

2. *SU*(2) phase operators

The complex extension of the $\mathfrak{su}(2)$ algebra is generated by the operators $\{\hat{h}, \hat{e}_+, \hat{e}_-\}$ with commutation relations

$$[\hat{h}, \hat{e}_\pm] = \pm\hat{e}_\pm, \qquad [\hat{e}_+, \hat{e}_-] = 2\hat{h}. \tag{2.1}$$

In the $(2j+1)$-dimensional space V_j spanned by the vectors $\{|jm\rangle : m = -j, \ldots, j\}$, which is the carrier of the irreducible representation (irrep) with spin j, the generators act in the standard way:

$$\hat{h}|jm\rangle = m|jm\rangle, \qquad \hat{e}_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|jm \pm 1\rangle. \tag{2.2}$$

The state $|\chi_j\rangle \equiv |jj\rangle$ is the highest weight of the irrep, so that

$$\hat{h}|\chi_j\rangle = j|\chi_j\rangle, \qquad \hat{e}_+|\chi_j\rangle = 0. \tag{2.3}$$

An advantageous realization providing a link with the two-mode relative phase is given by the Schwinger realization of $\mathfrak{su}(2)$ in terms of two bosonic fields $\hat{a}_1$ and $\hat{a}_2$ [41, 42]:

$$\hat{e}_+ \mapsto \hat{a}_1^\dagger \hat{a}_2, \qquad \hat{e}_- \mapsto \hat{a}_2^\dagger \hat{a}_1, \qquad \hat{h} \mapsto \tfrac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2\right), \tag{2.4}$$

which are the building blocks of the Stokes operators [27]. They act on the two-dimensional harmonic oscillator basis $|n_1, n_2\rangle$ related to the angular momentum states $|jm\rangle$ by $n_1 + n_2 = 2j$ and $n_1 - n_2 = 2m$.

If for a moment we interpret $\hat{a}_1$ and $\hat{a}_2$ as classical field amplitudes, it is apparent from (2.4) that the relative phase between the fields is encoded in $\hat{e}_\pm$. Therefore, it seems natural enough to look for a polar decomposition of the ladder operators [43]

$$\hat{e}_- = \hat{E}\,\hat{D}, \tag{2.5}$$

where, using (2.2), $\hat{D} = \sqrt{\hat{e}_-^\dagger \hat{e}_-}$ is a semi-positive Hermitian operator and $\hat{E}$ is a unitary matrix that can be interpreted as the exponential of a putative phase operator.

The rank of $\hat{e}_-$ is one less than the dimension of $\hat{e}_-$, so $\hat{E}$ is not completely specified. We remove the ambiguity by using cyclic boundary conditions so that $\hat{E}$ is the generator of an Abelian cyclic group [38]. The eigenvalues of $\hat{E}$ are then the quantized phases. As $\hat{E}^{2j+1} = \hat{\mathbb{1}}$, these eigenvalues are just ω^k, where $\omega = \exp[2\pi\mathrm{i}/(2j+1)]$ and $k = -j, \ldots, j$. Taking m modulo $2j+1$, we get in this way

$$\hat{E} = \sum_{m=-j}^{j} |jm-1\rangle\langle jm|, \tag{2.6}$$

or, equivalently,

$$\hat{E} = \begin{pmatrix} 0 & 0 & \ldots & & 0 & 1 \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix}. \tag{2.7}$$

The eigenstates of $\hat{E}$ are related to the basis states $|jm\rangle$ by a finite Fourier transform and are thus complementary to the $|jm\rangle$ states.

Observe that $\hat{E}$ is not an $SU(2)$ matrix. It can be written formally as the exponential of a Hermitian phase operator $\hat{\varphi}$, but $\hat{\varphi}$ is not in general an element of the $\mathfrak{su}(2)$ algebra. These observations hold (also in dimension 2 because of the choice of phase in completing $\hat{E}$), even though both have well-defined actions on the basis elements.

As heralded in the introduction, we now introduce as a tool of particular convenience a coherent state realization Γ for $\mathfrak{su}(2)$. To this end, we recall that the $SU(2)$ coherent states are defined by [44–46]

$$|\vartheta,\varphi\rangle = \hat{R}_z^{-1}(\varphi)\hat{R}_y^{-1}(\vartheta)|\chi_j\rangle, \tag{2.8}$$

where $\hat{R}_z$ and $\hat{R}_y$ represent rotations about the z and y axes, respectively. With any vector $|\Psi\rangle$, we associate the function

$$|\Psi\rangle \mapsto \Psi_\vartheta(\varphi) = \langle\chi_j|\hat{R}_y(\vartheta)\,\hat{R}_z(\varphi)|\Psi\rangle. \tag{2.9}$$

Note that $|\Psi_\vartheta(\varphi)|^2$ is precisely the Husimi Q-function for the corresponding (pure) state $|\Psi\rangle$. We can use the arbitrary nature of $|\Psi\rangle$ to define the action of $\hat{X}\in\mathfrak{su}(2)$ on $\Psi_\vartheta(\varphi)$ by

$$\hat{X}|\Psi\rangle \mapsto [\Gamma(\hat{X})\Psi]_\vartheta(\varphi) \equiv \langle\chi_j|\hat{R}_y(\vartheta)\hat{R}_z(\varphi)\hat{X}|\Psi\rangle. \tag{2.10}$$

Straightforward manipulations immediately produce the expressions

$$\begin{aligned}\hat{h} &\mapsto \Gamma(\hat{h}) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi},\\ \hat{e}_\pm &\mapsto \Gamma(\hat{e}_\pm) = -(\tan\vartheta)^{\mp 1}\,\mathrm{e}^{\pm\mathrm{i}\varphi}\left(j\mp\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi}\right).\end{aligned} \tag{2.11}$$

It is convenient to think of the coherent state $|\vartheta,\varphi\rangle$ as localized around the coordinates (ϑ,φ) on the Bloch sphere. The relative phase is linked to the azimuthal angle φ while the parameter ϑ is inessential for our purposes; to simplify the expressions for Γ, we choose ϑ so that $\tan\vartheta = -1$ and obtain

$$\begin{aligned}\hat{h} &\mapsto \Gamma(\hat{h}) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi},\\ \hat{e}_\pm &\mapsto \Gamma(\hat{e}_\pm) = \mathrm{e}^{\pm\mathrm{i}\varphi}\left(j\mp\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi}\right).\end{aligned} \tag{2.12}$$

One easily verifies that $\hat{X}\mapsto\Gamma(X)$ preserves the commutation relations (2.1) and is thus a realization of $\mathfrak{su}(2)$. Γ acts naturally in the infinite-dimensional space spanned by the exponential functions $\{\mathrm{e}^{im\varphi}:\,2m\in\mathbb{Z}\}$ and equipped with the scalar product

$$\langle f|g\rangle = \int_0^{2\pi} f^*(\varphi)\,g(\varphi)\,\mathrm{d}\varphi. \tag{2.13}$$

The subspace of states with $|m|\leqslant j$ is invariant under the action of Γ. The (normalized) basis elements of this invariant subspace are mapped to exponential functions $|jm\rangle \leftrightarrow \exp(\mathrm{i}m\varphi)/\sqrt{2\pi}$ and the action of (2.12) is

$$\Gamma(\hat{h})|jm\rangle = m|jm\rangle \qquad \Gamma(\hat{e}_\pm)|jm\rangle = (j\mp m)\,|jm\pm 1\rangle. \tag{2.14}$$

Under this inner product,

$$\langle jm'|\Gamma(\hat{e}_+)|jm\rangle \neq \langle jm'|\Gamma^\dagger(\hat{e}_-)|jm\rangle; \tag{2.15}$$

thus, the realization Γ is not Hermitian. However, we show in the appendix how, for fixed j, Γ is equivalent to the standard Hermitian representation given in equation (2.2).

The considerable merit of the realization Γ is that it is particularly well suited to analyze the polar decomposition: the unitary matrix $\hat{E}$ in V_j is immediately obtained from the action of $\mathrm{e}^{-\mathrm{i}\varphi}$, the 'phase' part of $\Gamma(\hat{e}_-)$:

$$\hat{E}_{m'm} = \Gamma(\hat{E})_{m'm} = \langle jm'|\mathrm{e}^{-\mathrm{i}\varphi}|jm\rangle, \qquad -j \leqslant m \leqslant j \quad \mathrm{mod}(2j+1). \tag{2.16}$$

Thus, $\mathrm{e}^{-\mathrm{i}\varphi}$ simply shifts the basis state $\mathrm{e}^{\mathrm{i}m\varphi}$ on the circle to its immediate neighbour $\mathrm{e}^{\mathrm{i}(m-1)\varphi}$, modulo $2j+1$. In addition, matrix elements of Γ become indistinguishable from those of the standard Hermitian representation in the limit where $m/j \to 0$: this makes Γ also very well suited to analyze some limits of large representations, and analyze a transition between the quantum and classical phase.

In the basis of exponential functions, the kth eigenstate $|\Phi_k\rangle$, corresponding to the eigenvalue ω^k, is (up to an overall phase)

$$\langle\varphi \mid \Phi_k\rangle = \frac{1}{\sqrt{2j+1}} \frac{\sin[(2j+1)\varphi]}{\sin\left(\varphi + \frac{\pi k}{2j+1}\right)}. \tag{2.17}$$

3. *SU*(3) phase operators

3.1. Polar decomposition for *SU*(3)

A basis for the complex extension of the Lie algebra $\mathfrak{u}(3)$ is given by the nine operators $\{\hat{C}_{ij} : i, j = 1, 2, 3\}$, with commutation relations

$$[\hat{C}_{ij}, \hat{C}_{k\ell}] = \delta_{jk}\hat{C}_{i\ell} - \delta_{i\ell}\hat{C}_{kj}. \tag{3.1}$$

The complex extension of $\mathfrak{su}(3)$ is obtained by restricting the $\mathfrak{u}(3)$ operators to $\{\hat{C}_{ij} : i \neq j\}$ and including two traceless linearly independent diagonal operators $\hat{h}_1$ and $\hat{h}_2$ that determine a Cartan subalgebra. A convenient choice of the latter is

$$\hat{h}_1 = \hat{C}_{11} - \hat{C}_{22}, \qquad \hat{h}_2 = \hat{C}_{22} - \hat{C}_{33}. \tag{3.2}$$

If one uses the boson realization

$$\hat{C}_{ij} = \hat{a}_i^\dagger \hat{a}_j, \qquad \hat{C}_{ij} = \hat{C}_{ji}^\dagger, \tag{3.3}$$

the $\frac{1}{2}(\lambda+1)(\lambda+2)$-dimensional set of harmonic oscillator states $\mathcal{S} = \{|n_1n_2n_3\rangle, n_1+n_2+n_3 = \lambda\}$ is invariant under the action of $\hat{C}_{ij}$ and is a basis for an irrep of $\mathfrak{su}(3)$ usually denoted $(\lambda, 0)$. The eigenvalues of $\hat{h}_1$ and $\hat{h}_2$ are directly related to population differences between levels 1 and 2, and 2 and 3, respectively.

The information in equation (3.1) is conveniently displayed using a root diagram [47]: with every $\mathfrak{su}(3)$ generator $\hat{C}_{ij}$, we associate the pair (x, y) of integers defined by

$$[\hat{h}_1, \hat{C}_{ij}] = x\,\hat{C}_{ij}, \qquad [\hat{h}_2, \hat{C}_{ij}] = y\,\hat{C}_{ij}, \tag{3.4}$$

and the root vector $\alpha = x\,\alpha_1 + y\,\alpha_2$, with basis α_1, α_2 having Cartesian components

$$\alpha_1 = (\sqrt{2}, 0), \qquad \alpha_2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right). \tag{3.5}$$

The root diagram for $\mathfrak{su}(3)$ is sketched in figure 1. Note that every generator is thus associated with a root vector and the diagonal operators are associated with vectors of length zero.

The commutation relations are given (up to a sign) by addition of the corresponding root vectors. If we label the operator $\hat{C}_{ij}$ by its root vector, $\hat{C}_{ij} \mapsto \hat{e}_\alpha$, then we have

$$[\hat{e}_\alpha, \hat{e}_\beta] \propto \begin{cases} \hat{e}_{\alpha+\beta}, & \text{if } \alpha+\beta \text{ is a root,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

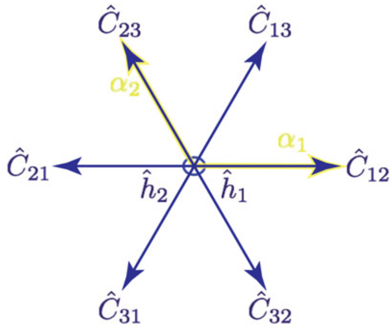

Figure 1. The root system for the complex extension of $\mathfrak{su}(3)$, showing (in yellow) the two fundamental positive roots.

The root diagram neatly shows, for instance, that $[\hat{C}_{23}, \hat{C}_{12}]$ is proportional to $\hat{C}_{13}$ in accordance with the vectorial addition of the appropriate roots.

Similarly, the weight diagram is a pictorial representation of the basis states of an irrep. The weight $w = x\,w^1 + y\,w^2$ of a basis state $|w\rangle$ is a vector with components related to the eigenvalues of the diagonal operators:

$$\hat{h}_1|w\rangle = x\,|w\rangle, \qquad \hat{h}_2|w\rangle = y\,|w\rangle. \tag{3.7}$$

The fundamental weights w^1 and w^2 have Cartesian coordinates

$$w^1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right), \qquad w^2 = \left(0, \sqrt{\frac{2}{3}}\right), \tag{3.8}$$

so that $\langle w^i|\alpha_j\rangle = \delta_{ij}$. A generator $\hat{C}_{ij}$ associated with the root α acts on a weight state $|w\rangle$ by translation on the hexagonal grid:

$$\hat{C}_{ij}|w\rangle \mapsto \hat{e}_\alpha|w\rangle \propto \begin{cases} |\alpha + w\rangle, & \text{if } \alpha + w \text{ is a weight,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

The operators $\hat{C}_{ij}$ acting on the $\frac{1}{2}(\lambda+1)(\lambda+2)$-dimensional space $\mathcal{S}$ can be represented by matrices. A polar decomposition of these matrices is given by

$$\hat{C}_{ij} = \hat{E}_{ij}\hat{D}_{ij}. \tag{3.10}$$

The operator $\hat{D}_{ij} = \sqrt{\hat{C}_{ij}^\dagger \hat{C}_{ij}}$ is non-negative definite while $\hat{E}_{ij}$ will be constructed as a unitary matrix, with $\hat{E}_{11} = \hat{E}_{22} = \hat{E}_{33} = \mathbb{1}$. It is easily shown that $[\hat{h}_1, \hat{D}_{ij}] = [\hat{h}_2, \hat{D}_{ij}] = 0$ and also that

$$\hat{D}_{12}^2 - \hat{D}_{21}^2 = \hat{h}_1, \qquad \hat{D}_{23}^2 - \hat{D}_{32}^2 = \hat{h}_2, \qquad \hat{D}_{13}^2 - \hat{D}_{31}^2 = \hat{h}_1 + \hat{h}_2. \tag{3.11}$$

The realization of $\mathfrak{su}(3)$ that optimally displays the polar decomposition is the $\mathfrak{su}(3)$ analog of (2.12). A coherent state $|\vartheta_1, \varphi_1, \vartheta_2, \varphi_2\rangle$ for the irrep $(\lambda, 0)$ is obtained by group action on the highest weight state; this highest weight state is the boson state $|\lambda, 0, 0\rangle$. A general quantum state $|\Psi\rangle$ is then represented by a function on $S^4 \sim SU(3)/U(2)$:

$$|\Psi\rangle \mapsto \Psi_{\vartheta_1,\vartheta_2}(\varphi_1, \varphi_2). \tag{3.12}$$

The two azimuthal angles φ_1, φ_2 control the relative phase between these populations, while two polar angles ϑ_1, ϑ_2 mix the number of excitations in each mode. For calculational convenience, the latter are chosen to simplify the coherent state representation of $\mathfrak{su}(3)$ elements, given by [39]

$$\begin{aligned}
\hat{h}_1 &\mapsto \Gamma(\hat{h}_1) = -\mathrm{i}\frac{\partial}{\partial\varphi_1}, \qquad \hat{h}_2 \mapsto \Gamma(\hat{h}_2) = -\mathrm{i}\frac{\partial}{\partial\varphi_2},\\
\hat{C}_{12} &\mapsto \Gamma(\hat{C}_{12}) = \frac{1}{3}\,\mathrm{e}^{\mathrm{i}(2\varphi_1-\varphi_2)}\left(\lambda + \mathrm{i}\frac{\partial}{\partial\varphi_1} - \mathrm{i}\frac{\partial}{\partial\varphi_2}\right),\\
\hat{C}_{23} &\mapsto \Gamma(\hat{C}_{13}) = \frac{1}{3}\,\mathrm{e}^{-\mathrm{i}(\varphi_1-2\varphi_2)}\left(\lambda + \mathrm{i}\frac{\partial}{\partial\varphi_1} + 2\mathrm{i}\frac{\partial}{\partial\varphi_2}\right).
\end{aligned} \tag{3.13}$$

This realization provides an obvious decomposition of the $\mathfrak{su}(3)$ raising operators. Much like $\mathfrak{su}(2)$, these operators act in a natural way on the Hilbert space spanned by the exponential functions $\{\mathrm{e}^{\mathrm{i}(w_1\varphi_1+w_2\varphi_2)}\}$. Again, Γ is not Hermitian although it is equivalent to a Hermitian representation, as indicated in [39] or as can be shown following the procedure of the appendix.

3.2. $SU(3)$ phase operators for the (1, 0) representation

We consider first the three-dimensional representation $(1,0)$, spanned by the boson states $\{|100\rangle, |010\rangle, |001\rangle\}$, where $|n_1n_2n_3\rangle$ denotes a state with population n_i in level i. This has been worked out in detail in [48] and from a different perspective in [49].

The components (x, y) of a weight are related to population differences by $x = n_1-n_2$, $y = n_2-n_3$. Explicitly, the weights of the basis vectors are $\{(1,0), (-1,0), (0,-1)\}$, respectively.

In this representation, we have the matrix realizations

$$\begin{aligned}
\hat{C}_{12} &= \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} = \hat{E}_{12}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\\
\hat{C}_{23} &= \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} = \hat{E}_{23}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.
\end{aligned} \tag{3.14}$$

The rank of $\hat{C}_{12}$ and $\hat{C}_{23}$ is 1, but their dimension is 3, which implies that the polar decomposition is (again) not completely specified. Indeed, one finds that the most general unitary $\hat{E}_{12}$ and $\hat{E}_{23}$ consistent with the matrix realization of $\hat{C}_{12}$ and $\hat{C}_{23}$ are

$$\begin{aligned}
\hat{E}_{12} &= \begin{pmatrix}0&1&0\\a&0&b\\b^*&0&-a^*\end{pmatrix}, \qquad aa^* + bb^* = 1,\\
\hat{E}_{23} &= \begin{pmatrix}c&d&0\\0&0&1\\d^*&-d^*&0\end{pmatrix}, \qquad cc^* + dd^* = 1.
\end{aligned} \tag{3.15}$$

Here, we have already restricted $\hat{E}_{ij}$ to be unitary so that a Hermitian phase operator can be properly defined. The issue is now to fix the unknown parameters a, b, c and d in (3.15).

3.2.1. $SU(2)$-invariant solution. The subset $\{\hat{h}_1, \hat{C}_{12}, \hat{C}_{21}\}$ of generators spans an $\mathfrak{su}(2)$ subalgebra of $\mathfrak{su}(3)$. The boson states $|100\rangle$ and $|010\rangle$ form a two-dimensional $\mathfrak{su}(2)$ subspace; the boson state $|001\rangle$ is an $\mathfrak{su}(2)$ singlet. Thus, one way of fixing the unitary matrix $\hat{E}_{12}$ is

Table 1. Notational details connecting the numbering of states, their weights, their boson representations and the expression of these states in terms of the angles φ_1, φ_2. In the polar representation, we omit for simplicity a factor 2π.

State	Weight	Boson state	Polar state
$\|1\rangle$	$(1, 0)$	$\|100\rangle$	$e^{i\varphi_1}$
$\|2\rangle$	$(-1, 1)$	$\|010\rangle$	$e^{i(\varphi_2-\varphi_1)}$
$\|3\rangle$	$(0, -1)$	$\|001\rangle$	$e^{-i\varphi_2}$

to require that $\hat{E}_{12}$ preserves the multiplet structure of this $\mathfrak{su}(2)$ subalgebra. This gives $a = -1, b = 0$, so that

$$\hat{E}_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.16}$$

The phase of a is inessential and has been chosen for convenience.

In the same way, the subset $\{\hat{h}_2, \hat{C}_{23}, \hat{C}_{32}\}$ spans a different $\mathfrak{su}(2)$ subalgebra, and we may also require that $\hat{E}_{23}$ acts within a multiplet of this subalgebra. This, in turn, implies

$$\hat{E}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{3.17}$$

For this solution, one obtains, from $\hat{E}_{ij} = e^{i\hat{\varphi}_{ij}}$,

$$\hat{\varphi}_{12} = i\frac{\pi}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{\varphi}_{23} = i\frac{\pi}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.18}$$

The notable feature of the matrices $\hat{E}_{12}$ and $\hat{E}_{23}$ is that they do not commute.

3.2.2. The complementary solution. An alternative choice of $\hat{E}_{ij}$ is obtained by using a different line of argument [50]. For $SU(2)$, the phase operator is thought to be complementary to the population difference $\hat{h}$. The generalization of this complementarity-based definition to $SU(3)$ can also be achieved for the irrep $(1, 0)$.

We recall the definition of the generalized Pauli matrices [51–53]. Let

$$\hat{Z} = \hat{h}_1 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \hat{X} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{pmatrix}. \tag{3.19}$$

Then,

$$\hat{X}^k \hat{Z}^\ell = \omega^{k\ell} \hat{Z}^\ell \hat{X}^k, \tag{3.20}$$

where $k, \ell \in \mathbb{Z}_3$ and $\omega = \exp(2\pi i/3)$. The subset $\{\hat{X}^k \hat{Z}^\ell\}$ of generalized Pauli matrices are elements of the finite Pauli subgroup $\wp_3$ of $SU(3)$ containing 27 elements and described elsewhere [54, 55]. $\{\hat{X}^k \hat{Z}^\ell\}$ also forms a basis for the $\mathfrak{su}(3)$ algebra, so that we can expand $\hat{E}_{12}$ as

$$\hat{E}_{12} = \sum_{k\ell} a_{k\ell} \hat{X}^k \hat{Z}^\ell. \tag{3.21}$$

A simple analysis shows that it is indeed possible to obtain a complementary solution, in the sense that

$$\hat{h}_1 \hat{E}_{12} = \omega^2 \hat{E}_{12} \hat{h}_1, \tag{3.22}$$

which then forces

$$\hat{E}_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & e^{i\beta} \\ e^{-i\beta} & 0 & 0 \end{pmatrix}. \tag{3.23}$$

A similar analysis for $\hat{E}_{23}$ and $\hat{h}_2$ produces the unitary solution

$$\hat{E}_{23} = \begin{pmatrix} 0 & e^{i\gamma} & 0 \\ 0 & 0 & 1 \\ e^{-i\gamma} & 0 & 0 \end{pmatrix}. \tag{3.24}$$

If, additionally, we insist that the $\hat{E}_{ij}$'s commute or, equivalently, $\hat{\varphi}_{13} = \hat{\varphi}_{12} + \hat{\varphi}_{23}$, we obtain the condition $\hat{E}_{13} = \hat{E}_{12}\hat{E}_{23}$, which implies

$$\beta + \gamma = 0, \pm 2\pi, \dots \quad 2\beta - \gamma = 0, \pm 2\pi, \dots \quad -\beta + 2\gamma = 0, \pm 2\pi, \dots, \tag{3.25}$$

wherefrom we get

$$3\beta = 0, \pm 2\pi, \dots, \qquad 3\gamma = 0, \pm 2\pi, \dots. \tag{3.26}$$

The solutions are found by fixing either β and deducing γ, or vice versa. The simplest nontrivial solution is found by choosing $\beta = 2\pi/3$ and $\gamma = -2\pi/3$. This produces

$$\hat{E}_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}, \quad \hat{E}_{23} = \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix}, \quad \hat{E}_{13} = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, \tag{3.27}$$

all of which are elements of the generalized Pauli group $\wp_3$.

3.3. $SU(3)$ *phase operators in the* $(\lambda, 0)$ *representation*

We now seek to generalize our discussion to irreps other than the simplest $(1, 0)$. The $d = \frac{1}{2}(\lambda + 1)(\lambda + 2)$ states of the form $|n_1 n_2 n_3\rangle$, where each n_i is a non-negative integer subject to the condition $n_1 + n_2 + n_3 = \lambda$, are transformed into one another under the action of any $\mathfrak{su}(3)$ generator and thus form an irrep of dimension d. The highest weight is $(\lambda, 0)$ and the highest weight state is $|\lambda 00\rangle$. We can again use the polar realization of equation (3.14) and express basis states of $(\lambda, 0)$ in terms of exponentials. Some closely related material has been presented in [56].

Consider, for instance, the case $\lambda = 2$, where the dimension of the space is 6. The operators $\hat{C}_{12}$ and $\hat{C}_{23}$ have matrix representation and polar decomposition of the general form

$$\begin{aligned} \hat{C}_{12} &= \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \hat{E}_{12} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \hat{C}_{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \hat{E}_{23} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}, \end{aligned} \tag{3.28}$$

Figure 2. The construction of $\mathfrak{su}(2)$-invariant phase operators for irreps of the $(\lambda, 0)$ type: equilateral triangles are turned into cones. The 'rings' are made from weights in $\mathfrak{su}(2)$-invariant subspaces.

with

$$\hat{E}_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ * & 0 & * & 0 & 0 & * \\ * & 0 & * & 0 & 0 & * \\ * & 0 & * & 0 & 0 & * \end{pmatrix}, \qquad \hat{E}_{23} = \begin{pmatrix} * & * & 0 & * & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ * & * & 0 & * & 0 & 0 \end{pmatrix}, \tag{3.29}$$

and the asterisks denoting here undetermined elements.

3.3.1. SU(2)-invariant solution. One way of fixing $\hat{E}_{12}$ and $\hat{E}_{23}$ so that they are unitary is to directly generalize the prescription of (3.16) and (3.17) and complete the matrices in an $\mathfrak{su}(2)$-invariant way. The $\mathfrak{su}(2)$-invariant solution, which always exists, can be obtained by transforming $\mathfrak{su}(2)$-invariant strings of weights parallel to a root into a circle, thus transforming the equilateral triangle formed by the weights into a cone, as illustrated in figure 2. The tip of the triangle is an $\mathfrak{su}(2)$ singlet while the base is made from the longest $\mathfrak{su}(2)$ string of weights.

For the case of $(2, 0)$, we obtain

$$\hat{E}_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \hat{E}_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{3.30}$$

This $\mathfrak{su}(2)$-preserving solution does not produce commuting matrices: the phases are not additive. This remains true for $(\lambda, 0)$-type of representations, where the weight diagram is an equilateral triangle with $\lambda + 1$ states on each side.

One could also search for a complementary-based solution. Now, the $(2, 0)$ irrep of $\mathfrak{su}(3)$ decomposes into a sum of two three-dimensional irreps of $\wp_3$, but the resulting matrices are incompatible with a polar decomposition of $\hat{C}_{12}$. Indeed, one shows that the

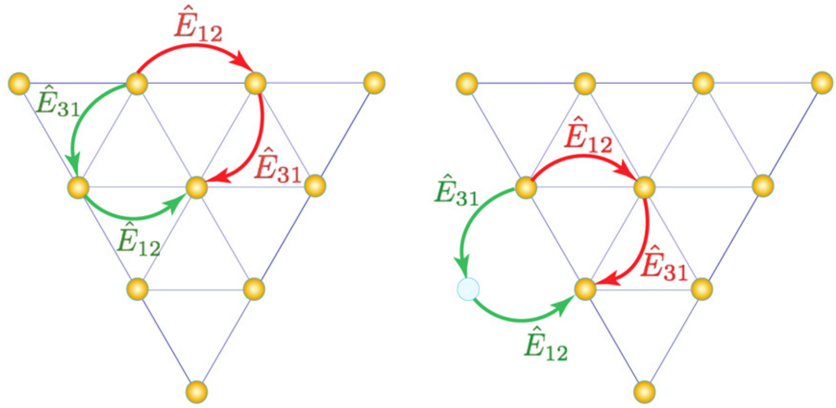

Figure 3. Left: illustrating how the operators $\hat{E}_{12}$ and $\hat{E}_{31}$ can commute. Red/darker lines: $\hat{E}_{31}\hat{E}_{12}$ acting on a state. Green/lighter lines: $\hat{E}_{12}\hat{E}_{31}$. Since the matrix elements of $\hat{E}_{ij}$ are always 1, the operators commute in the case illustrated here. Right: illustrating how the operators $\hat{E}_{12}$ and $\hat{E}_{31}$ can fail to commute when one or the other (or both) annihilate a state. Here, the initial state is killed by $\hat{E}_{31}$.

most general solution to the polar decompositions compatible with equation (3.29) cannot produce commuting matrices. This statement remains true for higher-dimensional irreps of the type $(\lambda, 0)$, provided that λ is finite.

3.3.2. Measuring non-commutativity. Quite generally the phase operators do not commute. To quantify the amount by which, say $\hat{E}_{12}\hat{E}_{31}$ fail to commute, we introduce the matrix norm $\|\hat{M}\|^2 = \mathrm{Tr}(\hat{M}^\dagger \hat{M})$ and define

$$\hat{M} = \hat{E}_{12}\hat{E}_{31}\,\hat{E}_{12}^{-1}\hat{E}_{31}^{-1} - \mathbb{1}. \tag{3.31}$$

Obviously, $\hat{M}$ should be the zero matrix if $\hat{E}_{12}$ and $\hat{E}_{31}$ commute. To compare values of $\|\hat{M}\|^2$ for different irreps $(\lambda, 0)$, it is convenient to normalize the length of $\hat{M}$ by dividing by the dimension $\frac{1}{2}(\lambda+1)(\lambda+2)$ of the irrep $(\lambda, 0)$. When this is done for the $\mathfrak{su}(2)$-invariant solutions, we find

$$\frac{\|\hat{M}\|^2}{\frac{1}{2}(\lambda+1)(\lambda+2)} = 2\frac{[2(\lambda+1)-1]}{\frac{1}{2}(\lambda+1)(\lambda+2)}. \tag{3.32}$$

This expression can be understood as follows. The action of $\hat{E}_{12}$ commutes with the action of $\hat{E}_{31}$ when the action of either is non-zero, as illustrated on the left of figure 3. On the other hand, $\hat{E}_{12}$ and $\hat{E}_{31}$ do not commute if one or the other acts on some suitable 'edge' state killed either by $\hat{E}_{12}$ or by $\hat{E}_{31}$. This is illustrated on the right of figure 3.

States killed by $\hat{E}_{12}$ or $\hat{E}_{31}$ are always located on the edge of the weight diagram for the irrep $(\lambda, 0)$. Any edge contains $(\lambda+1)$ states. There are two problematic edges (one for $\hat{E}_{12}$ and another for $\hat{E}_{31}$) so the number of problematic states is $2(\lambda+1)$. Since these two edges have a single state in common, the number of problematic states, adjusted for double counting, is just $2(\lambda+1)-1$. The overall multiplicative factor of 2 comes from the calculation of the trace, and the denominator is clearly just the normalization factor.

3.3.3. The $\lambda \to \infty$ limit and its solution. Equation (3.32) shows that for large λ, the amount by which $\hat{E}_{12}$ and $\hat{E}_{31}$ do not commute goes like $\sim\lambda^{-1}$: for large λ, the $\mathfrak{su}(2)$-invariant solutions commute and the phases φ_{12} and φ_{31} become additive. It is also clear that as $\lambda \to \infty$, the edge states become progressively displaced to infinity. In this limit, the finite triangular lattice of the weight diagram becomes a simple two-dimensional hexagonal crystal lattice.

In the polar realization (3.14), the generators acting on states having finite weight simplify to

$$\begin{aligned} &\hat{h}_1 = -\mathrm{i}\frac{\partial}{\partial\varphi_1}, \qquad \hat{h}_2 = -\mathrm{i}\frac{\partial}{\partial\varphi_2}, \\ &\hat{C}_{12} \sim \frac{1}{3}\lambda\,\mathrm{e}^{\mathrm{i}(2\varphi_1-\varphi_2)}, \quad \hat{C}_{23} \sim \frac{1}{3}\lambda\,\mathrm{e}^{-\mathrm{i}(\varphi_1-2\varphi_2)}, \quad \hat{C}_{13} \sim \frac{1}{3}\lambda\,\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)}. \end{aligned} \tag{3.33}$$

In this limit, Γ is Hermitian. The rescaling $\hat{C}_{ij} \to \hat{C}_{ij}/\lambda$ leads to commuting ladder operators.

In the $\lambda \to \infty$ limit, the phase operators $\hat{E}_{ij}$ act unitarily on every state of the form $\mathrm{e}^{\mathrm{i}(n\varphi_1+m\varphi_2)}$ with n, m finite integers. The common eigenstates of $\hat{E}_{12}$ and $\hat{E}_{23}$ are

$$|\varphi_1,\varphi_2\rangle = \frac{1}{2\pi}\sum_{n,m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}(2n-m)\varphi_1}\,\mathrm{e}^{\mathrm{i}(2m-n)\varphi_2}. \tag{3.34}$$

4. Concluding remarks

The general prescription provided for $\mathfrak{su}(3)$ can be extended to $\mathfrak{su}(n)$. For instance, let us look briefly at $\mathfrak{su}(4)$. The 12 roots corresponding to $\{\hat{C}_{ij} : i \neq j = 1, \ldots, 4\}$ are located at the vertices of a cuboctahedron, which is the intersection of a cube and an octahedron. There are three diagonal operators, represented by three roots of length 0 located at the center of the root diagram: $\hat{h}_k = \hat{C}_{kk} - \hat{C}_{k+1,k+1}$. Using again the boson realization $\hat{C}_{ij} = \hat{a}_i^\dagger\hat{a}_j$ and boson states $|n_1n_2n_3n_4\rangle$, the diagonal operators correspond to population differences between consecutive levels.

For such boson states, the weight diagram is a tetrahedron. Each slice parallel to a fundamental root of $\mathfrak{su}(4)$ is an $\mathfrak{su}(3)$ subspace. In particular, the action of some ladder operators will be undefined on one edge of the tetrahedron as the states on this edge are killed by them. Two polar operators will thus fail to commute when they act on states in some specific edge of the weight diagram.

There are $(\lambda+1)(\lambda+2)(\lambda+3)/6$ boson states of the form $|n_1n_2n_3n_4\rangle$ with $n_1 + n_2 + n_3 + n_4 = \lambda$. There are $(\lambda+1)(\lambda+2)/2$ states on each edge of the weight diagram. Two adjacent edges intersect on a line containing $\lambda + 1$ states, and they have one point in common. Defining $\hat{M}$ as in (3.31) for any pair of non-commuting roots and their phase operators, we found that, for $\mathfrak{su}(4)$,

$$\frac{\|M\|^2}{\frac{1}{6}(\lambda+1)(\lambda+2)(\lambda+3)} = \frac{[2\times(\lambda+1)(\lambda+2)-(\lambda+1)-1]}{\frac{1}{6}(\lambda+1)(\lambda+2)(\lambda+3)}. \tag{4.1}$$

Our interpretation is thus that the non-commutativity of phases is an 'edge' effect. Since the number of points on an edge of the weight diagram grows with a rate $\sim\lambda^{n-2}$ while the number of states in an $\mathfrak{su}(n)$ representation grows like $\sim\lambda^{n-1}$, phase operators constructed so as to preserve the $\mathfrak{su}(n-1)$ subalgebras of $\mathfrak{su}(n)$ will commute in the large λ limit.

As we have shown, in general, one cannot expect that phase operators will necessarily commute: they will commute in the limit of large representations where $\lambda \to \infty$, a limit which corresponds to a contraction of $\mathfrak{su}(n)$ for which Γ becomes Hermitian.

We have not investigated in detail the possibility of constructing commuting solutions which satisfy the complementary conditions for the special case of irreps of the type $(1, 0, \ldots, 0)$ of $\mathfrak{su}(n)$. However, it is probable that fully complementary solutions do not always exist. To find phase operators that are pairwise complementary is closely related to the existence problem for mutually unbiased bases; likely, when n is a prime, some elements of the generalized Pauli group are compatible with the polar decomposition of raising and lowering operators. When n is a power of a prime, it is not clear if the polar decomposition is compatible with the requirement of complementarity. When n is composite, the situation is even less clear as the construction of mutually unbiased bases remains an open problem.

In conclusion, the coherent state representation used in this paper exhibits two nice features particularly relevant to the analysis of phase operators. First, we have their geometrical interpretation in terms of azimuthal angles related to relative phases and associated with the ladder action of the appropriate generators. Second, the 'exponential part' of the realization naturally provides the unitary part of the polar decomposition of the generators. Note that the similarity transformation that maps the original Γ realization into a Hermitian one simply rescales the diagonal entries of the polar part of the matrix representation of the generator and thus has no effect on the interpretation of the phase part of the representation.

The kind of coherent state representation used in this paper would seem to form a natural gateway into understanding phases in the system described by algebras other than $\mathfrak{su}(n)$. Certainly the geometrical structure of coherent states should allow one to interpret the Â parameters of such a coherent state representation.

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Appendix Making the representation Γ Hermitian

In this appendix, we provide some mathematical details on the representation Γ. The discussion in this section is facilitated by denoting the representation of elements of $\mathfrak{su}(2)$ given in (2.2) by γ. In this notation, equation (2.10) takes the form

$$\gamma(\hat{X})|\Psi\rangle \mapsto [\Gamma(\hat{X})\Psi]_{\vartheta}(\varphi) = \langle \chi_j|\hat{R}_y(\vartheta)\hat{R}_z(\varphi)\gamma(\hat{X})|\Psi\rangle. \tag{A.1}$$

The representations γ and Γ are clearly isomorphic.

Since for $2j$ an integer every representation of $\mathfrak{su}(2)$ is equivalent to a Hermitian representation, there must be an intertwining operator that takes the non-Hermitian representation Γ in (2.12) to the Hermitian γ.

Following [57], we seek a similarity transformation

$$\gamma(\hat{X}) = \mathcal{K}^{-1}\,\Gamma(\hat{X})\mathcal{K} \tag{A.2}$$

to bring Γ to its Hermitian form. Note that under the inner product (2.13), $\Gamma(\hat{h})$ is already Hermitian so $\mathcal{K}$ commutes with $\Gamma(\hat{h})$. Next, from the requirement that $\gamma(\hat{e}_+) = \gamma^\dagger(\hat{e}_-)$, we find

$$\mathcal{K}^\dagger\Gamma^\dagger(\hat{e}_-)(\mathcal{K}^{-1})^\dagger = \mathcal{K}^{-1}\Gamma(\hat{e}_+)\mathcal{K} \Rightarrow \mathcal{K}\mathcal{K}^\dagger\Gamma^\dagger(\hat{e}_-) = \Gamma(\hat{e}_+)\mathcal{K}\mathcal{K}^\dagger. \tag{A.3}$$

Thus, the positive Hermitian operator $S = \mathcal{K}\mathcal{K}^\dagger$ is the intertwining operator for which

$$S\Gamma^\dagger(\hat{e}_-) = \Gamma(\hat{e}_+)S. \tag{A.4}$$

In addition, we observe that $\langle jm|\Gamma^\dagger(\hat{h})|jm\rangle = \langle jm|\Gamma(\hat{h})|jm\rangle$, so $\mathcal{K}$ and therefore S commute with $\Gamma(\hat{h})$ and we can write

$$S_{m'm} = \langle jm'|S|jm\rangle\delta_{mm'} \equiv S_m. \tag{A.5}$$

Because $\Gamma(\hat{e}_-)\mathrm{e}^{-\mathrm{i}j\varphi} = \Gamma(\hat{e}_+)\mathrm{e}^{\mathrm{i}j\varphi} = 0$, it is clear that if $\Psi(\varphi)$ is a function over the subspace of exponential functions spanned by $\{\mathrm{e}^{\mathrm{i}m\varphi}, -j \leqslant m \leqslant j\}$, then $S\Psi(\varphi)$ is also in that subspace.

The intertwining operator S is then obtained from the recursion relation that follows from equation (A.4):

$$S_{m+1}(j+m+1) = S_m(j-m). \tag{A.6}$$

The similarity transformation $\mathcal{K}$ is the positive Hermitian square root of S.

One may verify that the diagonal matrix $\mathcal{K}$ with elements

$$\mathcal{K}_{mm} = \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} \qquad \text{if } -j \leqslant m \leqslant j \tag{A.7}$$

satisfies the required recursion relation.

By simple inspection, one can backcheck the formalism and verify that this 'square root of a binomial' matrix, which we write as

$$\mathcal{K} = \sum_{m=-j}^{j} \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}\,|jm\rangle\langle jm|, \tag{A.8}$$

satisfies $[\mathcal{K}, \Gamma(\hat{h})] = [\mathcal{K}, \Gamma(\hat{e}_-)\Gamma(\hat{e}_+)] = [\mathcal{K}, \Gamma(\hat{e}_+)\Gamma(\hat{e}_-)] = 0$, and is such that

$$\gamma(\hat{h}) \equiv \mathcal{K}^{-1}\Gamma(\hat{h})\mathcal{K} = \Gamma(\hat{h}),$$

$$\gamma(\hat{e}_-) \equiv \mathcal{K}^{-1}\Gamma(\hat{e}_-)\mathcal{K}, \qquad \gamma(\hat{e}_+) \equiv \mathcal{K}^{-1}\Gamma(\hat{e}_+)\mathcal{K} = \gamma(\hat{e}_-)^\dagger.$$

One immediately checks that the resulting γ is indeed the standard Hermitian realization with action given in (2.2).

Having established that Γ is equivalent to (2.2), let us compare some aspects of their respective polar decomposition. The matrix realization of $\Gamma(\hat{e}_-)$ is factored as

$$\Gamma(\hat{e}_-) = \Gamma(\hat{E})\,\Gamma(\hat{D}), \tag{A.10}$$

where $\Gamma(\hat{D}) = [\Gamma(\hat{e}_-)^\dagger\,\Gamma(\hat{e}_-)]^{1/2}$. Following equations (2.15), we find

$$\Gamma(\hat{D}) = \sum_{m=-j}^{j}(j-m)|jm\rangle\langle jm|, \tag{A.11}$$

$$\Gamma(\hat{E}) = \sum_{m=-j}^{j}|jm-1\rangle\langle jm|. \tag{A.12}$$

In the same manner, using $\gamma(e_-) = \gamma(E)\sqrt{\gamma^\dagger(e_-)\gamma(e_-)}$, it is established that

$$\gamma(\hat{E}) = \sum_{m=-j}^{j}|jm-1\rangle\langle jm|. \tag{A.13}$$

Equations (A.13) and (A.12) are identical, but neither is completely defined because the rank of the matrix is one less than the dimension $2j+1$ of V_j: the matrix element of $\hat{E}$ calculated between the lowest weight state $|jj\rangle$ and any other state $|jm\rangle$ is not determined. We can make $\hat{E}$ into a unitary matrix by taking m modulo $2j+1$, and thus going from the finite line to the circle. If we also impose the same cyclicity condition for $\Gamma(E)$, we find $\Gamma(\hat{E})$ and $\gamma(\hat{E})$ coincide. With the cyclic boundary conditions, the final result is

$$\hat{E} = \sum_m |jm-1\rangle\langle jm|, \qquad -j \leqslant m \leqslant j. \tag{A.14}$$

A direct extension of this argument applies to the Hermitian extension of the analogous representation (3.13) for $\mathfrak{su}(3)$. Details on the construction of the $\mathcal{K}$ matrix for $\mathfrak{su}(3)$ can be found in [39].

References

[1] Scully M O and Zubairy M S 1997 *Quantum Optics* (Cambridge: Cambridge University Press)
[2] Hradil Z 1995 *Phys. Rev.* A **51** 1870–73
[3] Holland M J and Burnett K 1993 *Phys. Rev. Lett.* **71** 1355–8
[4] Giovannetti V, Lloyd S and Maccone L 2004 *Science* **306** 1330–6
[5] Pezzé L and Smerzi A 2008 *Phys. Rev. Lett.* **100** 073601
[6] Bahder T B 2011 *Phys. Rev.* A **83** 053601
[7] Demkowicz-Dobrzański R 2011 *Phys. Rev.* A **83** 061802
[8] Schleich W, Bandilla A and Paul H 1992 *Phys. Rev.* A **45** 6652–4
[9] Leonhardt U and Paul H 1993 *Phys. Rev.* A **47** R2460–3
[10] Freyberger M, Vogel K and Schleich W 1993 *Quantum Opt.* **5** 65–9
[11] Fiurášek J 2000 *Phys. Rev.* A **62** 013822
[12] Dubin D A, Hennings M A and Smith T B 2000 *Mathematical Aspects of Weyl Quantization and Phase* (Singapore: World Scientific)
[13] Schleich W P 2001 *Quantum Optics in Phase Space* (Berlin: Wiley-VCH)
[14] Shapiro J H and Wagner S S 1984 *IEEE J. Quantum Electron.* **20** 803–13
[15] Shapiro J H, Shepard S R and Wong W C 1989 *Phys. Rev. Lett.* **62** 2377–80
[16] Noh J W, Fougères A and Mandel L 1991 *Phys. Rev. Lett.* **67** 1426–9
[17] Noh J W, Fougères A and Mandel L 1992 *Phys. Rev.* A **46** 2840–52
[18] Noh J W, Fougères A and Mandel L 1992 *Phys. Rev.* A **45** 424–42
[19] London F 1926 *Z. Phys.* **37** 915–25
[20] Dirac P A M 1927 *Proc. R. Soc.* A **114** 243–65
[21] Heitler W 1954 *The Quantum Theory of Radiation* 3rd edn (Oxford: Oxford University Press)
[22] Lynch R 1995 *Phys. Rep.* **256** 368
[23] Tanas R, Miranowicz A and Gantsog T 1996 *Quantum Phase Properties of Nonlinear Optical Phenomena* (*Progress in Optics* vol 35) ed E Wolf (Amsterdam: Elsevier) pp 355–446
[24] Peřinová V, Lukš A and Peřina J 1998 *Phase in Optics* (Singapore: World Scientific)
[25] Barnett S M and Vaccaro J A 2007 *The Quantum Phase Operator: A Review* (London: Chapman and Hall)
[26] Luis A and Sanchez-Soto L L 1996 *Phys. Rev.* A **53** 495–501
[27] Luis A and Sánchez-Soto L L 2000 *Quantum Phase Difference, Phase Measurements and Quantum Stokes Parameters* (*Progress in Optics* vol 41) ed E Wolf (Amsterdam: North-Holland) pp 421–81
[28] Chuang I and Nielsen M 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[29] Daoud M and Kibler M R 2010 *J. Phys. A: Math. Theor.* **43** 115303
[30] Kibler M R 2008 *J. Phys. A: Math. Theor.* **41** 375302
[31] Lévy-Leblond J M 1973 *Rev. Mex. Fis.* **22** 15–24
[32] Vourdas A 1990 *Phys. Rev.* A **41** 1653–61
[33] Vourdas A 1993 *Phys. Scr.* T **48** 84–7
[34] Vourdas A 1993 *Phys. Rev.* A **45** 1943–50
[35] Chaichian M and Ellinas D 1990 *J. Phys. A: Math. Gen.* **23** L291–6
[36] Luis A and Sánchez-Soto L L 1993 *Phys. Rev.* A **48** 4702–08
[37] Sánchez-Soto L L and Luis A 1994 *Opt. Commun.* **105** 84–8
[38] Luis A and Sánchez-Soto L L 1997 *Phys. Rev.* A **56** 994–1006
[39] de Guise H and Bertola M 2002 *J. Math. Phys.* **43** 3425–44
[40] de León P L G and Gazeau J P 2007 *Phys. Lett.* A **361** 301–4
[41] Schwinger J 1965 *Quantum Theory of Angular Momentum* (New York: Academic)
[42] Chaturvedi S, Marmo G and Mukunda N 2006 *Rev. Math. Phys.* **18** 887–912
[43] Lancaster P and Tismenetsky M 1985 *The Theory of Matrices* (London: Academic)
[44] Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* A **6** 2211–37
[45] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[46] Gazeau J P 2009 *Coherent States in Quantum Physics* (Weinheim: Wiley-VCH)
[47] Hall B C 2003 *Lie Groups, Lie Algebras, and Representations* (New York: Springer)

[48] Klimov A B, Sánchez-Soto L L, de Guise H and Björk G 2004 *J. Phys. A: Math. Gen.* **37** 4097–106
[49] Daoud M and Kibler M R 2011 *J. Math. Phys.* **52** 082101
[50] Klimov A B, Sanchez-Soto L L and de Guise H 2005 *J. Opt. B: Quantum Semiclass. Opt.* **7** 283–7
[51] Schwinger J 1960 *Proc. Natl Acad. Sci. USA* **46** 570–6
[52] Schwinger J 1960 *Proc. Natl Acad. Sci. USA* **46** 883–97
[53] Schwinger J 1960 *Proc. Natl Acad. Sci. USA* **46** 1401–15
[54] Patera J and Zassenhaus H 1988 *J. Math. Phys.* **29** 665–73
[55] Patera J 1989 *J. Math. Phys.* **30** 2756–62
[56] Zanette B and de Guise H 2012 *Phys. Scr.* **2012** 014032
[57] Rowe D J, Blanc R L and Repka J 1989 *J. Phys. A: Math. Gen.* **22** L309–16